SECOND EDITION

Operational Research

W. M. Harper & H. C. Lim

THE M & E HANDBOOK SERIES

Operational Research

W. M. Harper
ACMA, MBIM

H. C. Lim
MBA, ACA

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Preface to the Second Edition

More and more the operations of business concerns are being subjected to mathematical analysis. Inevitably, therefore, the student of business is required to look ever more closely at the concepts and techniques that have emerged from prior studies of this sort, and examining bodies expect ever-increasing knowledge and comprehension in this field. For the mathematically-minded young full-time student with an extensive academic background this new demand by business presents little in the way of problems, but for his more practical contemporary whose time and effort have been more directed towards doing a job than researching it, it is a totally different matter. With only a relatively elementary mathematical education (and one often concluded many years previously at that) and a lack of familiarity with "throwing around" mathematical concepts, expressions and formulae, these new methods of analysing the business he understands only too well in an empirical way appear incomprehensible and impractical and, worst of all, lacking any clearly defined and easily found logical path that will take him to a point where he can obtain a confident grasp of their purpose and relevance.

It is primarily for this student that this book has been written. In the first edition all the basic mathematical theory was covered in an opening part but the continued and remorseless increase in the sophistication of the examination questions coupled with the rising mathematical standard of new students has made this practice unrealistic—the former development causing a problem regarding the book's length and the latter resulting in many students not now needing quite such a basic introduction. As a result the required mathematics is merely revised in this edition, although for those students who lacked the opportunity to reach the appropriate level the HANDBOOK *Basic Mathematics* has been published in parallel in this series. This latter text fully covers the introductory mathematics—the one or two necessary topics not dealt with there (incremental techniques and probability distributions) being included in this book alongside the revision material in Part I.

What's in a name? The introduction of relatively advanced mathematics into business analysis is, on an historical timescale, so new that the terminology is still unstandardised. As will be clear from Chapter V, it is arguable that this book would be more appropriately titled "Quantitative Techniques", since it is this aspect of the broader topic of operational research that is discussed. Nevertheless, quantitative techniques are so widely believed to represent the whole of operational research that a person looking for a first text on the subject of quantitative techniques would be more likely to expect to find what he was looking for under the title "Operational Research" than any other. For this reason this book has been given the title it has. While possibly academically inaccurate, it does at least describe the book in terms the potential reader would probably use himself and in this sense is an accurate indication of the book's contents.

"Ballads, songs and snatches." It should, perhaps, be made clear that operational research is essentially practical even though mathematical theory is used in its application. This means the subject grew as a series of more or less disconnected techniques. In consequence, one chapter does not necessarily lead logically to the next—rather each chapter stands relatively independent of whatever preceded it (apart, that is, from the mathematics covered in Part I). As a result it is not necessary to understand fully one chapter before reading further in the book. Some chapters are relatively more difficult than others and if the student finds one that is rather hard to grasp at first reading he should nevertheless proceed to the next chapter in good heart for in all probability the earlier insurmountable hurdles will not again appear in the book.

Plan of book. The fragmented nature of the subject of operational research has made the book rather hard to plan. However, after due thought it was decided that, as in the first edition, there should be a first part dealing with the mathematics employed in operational research and after that a second part covering those operational techniques that could be used in the application of the relatively more specialised techniques. Thus, the approach to operational research, modelling, decision theory and simulation may all play a part in a replacement study or a stock control decision. And these more specialised techniques naturally then fall into the third Part of the book—though their order is somewhat arbitrary.

Progress Tests. Most of the chapters conclude with a Progress Test which usually involves practical work. This work is very often designed not only to test the student's comprehension but also to illustrate *additional* points relevant to the topic dealt with in the chapter, particularly in connection to the approach to problem solutions. Ideally the student should attempt all these tests but if he feels he has grasped the main points in the chapter and time is short then although he may not attempt the tests he should at least study the questions and answers to ensure this additional material is not overlooked.

