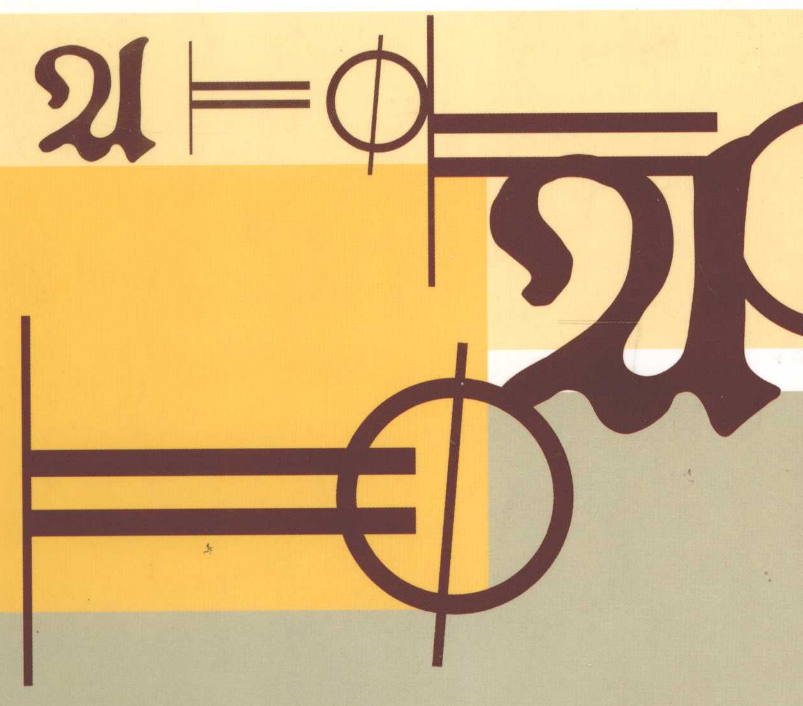
Proceedings of the 9th Asian Logic Conference

MATHEMATICAL LOGIC IN ASIA



Editors

Sergey Goncharov ϕ Rod Downey ϕ Hiroakira Ono

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MATHEMATICAL LOGIC IN ASIA

Novosibirsk, Russia 16 – 19 August 2005

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PREFACE

The Asian Logic Conference has occurred every three years since its inception in Singapore in 1981. It rotates among countries in the Asia Pacific region with interests in the broad area of logic including theoretical computer science. It is now considered a major conference in this field and is regularly sponsored by the Association for Symbolic Logic. This volume contains papers, many of them surveys by leading experts, of the 9th meeting in Novosibirsk, Russia.

We were very pleased to find that *World Scientific* were enthusiastic to support this venture. Authors were invited to submit articles to the present volume, based around talks given at either meeting. The editors were very concerned to make sure that the planned volume was of high quality. We think the resulting volume is fairly representative of the thriving logic groups in the Asia-Pacific region, and also fairly representative of the meetings themselves.

The Ninth Asian Logic Conference was organised by Sobolev Institute of Mathematics of the Siberian Branch of the Russian Academy of Sciences and Novosibirsk State University under the sponsorship of Russian Foundation for Basic Research, Association for Symbolic Logic, Department of Mechanics and Mathematics of Novosibirsk State University, Siberian Foundation for Algebra and Logic, Novosibirsk Center of Information Technologies UniPro Co., Ltd., LLC Alekta, and Transtext Co. Ltd.

The conference took place in Novosibirsk, Akademgorodok, Russia, from August 16 to August 19, 2005.

The programme consisted of plenary lectures delivered by invited speakers and contributions in four sections.

