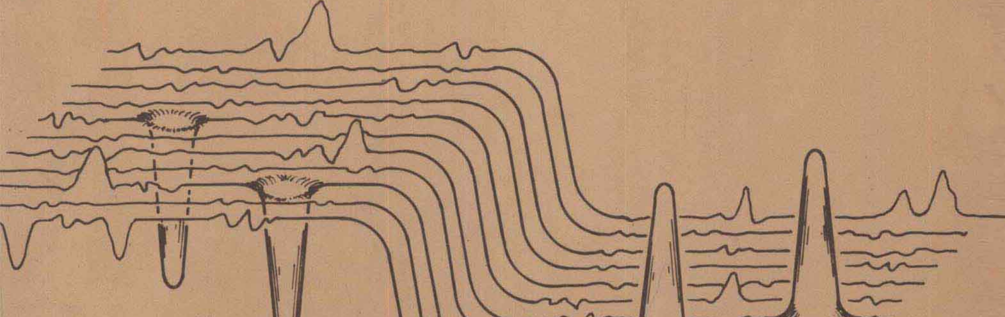
James Glimm
Arthur Jaffe

Quantum Physics

A Functional Integral Point of View



Springer-Verlag
New York Heidelberg Berlin

James Glimm
Arthur Jaffe

Quantum Physics

A Functional Integral Point of View

With 43 Illustrations

Springer-Verlag
New York Heidelberg Berlin

James Glimm
The Rockefeller University
New York, NY 10021
USA

Arthur Jaffe
Harvard University
Cambridge, MA 02138
USA

Library of Congress Cataloging in Publication Data

Glimm, James,
Quantum physics.

Bibliography: p.
Includes index.
1. Quantum field theory. 2. Quantum theory.
3. Statistical physics. I. Jaffe, Arthur, joint
author. II. Title.
QC174.45.J33 530.1'43 81-17

Printed in the United States of America.

9 8 7 6 5 4 3 2 1

ISBN 0-387-90551-0 Springer-Verlag New York Heidelberg Berlin (hard cover)
ISBN 3-540-90551-0 Springer-Verlag Berlin Heidelberg New York (hard cover)
ISBN 0-387-90562-6 Springer-Verlag New York Heidelberg Berlin (soft cover)
ISBN 3-540-90562-6 Springer-Verlag New York Heidelberg Berlin (soft cover)

Quantum Physics

Introduction

This book is addressed to one problem and to three audiences.

The problem is the mathematical structure of modern physics: statistical physics, quantum mechanics, and quantum fields. The unity of mathematical structure for problems of diverse origin in physics should be no surprise. For classical physics it is provided, for example, by a common mathematical formalism based on the wave equation and Laplace's equation. The unity transcends mathematical structure and encompasses basic phenomena as well. Thus particle physicists, nuclear physicists, and condensed matter physicists have considered similar scientific problems from complementary points of view.

The mathematical structure presented here can be described in various terms: partial differential equations in an infinite number of independent variables, linear operators on infinite dimensional spaces, or probability theory and analysis over function spaces. This mathematical structure of quantization is a generalization of the theory of partial differential equations, very much as the latter generalizes the theory of ordinary differential equations. Our central theme is the quantization of a nonlinear partial differential equation and the physics of systems with an infinite number of degrees of freedom.

Mathematicians, theoretical physicists, and specialists in mathematical physics are the three audiences to which the book is addressed.

Each of the three parts is written with a different scientific perspective. Part I is an introduction to modern physics. It is designed to make the treatment of physics self-contained for a mathematical audience; it covers quantum theory, statistical mechanics, and quantum fields. Since it is addressed primarily to mathematicians, it emphasizes conceptual structure—the definition and formulation of the problem and the meaning of the answer—rather than techniques of solution. Because the emphasis differs from that of conventional physics texts, physics students may find this part a useful supplement to their normal texts. In particular, the development of quantum mechanics through the Feynman–Kac formula and the use of function space integration may appeal to physicists who want an introduction to these methods.