Examination questions. This edition concludes with a completely new set of examination questions. Although it is not necessarily conceded that the later a question the better it tests the student's ability, it is accepted that the examination language does change over the years and familiarity with the language is a relevant factor to students taking a forthcoming examination. Consequently no questions in Appendix V are taken from papers prior to 1976.

Bibliography. For further reading the student is referred to the following books:

For the required basic mathematical theory:

W. M. Harper and L. W. T. Stafford, *Basic Mathematics* (M & E HANDBOOK series)

For further general reading which parallels this book:

L. W. T. Stafford, *Business Mathematics* (M & E HANDBOOK series)

M. S. Makower and E. Williamson, *Teach Yourself Operational Research* (English Universities Press)

Albert Battersby, *Mathematics in Management* (Pelican)

N. W. Marsland, *Quantitative Techniques for Business* (Polytech Publishers)

Kemeny, Schleifer, Snell, Thompson, *Finite Mathematics with Business Applications* (Prentice Hall)

For a greater emphasis on probability and statistics in quantitative techniques:

Bierman, Bonini, Fouraker and Jaedicke, *Quantitative Analysis for Business Decisions* (Richard D. Irwin)

R. Schlaifer, *Probability and Statistics for Business Decisions* (McGraw-Hill)

For a more detailed treatment of stock control:
Albert Battersby, *A Guide to Stock Control* (Pitman)

For a more detailed treatment of network analysis:
K. G. Lockyer, *An Introduction to Critical Path Analysis* (Pitman)

For a set of case studies illustrating the use of quantitative techniques:
Phillip G. Carlson, *Quantitative Techniques for Management* (Harper & Row)

Acknowledgments. We gratefully acknowledge permission to quote from the past examination papers of the following bodies:

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Institute of Chartered Accountants in England and Wales (CA)
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W.M.H.
H.C.L.

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PART ONE

FUNDAMENTAL MATHEMATICAL TECHNIQUES

CHAPTER I

Basic Mathematics

Students hardly need to be told that operational research leans heavily on mathematical techniques and to follow a book on the subject requires a knowledge of the more basic elements of business mathematics. In the HANDBOOK series this need is catered for by the book *Basic Mathematics*. However, many students will almost certainly approach operational research with sufficient mathematical background to make the reading of that book unnecessary. For those students, and also those who having read *Basic Mathematics* would find a summary of the main points useful, this chapter summarises the mathematics needed for the rest of this book.

It must be emphasised, however, that this is *only* a summary and so any student who finds himself struggling with any part of it is advised to return to his other mathematical texts and revise the topic that is giving him trouble.

1. Reading algebraic expressions. When reading algebraic expressions note the following points.

(*a*) *Variables, coefficients and constants.* A *variable* is a symbol that in the context involved varies in value from one situation to another; a *coefficient* is any value by which a variable is multiplied; a *constant* is any value that remains unchanged throughout. Thus, in the expression $3x + 6$, x is a variable, "3" is a coefficient and "6" is a constant.

(*b*) *Limits.* The concept of "limit" is used in a context in which a variable increasingly approaches a given value or limit. Thus, if one progressively added to a cumulative length one-half the length previously added, then after an infinite number of additions one would have a cumulative length equal to twice the starting length (as a moment's thought will show). This can be shown symbolically as $\underset{x \to \infty}{L}\left(y + \frac{y}{2^1} + \frac{y}{2^2} + \frac{y}{2^3} \ldots \frac{y}{2^x}\right) = 2y$, where $\underset{x \to \infty}{L}$ reads "at the limit when $x = \infty$".

(*c*) *Indices*. An *index* number symbolises the number of times a given figure is multiplied by itself. A *negative index* symbolises the reciprocal of the value one would have obtained had the index been positive: e.g. $x^{-2} = \frac{1}{x^2}$. The denominator of a *fractional index* symbolises the root of the given figure—the numerator being a normal index number (e.g. $x^{1/3} = \sqrt[3]{x^1}$). A *mixed index* must be read as a combination of the simpler indices: e.g. $x^{-2/3} = \frac{1}{\sqrt[3]{x^2}}$

(*d*) *Suffixes*. A suffix merely *identifies* the value to which it is attached. Thus, x_3 means "the third x value"; x_i "the ith x value"; σ_x "the standard deviation of x"; x_{t-4} "the value of x at $t - 4$" (and since t usually identifies a given time period, $t - 4$ identifies the time period four periods before t).