Plenary speakers were Pavel Alaev (Russia), Lev Beklemishev (Russia, Netherlands), Su Gao (USA), Yurii Ershov (Russia), Sanjay Jain (Singapore), Vladimir Kanovei (Russia), Bakhadyr Khoussainov (New Zealand), Andrei Mantsivoda (Russia), Joe Miller (USA), Hiroakira Ono (Japan), Vladimir Rybakov (Russia, Great Britain), Masahiko Sato (Japan), Moshe Vardi (USA), Andrei Voronkov (Great Britain), Xishun Zhao (China). The total number of plenary lectures was 15. Contributed lectures on recursion theory, set theory, proof theory, model theory and universal algebra, non-classical logic, and logic in computer science were presented in the following sections: Computability theory, Model theory and Set theory, Non-classical logics, Proof theory, and universal algebra, and Applications of logic in computer science. The total number of contributed talks was 58.

The geography of the event included Russia, China, Japan, Singapore, USA, New Zealand, Great Britain, Korea, Canada, Germany, Greece, Kazakhstan. The number of participants was about 100 scientists.

We are grateful to Ekaterina Fokina for the great work with authors and referees while preparing the Proceedings and to Vladimir Vlasov for making the camera-ready manuscript.

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Sincerely yours, the editors:
Rod Downey, Sergey Goncharov, and Hiroakira Ono.

CONTENTS

ANOTHER CHARACTERIZATION OF THE DEDUCTION-DETACHMENT THEOREM

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In Abstract Algebraic Logic, a Hilbert-style deductive system is identified with the set of its theories. This set of theories must be algebraic and must be closed under arbitrary intersections and inverse substitutions. Similarly, a Gentzen-style deductive system can be defined by providing a set of theories with similar properties, but now each theory must be a set of sequents, not just formulas. There are various kinds of Gentzen-style structures that naturally arise in connection with Hilbert systems, but in generally they fall short of being Gentzen systems. One of such structures is a family of axiomatic closure relations. Each of axiomatic closure relations is defined as a set of consequences that can be derived in the Hilbert system by modulo of some its theory, taken as the set of additional axioms. The main result of this work is the proof that a Hilbert system $\mathcal{S}$ admits the Deduction-Detachment Theorem if and only if the set of all axiomatic closure relations for $\mathcal{S}$ forms a Gentzen system.

1. Introduction

In Algebraic Logic, an abstract Hilbert-style deductive system $\mathcal{S}$ is identified with a family of sets, called $\mathcal{S}$-theories, of formulas of a given propositional language, such that this family, usually denoted by $\mathrm{Th}\,\mathcal{S}$, is

1) closed under arbitrary intersections, i.e., $\mathrm{Th}\,\mathcal{S}$ is a *closure system*,

2) closed under unions of upward-directed families, i.e., $\mathrm{Th}\,\mathcal{S}$ is *algebraic*,

3) closed under inverse substitutions, i.e., a preimage of any $\mathcal{S}$-theory under an arbitrary substitution is an $\mathcal{S}$-theory again.

We call $\mathcal{S}$ "abstract" because its definition does not refer to any particular axiomatization. It is easy to see that the closure operator

$$(.)^{\mathcal{S}} : X \mapsto \bigcap\{T \in \mathrm{Th}\,\mathcal{S} \mid X \subset T\},$$

associated with such closure system $\mathrm{Th}\,\mathcal{S}$, defines a finitary, substitutional consequence relation as follows

$$X \vdash_{\mathcal{S}} \alpha \iff \alpha \in X^{\mathcal{S}}.$$

It was suggested by researchers of Barcelona group to treat the Gentzen case similarly, and identify an abstract Gentzen-style deductive system with the set of its theories with two distinctive features

1) a theory is a set of *sequents*, i.e., sequences of formulas,

2) a substitution acts on sequents componentwise.

Out of numerous and intricate connections between Hilbert- and Gentzen-style deductive systems we will consider in this paper just one: *the deduction-detachment theorem*, discovered independently by Tarski and Herbrand. We define it in a slightly more general form.