Part II presents quantum fields. Boson fields with polynomial self-interaction in two space-time dimensions—$P(\phi)_2$ fields—are constructed. This treatment is mathematically complete and self-contained, assuming some knowledge of Hilbert space operators and of function space integrals. The original construction of the authors has been replaced by successive improvements and simplifications accumulated for more than a decade. This development is due to the efforts of a small and dedicated group of some thirty constructive field theorists including Fröhlich, Guerra, Nelson, Osterwalder, Rosen, Schrader, Simon, Spencer, and Symanzik, as well as the authors. Physicists may find Part II useful as a supplement to a conventional quantum field text, since the mathematical structure (normally omitted from such texts) is developed here.

Part II contains the resolution of a scientific controversy. For years physicists and mathematicians questioned whether nonlinear field theory is compatible with relativistic quantum mechanics. Could quantization defined by renormalized perturbation theory be implemented mathematically? The mathematically complete construction of $P(\phi)_2$ fields presented here and the construction of Yukawa$_{2,3}$, ϕ_3^4, sine-Gordon$_2$, Higgs$_2$, etc., fields in the literature provide the proof. Central among the issues resolved by this work is the meaning of renormalization outside perturbation theory. The mathematical framework for this analysis includes the theory of renormalization of function space integrals. From the viewpoint of mathematics the implementation of these ideas has involved essentially the creation of a new branch of mathematics.

Whether the equations are mathematically consistent in four space-time dimensions has not been resolved. There is speculation, for example, that the equations for coupled photons and electrons (in isolation from other particles) may be inconsistent, but that the inclusion of coupling to the quark field may give a consistent set of equations. A proper discussion of this issue is beyond the scope of this book, but is alluded to in Chapters 6 and 17.

Particle interaction, scattering, bound states, phase transitions, and critical point theory form the subject of Part III. Here we develop the consequences of the Part II existence theory and make contact with issues of broad concern to physics. This part of the book is written at a more advanced level, and is addressed mainly to theoretical and mathematical physicists. It is neither self-contained nor complete, but is intended to develop central ideas, explain main results of a mathematical nature, and provide an introduction to the literature.

Condensed matter physicists may find interesting the discussion of phase transitions and critical phenomena. The central matters are series expansions and correlation bounds. These methods find application in diverse areas. We give detailed justification of the connection (by analytic continuation) between quantum fields and classical statistical mechanics. Professional physicists could well start directly in Part III, returning to earlier material only as necessary.

Readers interested in the historical development of constructive quantum field theory are referred to the various survey articles of the authors and others. In this book the specific, detailed references are minimized, especially in the self-contained Parts I and II. A large bibliography has been included; we apologize for the inevitable omissions.

Numerous colleagues, students, and friends helped make this book possible. Of particular importance were R. D'Arcangelo, R. Brandenberger, B. Drauschke, J.-P. Eckmann, J. Gonzalez, W. Minty, K. Peterson, P. Petti, the staff at Springer Verlag, and especially our wives Adele and Nora. We are also grateful to the ETH, the IHES, the University of Marseilles and the CEN Saclay for hospitality as well as to the Guggenheim Foundation and the NSF for support.

Conventions and Formulas

Fourier transforms

$$f(x) = (2\pi)^{-d/2} \int e^{ipx} f^{\sim}(p)\, dp,$$

$$f^{\sim}(p) = (2\pi)^{-d/2} \int e^{-ipx} f(x)\, dx,$$

$$f(\theta) = (2\pi)^{-d/2} \sum e^{in\theta} f^{\sim}(n),$$

$$f^{\sim}(n) = (2\pi)^{-d/2} \int_0^{2\pi} e^{-in\theta} f(\theta)\, d\theta.$$

Minkowski vectors

$$x = (x_0, \mathbf{x}) = (x_0, \ldots, x_{d-1}),$$

$$x^2 = x \cdot x = -x_0^2 + \mathbf{x}^2, \qquad p^2 = p \cdot p = -p_0^2 + \mathbf{p}^2,$$

$$x \cdot p = \sum x_i p^i = -x_0 p_0 + \mathbf{x} \cdot \mathbf{p},$$

$$\Box = -\partial_t^2 + \Delta = -\partial x_0^2 + \sum_{i=1}^{d-1} \partial x_i^2.$$