(*e*) *The* $\sum$ *notation*. $\sum$ means "add together". Thus, $\sum x$ means "add together all the x's". Often the start and end of the addition is indicated by writing the start at the bottom of the $\sum$ and the end at the top. So $\sum_{i=5}^{10} x_i$ reads "add together all the x's starting with x_5 and ending with x_{10}". Note that the starting and ending symbol can form part of the computations—e.g. $\sum_{i=5}^{10} i$ reads "Add together all the values of i starting at $i = 5$ and ending at $i = 10$" (i.e. $5 + 6 + 7 + 8 + 9 + 10$).

(*f*) *Factorial !* The symbol ! after a value means "multiply together all the numbers found by starting with the given value and progressively reducing the value by one until 1 itself is reached" (e.g. $5! = 5 \times 4 \times 3 \times 2 \times 1$). Note that $0! = 1$.

(*g*) *Exponential* e. The symbol e represents a constant value given by the expression $\underset{x \to \infty}{L}\left(\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} \ldots \frac{1}{x!}\right)$. This value to five decimal places is 2.71828 and a table of other values of e is given in Appendix I.

2. Solving equations. Solving an ordinary equation simply involves manipulating the symbols until the unknown variable stands by itself on one side of the equation, whereas:

(*a*) *Solving simultaneous equations* can be achieved by multiplying the whole of one equation by a number which results in one of the unknown variables having exactly the same coefficient in both equations, then adding or subtracting one equation to or from the other so that a third equation results which does *not* include this variable. Then this equation is solved for the remaining variable in the normal way and the value of this variable substituted in one of the original equations so that that, too, can be solved in the normal way to give the previously eliminated variable.

(*b*) *Solving quadratic equations* of the form $ax^2 + bx + c = 0$ can be achieved by application of the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

3. Progressions. Normally speaking, a progression is a series of numbers having a common difference between terms (when it is an *arithmetical progression*) or a common ratio between terms (when it is a *geometrical progression*). The *sums* of these progressions, where these are *n* terms and the first term is *a*, are given by the formulae:

(*a*) Sum of an arithmetical progression

$$= \frac{n}{2}[2a + (n - 1)\,d]$$

where d = common difference

(*b*) Sum of a geometrical progression

$$= \frac{a(1 - r^n)}{1 - r}$$

where *r* = common ratio

EXAMPLES

Sum of the progression

$$15 + 18 + 21 + 24 \ldots 39 = \frac{9}{2}[2 \times 15 + (9 - 1) \times 3] = \underline{\underline{243}}.$$

Sum of the progression

$$9 + 3 + 1 \ldots \tfrac{1}{9} = \frac{9\,[1 - (\frac{1}{3})^5]}{1 - \frac{1}{3}} = \underline{\underline{13.444}}$$

4. Permutations and combinations. *Permutations* are arrangements of items where the *order* of the items is relevant. *Combinations* are arrangements of items where the order of the items is not relevant. The number of permutations and combinations is symbolised by ^nP_x and $\binom{n}{x}$ respectively, where n = total number of items available to make the arrangements and x = number of items actually taken to make up each arrangement; these symbols being read as "number of permutations/combinations of n items taking x at a time". The formulae for these are as follows:

$^nP_x = n^x$ where there is "selection with replacement", i.e. each item selected for inclusion in an arrangement is replaced after selection back among the items available for selection so that it can be selected again.

$$^nP_x = \frac{n!}{(n-x)!}$$ where there is "selection without replacement".

$$\binom{n}{x} = \frac{n!}{(n-x)!x!}$$

5. Calculus. Essentially calculus is used in operational research to find the slope of a curve—when *differential calculus* is used—or the area under a curve—when *integral calculus* is used. Note the following points in respect of these two techniques:

(*a*) *Differential calculus.* In differential calculus the following formulae hold:

(*i*) If $y = ax^n$ then $\frac{dy}{dx} = anx^{n-1}$.