A Hilbert-style deductive system admits the multiterm deduction-detachment theorem if there is a finite set of formulas $\Delta = \{\delta_i(x, y)\}_{i \in I}$ *such that for all formulas* α, β *and every set of formulas* Γ

$$\Gamma, \alpha \vdash_{\mathcal{S}} \beta \iff (\forall \delta \in \Delta)\ \Gamma \vdash_{\mathcal{S}} \delta(\alpha, \beta).$$

Even though the deduction-detachment theorem can and usually is formulated by the Gentzen rules, it was not known what abstract Gentzen-style deductive system corresponds to this axiomatization. It turns out that the key to this correspondence is axiomatic closure relations:

Let $\mathcal{S}$ *be an abstract Hilbert-style deductive system and* $T \in \mathrm{Th}\,\mathcal{S}$. *Then*

$$\{\langle \alpha_1, \ldots, \alpha_n, \beta \rangle \mid T, \alpha_1, \ldots, \alpha_n \vdash_{\mathcal{S}} \beta\}$$

is called an axiomatic closure relation *for* $\mathcal{S}$.

In other words, an axiomatic closure relation list all consequences that are possible in $\mathcal{S}$ if we add all formulas from some $\mathcal{S}$-theory as axioms (not axiom schemes). In this paper we will show that

An abstract Hilbert-style deductive system $\mathcal{S}$ *admits the multiterm deduction-detachment theorem if and only if the set of axiomatic closure relations for* $\mathcal{S}$ *forms an abstract Gentzen-style deductive system, i.e., it is*

1) *closed under arbitrary intersections,*

2) *closed under unions of upward-directed families,*

3) *closed under inverse substitutions.*

In the following, abstract Hilbert-style deductive systems will be referred to as simply Hilbert systems, and similarly for abstract Gentzen-style deductive systems.

2. Definitions and Preliminaries

Sometimes the contraction "iff" or the symbol " $\iff$ " will be used for the phrase "if and only if", "□" for "the end of proof" or "the end of definition",

"$\stackrel{def}{=}$" for "equals by definition", "$\forall$" stands for "for all", " $\Longrightarrow$ " for "implies".

Suppose A is a set. Then $\mathcal{P}(A) := \{X \mid X \subseteq A\}$ is the *power-set* of A. We write $X \subseteq_\omega A$ if X is a finite subset of A, furthermore $\mathcal{P}_\omega(A) := \{X \mid X \subseteq_\omega A\}$. For a family of sets $\mathcal{C} \subseteq \mathcal{P}(A)$, we define $\bigcup \mathcal{C} := \bigcup_{X \in \mathcal{C}} X$, $\bigcap \mathcal{C} := \bigcap_{X \in \mathcal{C}} X$. The *n-th cartesian power* of a non-empty set A is the set $A^n := \prod_{i \in n} A$ of all vectors of length n with elements from A. A^+ denotes $\bigcup_{n=1}^{\infty} A^n$, the set of all non-empty finite sequences of elements of A. An arbitrary element of A^+ we write as $\bar{a}$. If $\bar{a} = \langle a_1, \ldots, a_n \rangle$, we also write $\{\bar{a}\}$ for $\{a_1, \ldots, a_n\}$. A function $f : A^n \to A$ is called an n-ary *operation on* A. Instead of $f\langle \bar{a} \rangle$ or $f(\langle \bar{a} \rangle)$ we will often write $f(\bar{a})$. A *unary operation* $f : A \to A$ is also called a *mapping* on A.

A binary relation $R \subseteq A \times A$ is *reflexive* if for all $a \in A$, aRa; *symmetric* if for all $a, b \in A$, aRb implies bRa; *transitive* if for all $a, b, c \in A$, from aRb and bRc it follows that aRc; *antisymmetric* if for all $a, b \in A$, aRb and bRa implies that $a = b$. We call $R \subseteq A \times A$ a *partial order on* A if R is reflexive, transitive and antisymmetric.