Euclidean vectors

$$x_d = ix_0,$$

$$x^2 = x \cdot x = \sum_{i=1}^{d} x_i^2,$$

$$\Delta = \sum_{i=1}^{d} \partial x_i^2.$$

Schrödinger's equation

$$\hbar = h/2\pi,$$

$$i\hbar\dot{\theta} = H\theta, \qquad \theta(t) = e^{-itH/\hbar}\theta(0),$$

$$p = -i\hbar\,\frac{\partial}{\partial q}, \qquad [p(x), q(y)] = -i\hbar\,\delta(x-y)$$

Covariance operators $C_m \in \mathscr{C}_m$ satisfy

$$(-\Delta + m^2)C_m = \delta.$$

σ and γ matrices

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},$$

$$\sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

$$\gamma_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \qquad i = 1, 2, 3,$$

$$\gamma_0 = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix}, \qquad \gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3 = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix},$$

$$\not{a} = \sum a_\mu \gamma_\mu,$$

$$\not{a}^2 = \sum a_\mu^2 = a^2.$$

Dirac equation (zero field)

$$(h\not{\partial} - mc)\psi = 0.$$

Dirac equation in external field A

$$\left(h\not{\partial} + i\frac{e}{c}\not{A} - mc\right)\psi = 0.$$

List of Symbols

a, a^*, A, A^*	annihilation and creation operators
a, A	free energy
A	antisymmetrization operator
$\mathscr{A}$	action
$\mathfrak{A}, \mathscr{A}$	algebra of operators
b	bond
B	observable; region in space-time
$\mathscr{B}$	set of bonds
c	diagonal values of C, $c(x) = C(x, x)$; critical (as a subscript); constant
cr	critical
cl	classical
C	covariance; (Chap. 7) complex numbers
$\mathscr{C}, \mathscr{C}_m$	a class of covariance operators (Sec. 7.9)
d	dimension of space-time
D	Dirichlet boundary conditions
$\mathscr{D}$	domain of an operator; C_0^∞ test function space
$\mathscr{D}'$	Schwartz distribution space
$\mathscr{D}^{(j)}$	domain for irreducible spin j representation of SU(2, C)
E	energy level; eigenvalue for H; Euclidean transformation; Euclidean group

$\mathscr{E}$	Euclidean group; Euclidean Hilbert space; time strip (Section 10.5)
f	test function; free energy
$\mathscr{F}$	Fock space
g	test function
$\mathscr{G}$	group
h, $\hbar$	Planck's constant
h	external field
H	Hamiltonian
HS	Hilbert–Schmidt
$\mathscr{H}(\mathbf{x})$	Hamiltonian density
$\mathscr{H}$	Hilbert space of quantum states
I	identity operator
J	interaction strength for Ising ferromagnet
j, $\mathbf{J}$	angular momentum
k	Boltzmann's constant
$\mathscr{K}$	kernel of semigroup
K	kernel of Bethe–Salpeter equation
L	angular momentum (Section 15.1)
L_s, L_i, L	lines in Feynman graphs (self-interacting, interacting)
$\mathscr{L}$	Lagrangian; lattice; multiple reflection norm (Section 10.5); Lorentz group
m, M	mass; magnetization; multiple reflection norm (Section 10.5)
n	number of field components; degree of polynomial P
nn	nearest neighbor
N	Neumann boundary condition; $N(f)$ = norm of f.
$\mathscr{N}$	null space for inner product
p	period boundary conditions; pressure; Lebesgue index; degree of polynomial P
p, P	momenta; momentum operator; momentum space
P	polynomial interaction; projection operator
P_n	Hermite polynomial
q, Q	configuration; configuration space; Lebesgue index
R	real numbers; multiple reflection norm (Section 10.5)
R^d	Euclidean d-space
s	time