(*ii*) If y comprises two added functions, u and v, of x (i.e. $y = u + v$), then $\frac{dy}{dx} = \frac{du}{dx} + \frac{dv}{dx}$ (e.g. $y = 5x^3 + 8x^2$ so that $u = 5x^3$ and $v = 8x^2$, then $\frac{dy}{dx} = 15x^2 + 16x$).

(*iii*) If $y = uv$ then $\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$ (e.g. if $y = uv = (x^2 - 5)(3x + 2)$ then $\frac{dy}{dx} = (x^2 - 5) \times 3 + (3x + 2) \times 2x$

(*iv*) If $y = \frac{u}{v}$ then $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dy}{dx}}{v^2}$ e.g. if $y = \frac{u}{v} = \frac{x^3 + 9}{x^{1/2}}$

then $\frac{dy}{dx} = \frac{x^{1/2} \times 3x^2 - (x^3 + 9) \times \frac{1}{2}x^{-\frac{1}{2}}}{(x^{1/2})^2}$.

Differential calculus can be used to find the maximum and/or minimum of an expression, for since the slope at these points is zero all one needs to do is to differentiate the expression, set the result equal to 0 and solve as usual. (E.g. maximum of $50 + 12x - 3x^2$ is given by $\frac{d(50 + 12x - 3x^2)}{dx} = 0$, i.e. $12 - 6x = 0$ and so $x = 2$.)

So the expression is at a maximum when $x = 2$.

(*b*) *Integral calculus*. The symbol for integration is $\int$. When the limits of a variable are written above and below this symbol, the relevant expression after the symbol and the variable (preceded by a d) written after that again, then integration "in respect of" the variable enables the area between the curve, the variable axis and the two given limits to be found (e.g. $\int_2^5 (x^2 + 1)\,dx$ enables the area between the curve $x^2 + 1$, the x axis and the limits $x = 2$ and $x = 5$ to be found).

The following formulae apply in integration:

(*i*) $\int x^n dx = \frac{1}{n + 1}x^{n+1} + A$, where A is the "constant of integration", i.e. an unknown constant (e.g. $\int x^3\,dx = \frac{1}{4}x^4 + A$).

(*ii*) $\int (u + v)\,dx = \int u\,dx + \int v\,dx$

(e.g. $\int (3x^2 + 4)\,dx = \int 3x^2\,dx + \int 4\,dx = x^3 + 4x + A$, since $4 = 4x^0$).

To find the area under the curve from the integration, simply substitute the two limiting values and subtract the second of the resulting amounts from the first (note that this will eliminate the constant of integration).

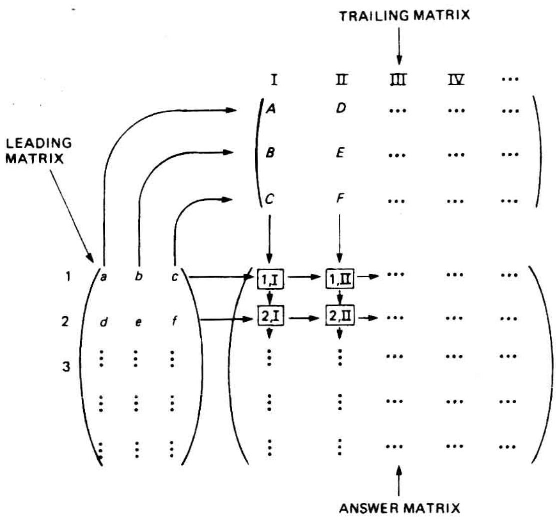
EXAMPLE: $\int_2^5 (x^2 + 1)\,dx = {}_2^5\left[\frac{1}{3}x^3 + x + A\right]$

$$= \left(\frac{125}{3} + 5 + A\right) - \left(\frac{8}{3} + 2 + A\right)$$

$$= \frac{125}{3} + 5 + A - \frac{8}{3} - 2 - A = \underline{\underline{42}}$$

6. Matrix algebra. Matrix algebra is a set of mathematical procedures in which numbers and expressions are manipulated in the form of layouts of rows and columns. Note that the number or symbol at the intersection of any row and column is referred to as an *element*, and that the term *conformable* means that the matrices to which the term refers are in a form which enables them to be manipulated together.