If $\leqslant$ is a partial order on A and $X \subseteq A$, an element $a \in A$ such that for all $x \in X$, $x \leqslant a$ is called an *upper boundary of* X; dually, an element $a \in A$ such that for all $x \in X$, $x \geqslant a$ is called a *lower boundary of* X; $\inf X$ is the largest (if it exists) element of A among the lower boundaries of X; similarly, $\sup X$ is the smallest (if it exists) element of A among the upper boundaries of X. If inf (sup) exists for any two-element subset of A, A is called a *lower (upper) semi-lattice*. In that case, $\inf\{a, b\}$ is usually denoted by $a \wedge b$, and $\sup\{a, b\}$ as $a \vee b$, and interpreted as binary operations on A. If both $\wedge$ and $\vee$ defined for any pair of elements of A, A is called a *lattice*. If inf and sup exists for any non-empty subset of A, A is called a *complete lattice*.

For a mapping $h : A \to A$ the operator-style notation ha will be routinely used instead of the function-style notation $h(a)$. Also any mapping h defined on A can be uniquely extended to a mapping on A^+ by the following definition:

$$h\langle a_1, \ldots, a_n \rangle = \langle ha_1, \ldots, ha_n \rangle, \quad \text{for all } \langle a_1, \ldots, a_n \rangle \in A^+.$$

The latter defines a *complex* (defined on sets of elements) mapping on $\mathcal{P}(A^+)$ as follows,

$$hX = \{h\langle \bar{a} \rangle \mid \langle \bar{a} \rangle \in X\}, \quad \text{for all } X \subseteq A^+.$$

Note that the same symbol h will be used routinely for all these mappings.

A *propositional language type* is any non-empty set $\mathcal{L}$. The elements of $\mathcal{L}$ are called *functional symbols* in an algebraic context or *logical connectives* in a logical context. With $\mathcal{L}$ is associated an *arity* function $\rho : \mathcal{L} \to \omega$ such that ρf is the *arity* or *rank* of the functional symbol $f \in \mathcal{L}$. For each $n \in \omega$: $\mathcal{L}_n := \{f \in \mathcal{L} \mid \rho f = n\}$. An *algebra* $\mathbf{A}$ of type $\mathcal{L}$ is a pair $\langle A, \mathcal{L}^{\mathbf{A}}\rangle$, where A is a non-empty set called *universe* of $\mathbf{A}$ and $\mathcal{L}^{\mathbf{A}} = \{f^{\mathbf{A}} \mid f \in \mathcal{L}\}$ is a list of operations over the set A such that for every $f \in \mathcal{L}_n$, $f^{\mathbf{A}} : A^n \to A$. Members of $\mathcal{L}^{\mathbf{A}}$ are called *basic operations* of $\mathbf{A}$. If $\mathbf{A}$, $\mathbf{B}$ are algebras of the same type, then a mapping $h : A \to B$ is called a homomorphism of $\mathbf{A}$ into $\mathbf{B}$ (written $h : \mathbf{A} \to \mathbf{B}$), if for every $f \in \mathcal{L}_n$ and every $\langle \bar{a}\rangle \in A^n$, $hf^{\mathbf{A}}\langle \bar{a}\rangle = f^{\mathbf{B}}h\langle \bar{a}\rangle$. A homomorphism $h : \mathbf{A} \to \mathbf{A}$ is called an *endomorphism of* $\mathbf{A}$; if h is also surjective and injective, then h is an *automorphism of* $\mathbf{A}$.

Let $X = \{x_i\}_{i\in I}$ be a non-empty set. The set $\mathrm{Fm}_{\mathcal{L}}\, X$ of *formulas (or terms) of type* $\mathcal{L}$ *over the set of generators* X is defined recursively as follows

1. $X \subseteq \mathrm{Fm}_{\mathcal{L}}\, X$,

2. if $f \in \mathcal{L}_n$ and $\alpha_1, \ldots, \alpha_n \in \mathrm{Fm}_{\mathcal{L}}\, X$, then $\langle f, \alpha_1, \ldots, \alpha_n\rangle \in \mathrm{Fm}_{\mathcal{L}}\, X$.