s, S	entropy
S	generating function; Schwinger function; sphere; symmetrization operator
$\|S^n\|$	volume of n-sphere
$\mathscr{S}$	Schwartz space of rapidly decreasing test functions
$\mathscr{S}'$	Schwartz space of tempered distributions
$\mathfrak{S}_n$	symmetric group on n elements (permutation group)
t	Euclidean time $(=x_d)$; Minkowski time $(=x_0)$
T	time ordering; truncation
U, V	unitary operator on Hilbert space
V	potential
W	Wightman function
dW	Wiener measure
$\mathscr{W}$	Wiener path space
$\mathscr{X}$	phase space
x	point in space time
$\mathbf{x}$	point in space
z	fugacity; activity
Z	partition function; field strength renormalization constant; integers
Z_+	nonnegative integers; partition function
β	$(1/kT)$ inverse temperature
γ	critical exponent
γ, Γ	boundary; phase boundary
Γ	Dirichlet boundary conditions on Γ; inverse to propagator or two point function
$\|\Gamma\|$	length or area of Γ
δ	Dirac δ function; Kronecker δ function; lattice spacing; critical exponent
Δ	Laplacian; special solution of wave or Laplace equation (propagator) also unit square
ε	$2\theta - 1$ (a type of Heaviside function); lattice spacing; reduced temperature $(T - T_c)/T_c$
ζ, η	critical exponents
θ	reflection operator; Heaviside function; state in $\mathscr{H}$
κ	momentum cutoff
λ	coupling constant
Λ	bounded region of space

$\|\Lambda\|$	area or volume of Λ
μ	$(-\Delta + m^2)^{1/2} = (p^2 + m^2)^{1/2}$; chemical potential; external field
$d\mu$	statistical weight or ensemble
ν	frequency; critical exponent
$d\nu$	statistical weight or ensemble
ξ	random variable
Ξ	partition function
π	3.14159; momentum conjugate to field ϕ
Π	projection operator; hyperplane
$\Pi_\pm$	half spaces of $R^d\backslash\Pi$
ρ	density
σ	mass^2; Ising spin variable; time
Σ	proper self-energy
ϕ, Φ	quantum field; configuration of classical field
$d\Phi_C$	Gaussian measure, covariance C
χ	susceptibility; random variable; state in $\mathscr{H}$; characteristic function
ψ	quantum field; state in $\mathscr{H}$
ω	frequency; Wiener path; angular integration variable
Ω	vacuum state; ground state; equilibrium state
∂	derivative; boundary operator
∇	gradient; divergence
$\|\cdot\|$	absolute value; area, volume or number of $\cdot$; norm
$\hat{}$	projection operator from the Euclidean path space to the Hilbert space of quantum states
$\tilde{}$	Fourier transform
$\langle\ ,\ \rangle$	inner product
$\langle\ \rangle$	expectation; integral with respect to $d\mu$
$[\ ,\ ]$	commutator: $[a, b] = ab - ba$
$\{\ ,\ \}$	anticommutator: $\{a, b\} = ab + ba$
$\varnothing$	free boundary conditions; empty set
$\times$	vector product
$\cdot$	time derivative; position for missing variable as in $f(\cdot) = f$ for a function f.
$\backslash$	set theoretic difference: $A\backslash B = \{x: x \in A, x \notin B\}$
$\bar{}$	complex conjugation; closure

Contents

Introduction
Conventions and Formulas
List of Symbols

PART I An Introduction to Modern Physics

1	**Quantum Theory**	3
	1.1 Overview	3
	1.2 Classical Mechanics	3
	1.3 Quantum Mechanics	7
	1.4 Interpretation	11
	1.5 The Simple Harmonic Oscillator	12
	1.6 Coulomb Potentials	20
	1.7 The Hydrogen Atom	24
	1.8 The Need for Quantum Fields	27
2	**Classical Statistical Physics**	28
	2.1 Introduction	28
	2.2 The Classical Ensembles	30
	2.3 The Ising Model and Lattice Fields	36
	2.4 Series Expansion Methods	37

3 **The Feynman–Kac Formula** 42

3.1 Wiener Measure 42
3.2 The Feynman–Kac Formula 46
3.3 Uniqueness of the Ground State 49
3.4 The Renormalized Feynman–Kac Formula 51

4 **Correlation Inequalities and the Lee–Yang Theorem** 55

4.1 Griffiths Inequalities 55
4.2 The Infinite Volume Limit 58
4.3 ξ^4 Inequalities 59
4.4 The FKG Inequality 63
4.5 The Lee–Yang Theorem 64
4.6 Analyticity of the Free Energy 67
4.7 Two Component Spins 70