Given that A, B and C symbolise matrices A, B and C, and that A_{ij}, B_{ij} and C_{ij} represent the values of the elements in the ith row and jth column of A, B and C respectively, the following formulae



$$1, \mathrm{I} = aA + bB + cC \qquad 2, \mathrm{I} = dA + eB + fC$$
$$1, \mathrm{II} = aD + bE + cF \cdots 2, \mathrm{II} = dD + eE + fF$$

FIG. 1 *Summary of interconnections of elements in leading, trailing and answer matrices when carrying out matrix multiplication.*

apply to addition, subtraction and multiplication in matrix algebra (there is no division):

(*a*) *$A + B$ and $A - B$*. To be conformable A and B must have the same number of rows and the same number of columns as each other. Given this, $A + B$ results in an answer matrix C in which each element is computed as $C_{ij} = A_{ij} + B_{ij}$, while $A - B$ results in each element in answer matrix C being computed as $C_{ij} = A_{ij} - B_{ij}$.

(*b*) $A \times B$. To be conformable, the first matrix (termed the *leading matrix*) must have the same number of columns as the second matrix (called the *trailing matrix*) has rows. Given this, $A \times B$ results in an answer matrix C in which each element is computed as $C_{ij} = \sum_{k=1}^{n} A_{ik} \times B_{kj}$, where n = number of columns in A = number of rows in B (*see* Fig. 1). Note, incidentally, that where A and B are square matrices (a *square matrix* being one having the same number of rows as columns) then $A \times B \neq B \times A$.

Finally, note the following points:

When matrix A is multiplied by a single value K (called a *scalar*) then $C_{ij} = K \times A_{ij}$.

An *identity* (or *unit*) *matrix*, symbolised as I, is a square matrix with each element on the main diagonal (i.e. the diagonal running from the top left to the bottom right) a 1 and all the other elements 0's.

A *zero matrix* is a matrix in which every element is zero.

The *inverse matrix*, symbolised as A^{-1}, of a matrix A is a matrix such that $A^{-1}A = I$.

PROGRESS TEST 1

1. Find:

(*a*) $\underset{n \to \infty}{\mathrm{L}} \sum_{i=1}^{n} e^{-1} i^{-\frac{1}{2}} \frac{i}{i!}$ to three decimal places.

(*b*) $\sum_{i=3}^{10} (2i + 2^{i})$.

(*c*) p_0 in terms of T where

$$p_0 = \frac{C!(1 - T)}{(CT)^{C} + C!(1 - T)\left(\sum_{n=0}^{C-1} \frac{1}{n!}(CT)^{n}\right)}$$

and when $C = 3$.

2. A holiday schedule has to be prepared for six employees. Find the number of different schedules that can be prepared in each of the following cases:

(*a*) There are six holiday months and never more than one employee must be scheduled in any month. Consider where:

(*i*) All must take their holidays during the holiday months.

(*ii*) Five employees must take their holidays during the holiday months and one outside that period.

(*b*) There are four holiday months and three employees must take their holidays in a single month while the others must be scheduled so that there is never more than one employee scheduled in each of the three remaining months.

(*c*) There are two holiday months and three men must be scheduled for each of these months.

3. Find:

(*a*) the value of x that maximises the expression $(x^2 - 1)(1 - x^2)$.

(*b*) the area under the curve of the expression $6x^3 + 6x^2 - 6x + 3$ between the limits $x = 1$ and $x = 2$.

4. If $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$ and $B = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}$ find:

(*a*) $A - B$.

(*b*) $A \times B$.