Traditionally a formula $\langle f, \alpha_1, \ldots, \alpha_n\rangle$ is written as $f(\alpha_1, \ldots, \alpha_n)$. Formulas will be denoted usually by small Greek letters. We write $\alpha(p_1, \ldots, p_n)$ or $\mathrm{Var}(\alpha) \subseteq \{p_1, \ldots, p_n\}$, if $\alpha \in \mathrm{Fm}_{\mathcal{L}}\{p_1, \ldots, p_n\}$. A vector $\langle \alpha_1, \ldots, \alpha_k\rangle$ of $\mathrm{Fm}_{\mathcal{L}}^{+}$ is called a *sequent* and will be written usually in the form $\alpha_1, \ldots, \alpha_{k-1} \rhd \alpha_k$.

We can induce the structure of an algebra on $\mathrm{Fm}_{\mathcal{L}}\, X$ by associating with each $f \in \mathcal{L}_n$ a n-ary operation $f^{\mathbf{Fm}_{\mathcal{L}}\, X}$ on the set $\mathrm{Fm}_{\mathcal{L}}\, X$ defined by $f^{\mathbf{Fm}_{\mathcal{L}}\, X}\langle \bar{\alpha}\rangle = f(\bar{\alpha})$. The superscript in this case is usually omitted. This algebra $\mathbf{Fm}_{\mathcal{L}}\, X$ is called the *algebra of formulas (terms) of type* $\mathcal{L}$ *over the set of variables* X. We fix a countable set $\mathrm{Var} = \{x_0, x_1, x_2, ...\}$ of *propositional variables*. Then $\mathbf{Fm}_{\mathcal{L}}\,\mathrm{Var}$ is called *the formula algebra over the language of type* $\mathcal{L}$ and will be denoted $\mathbf{Fm}_{\mathcal{L}}$. The universe of $\mathbf{Fm}_{\mathcal{L}}$ is denoted as $\mathrm{Fm}_{\mathcal{L}}$.

An algebra $\mathbf{Fm}_{\mathcal{L}}\, X$ is an *absolutely free algebra over the set* X in the class of all algebras of type $\mathcal{L}$. This means that, for every algebra $\mathbf{A}$ of type $\mathcal{L}$, an arbitrary mapping $h : X \to A$ can be uniquely extended to a homomorphism $h : \mathbf{Fm}_{\mathcal{L}}\, X \to A$. In particular any homomorphism $h : \mathbf{Fm}_{\mathcal{L}}\, X \to \mathbf{A}$ is determined by the mapping $h : X \to A$. A homomorphism $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{A}$ is called an *evaluation*; a homomorphism $h : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{Fm}_{\mathcal{L}}$ is called a *substitution*.

A family $\mathcal{C} \subseteq \mathcal{P}(A)$ is *upward-directed* if for every pair $X, Y \in \mathcal{C}$ there is $Z \in \mathcal{C}$ such that $X, Y \subseteq Z$. A subset $\mathcal{C} \subseteq \mathcal{P}(A)$ is *algebraic* if $\bigcup \mathcal{D} \in \mathcal{C}$ for every upward-directed subfamily $\mathcal{D} \subseteq \mathcal{C}$. A family $\mathcal{C} \subseteq \mathcal{P}(A)$ is called a *closure system on* A if $A \in \mathcal{C}$ and $\bigcap \mathcal{D} \in \mathcal{C}$ for every non-empty subfamily $\mathcal{D} \subseteq \mathcal{C}$. A closure system $\mathcal{C}$ on $\mathrm{Fm}_{\mathcal{L}}$ is *(surjectively) invariant* if for any (surjective) substitution σ and any $T \in \mathcal{C}$, $\sigma^{-1}T := \{\alpha \mid \sigma\alpha \in T\} \in \mathcal{C}$, or, in other words, if $\sigma^{-1}\mathcal{C} \subseteq \mathcal{C}$ for all (surjective) $\sigma : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{Fm}_{\mathcal{L}}$. Similarly, a closure system $\mathcal{C}$ on $\mathrm{Fm}^{+}_{\mathcal{L}}$ is *(surjectively) invariant* if for any (surjective) substitution σ and any $T \in \mathcal{C}$, $\sigma^{-1}T = \{\bar{\alpha} \rhd \alpha \mid \sigma(\bar{\alpha} \rhd \alpha) \in T\} \in \mathcal{C}$.