5 **Phase Transitions and Critical Points** 72

5.1 Pure and Mixed Phases 72
5.2 The Mean Field Picture 74
5.3 Symmetry Breaking 77
5.4 The Droplet Model and Peierls' Argument 80
5.5 An Example 85

6 **Field Theory** 88

6.1 Axioms 88
(i) Euclidean Axioms 88
(ii) Minkowski Space Axioms 95
6.2 The Free Field 98
6.3 Fock Space and Wick Ordering 104
6.4 Canonical Quantization 109
6.5 Fermions 113
6.6 Interacting Fields 116

PART II Function Space Integrals

7 **Covariance Operator = Green's Function = Resolvent Kernel = Euclidean Propagator = Fundamental Solution** 123

7.1 Introduction 123
7.2 The Free Covariance 125
7.3 Periodic Boundary Conditions 127
7.4 Neumann Boundary Conditions 128
7.5 Dirichlet Boundary Conditions 129

7.6 Change of Boundary Conditions 130
7.7 Covariance Operator Inequalities 130
7.8 More General Dirichlet Data 132
7.9 Regularity of C_B 137
7.10 Reflection Positivity 141

8 Quantization = Integration over Function Space 144

8.1 Introduction 144
8.2 Feynman Graphs 145
8.3 Wick Products 147
8.4 Formal Perturbation Theory 150
8.5 Estimates on Gaussian Integrals 152
8.6 Non-Gaussian Integrals, $d = 2$ 157
8.7 Finite Dimensional Approximations 163

9 Calculus and Renormalization on Function Space 166

9.1 A Compilation of Useful Formulas 166
(i) Wick Product Identities 166
(ii) Gaussian Integrals 169
(iii) Integration by Parts 171
(iv) Limits of Measures 172
9.2 Infinitesimal Change of Covariance 174
9.3 Quadratic Perturbations 175
9.4 Perturbative Renormalization 179
9.5 Lattice Laplace and Covariance Operators 183
9.6 Lattice Approximation of $P(\phi)_2$ Measures 189

10 Estimates Independent of Dimension 193

10.1 Introduction 193
10.2 Correlation Inequalities for $P(\phi)_2$ Fields 193
10.3 Dirichlet or Neumann Monotonicity and Decoupling 195
10.4 Reflection Positivity 198
10.5 Multiple Reflections 200
10.6 Nonsymmetric Reflections 206

11 Fields Without Cutoffs 213

11.1 Introduction 213
11.2 Monotone Convergence 213
11.3 Upper Bounds 215

12 Regularity and Axioms 219

12.1 Introduction 219
12.2 Integration by Parts 220

12.3 Nonlocal ϕ^j Bounds 223
12.4 Uniformity in the Volume 225
12.5 Regularity of the $P(\phi)_2$ Field 229

PART III The Physics of Quantum Fields

13 Scattering Theory: Time-Dependent Methods 237

13.1 Introduction 237
13.2 Multiparticle Potential Scattering 240
13.3 The Wave Operator for Quantum Fields 244
13.4 Wave Packets for Free Particles 247
13.5 The Haag–Ruelle Theory 250

14 Scattering Theory: Time-Independent Methods 256

14.1 Time-Ordered Correlation Functions 256
14.2 The S Matrix 259
14.3 Renormalization 260
14.4 The Bethe–Salpeter Kernel 264

15 The Magnetic Moment of the Electron 270

15.1 Classical Magnetic Moments 270
15.2 The Fine Structure of the Hydrogen Atom and the Dirac Equation 272
15.3 The Dirac Theory 274
15.4 The Anomalous Moment 276
15.5 The Hyperfine Structure and the Lamb Shift of the Hydrogen Atom 279

16 Phase Transitions 280

16.1 Introduction 280
16.2 The Two Phase Region 284
16.3 Symmetry Unbroken, $d = 2$ 294
16.4 Symmetry Broken, $3 \le d$ 297

17 The ϕ^4 Critical Point 304

17.1 Elementary Considerations 304
17.2 The Absence of Even Bound States 305
17.3 A Bound on the Coupling Constant λ_{phys} 307
17.4 Existence of Particles and a Bound on $dm^2/d\sigma^2$ 309
17.5 Existence of the ϕ^4 Critical Point 310