A *closure operator on* A is a mapping $\mathbf{C} : \mathcal{P}(A) \to \mathcal{P}(A)$ such that for any $X, Y \subseteq A$, $X \subseteq \mathbf{C}(X) = \mathbf{C}(\mathbf{C}(X)) \subseteq \mathbf{C}(X \cup Y)$. A set $X \in \mathcal{P}(A)$ such that $\mathbf{C}(X) = X$ is called a *closed set of* $\mathbf{C}$. A closure operator $\mathbf{C}$ is *finitary* if for any $X \subseteq A$, $\mathbf{C}(X) = \bigcup\{\mathbf{C}(Y) \mid Y \subseteq_{\omega} X\}$. The following relations between closure systems and closure operators are well known: 1) if $\mathbf{C}$ is a closure operator on A, then the family of its closed sets is a closure system on A; 2) if $\mathcal{C}$ ia a closure system on A, then the mapping $\mathbf{C}_{\mathcal{C}} : \mathcal{P}(A) \to \mathcal{P}(A)$ defined for each $X \subseteq A$ as $\mathbf{C}_{\mathcal{C}}X := \bigcap\{Y \in \mathcal{C} \mid X \subseteq Y\}$ is a closure operator on A; 3) $\mathcal{C}$ is algebraic iff $\mathbf{C}_{\mathcal{C}}$ is finitary. We use interchangeably the exponential and prefix notations for closure operators, thus $X^{\mathcal{C}} = \mathbf{C}_{\mathcal{C}}X$.

Every closure system $\mathcal{C}$, as a family of subsets ordered under set-inclusion, is a complete lattice. The infimum of a family $\{X_i\}_{i \in I} \subseteq \mathcal{C}$ is its intersection $\bigcap_{i \in I} X_i$, and its supremum is $\bigvee^{\mathcal{C}}_{i \in I} X_i := \mathbf{C}_{\mathcal{C}}(\bigcup_{i \in I} X_i)$; its largest element is A, and its smallest element is $\mathbf{C}_{\mathcal{C}}(\emptyset) = \bigcap \mathcal{C}$.

A *Hilbert system* is a pair $\mathcal{S} = \langle \mathbf{Fm}_{\mathcal{L}}, \mathrm{Th}\,\mathcal{S}\rangle$ such that $\mathrm{Th}\,\mathcal{S} \subseteq \mathcal{P}(\mathrm{Fm}_{\mathcal{L}})$ is an algebraic invariant closure system on $\mathrm{Fm}_{\mathcal{L}}$. A *Gentzen system* is a pair $\mathcal{G} = \langle \mathbf{Fm}_{\mathcal{L}}, \mathrm{Th}\,\mathcal{G}\rangle$ such that $\mathrm{Th}\,\mathcal{G} \subseteq \mathcal{P}(\mathrm{Fm}^{+}_{\mathcal{L}})$ is an algebraic invariant closure system on $\mathrm{Fm}^{+}_{\mathcal{L}}$. For a Hilbert system $\mathcal{S}$ and all $T \in \mathrm{Th}\,\mathcal{S}$, $[T)_{\mathrm{Th}\,\mathcal{S}} := \{U \in \mathrm{Th}\,\mathcal{S} \mid T \subseteq U\}$ denote a principal filter of the lattice $\mathrm{Th}\,\mathcal{S}$ generated by T. If $\mathcal{R}$ is a Hilbert or Gentzen system, we denote $\mathrm{Thm}\,\mathcal{R} := \bigcap \mathrm{Th}\,\mathcal{R}$.

We take a Cantor-style approach towards Gentzen rules: we view a rule not as a "rule"— description of an action, but as a list of all its applications.

A *Gentzen sequent* is a sequence $\bar{s} \rhd s$ of sequents. A *Gentzen rule* $\bar{s} \vdash s$ is a set of all substitution instances of the Gentzen sequent $\bar{s} \rhd s$, i.e.,

$$\bar{s} \vdash s := \{\sigma(\bar{s} \rhd s) \mid \sigma : \mathbf{Fm}_{\mathcal{L}} \to \mathbf{Fm}_{\mathcal{L}}\}.$$

A Gentzen rule $s_1, \ldots, s_n \vdash s$ can also be written as $\dfrac{s_1, \ldots, s_n}{s}$.

Let x, y, z be variables. *Standard* rules (sometimes called *structural*) are rules of the form

(Ax) $\vdash \Gamma, x, \Sigma \rhd x$	Axioms
(Ex) $\Gamma, x, y, \Sigma \rhd z \vdash \Gamma, y, x, \Sigma \rhd z$	Exchange
(W) $\Gamma, \Sigma \rhd y \vdash \Gamma, x, \Sigma \rhd y$	Weakening

$$\begin{array}{ll}\text{(Con) } \Gamma, x, x, \Sigma \rhd y \vdash \Gamma, x, \Sigma \rhd y & \text{Contraction}\\ \text{(Cut) } \Gamma, x, \Sigma \rhd y;\ \Theta \rhd x \vdash \Gamma, \Theta, \Sigma \rhd y & \text{Cut}\end{array}$$

where Γ, Σ, Θ range over the set of finite, possibly empty, sequences of variables of $\mathrm{Fm}_{\mathcal{L}}$.

Suppose $\mathcal{G} = \langle \mathbf{Fm}_{\mathcal{L}}, \mathrm{Th}\,\mathcal{G}\rangle$ is a Gentzen system. We say that a Gentzen rule $\bar{s} \vdash s$ *holds in* $\mathcal{G}$ (we write it as $\bar{s} \vdash_{\mathcal{G}} s$) if for every substitution σ and every $\mathcal{G}$-theory T

$$\sigma\{\bar{s}\} \subseteq T \implies \sigma s \in T.$$

3. Closure Relations

Definition 3.1. Let $\mathcal{C}$ be a closure system on $\mathrm{Fm}_{\mathcal{L}}$. Define

$$\mathbf{R}_{\mathcal{L}}\,\mathcal{C} = \{\bar{\alpha} \rhd \alpha \in \mathrm{Fm}^{+}_{\mathcal{L}} \mid (\forall X \in \mathcal{C})\ \{\bar{\alpha}\} \subseteq X \Longrightarrow \alpha \in X\}.$$

Definition 3.2. Let $\mathcal{S}$ be a Hilbert system. If $\mathcal{C} \subseteq \mathrm{Th}\,\mathcal{S}$ is an algebraic closure system on $\mathrm{Fm}_{\mathcal{L}}$, then $\mathbf{R}_{\mathcal{L}}\,\mathcal{C}$ is called *a general finite closure relation for* $\mathcal{S}$ or simply *a general closure relation for* $\mathcal{S}$. The set of all general closure relations for $\mathcal{S}$ will be denoted by $\mathbf{Gcr}\,\mathcal{S}$. □

For every Hilbert system $\mathcal{S}$ of type $\mathcal{L}$ there is a distinguished general closure relation $\mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S}$, which in its turn defines a Gentzen axiomatization for a Gentzen system:

$$\vdash \mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S} := \bigcup\{\vdash \bar{\alpha} \rhd \alpha \mid \bar{\alpha} \rhd \alpha \in \mathbf{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S}\}.$$

Proposition 3.3. [1, Theorem 2.2.10]
For any Hilbert system $\mathcal{S}$ of type $\mathcal{L}$

(1) $\boldsymbol{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S} = \{\bar{\alpha} \rhd \alpha \mid \bar{\alpha} \vdash_{\mathcal{S}} \alpha\}$,
(2) $\boldsymbol{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S}$ *is invariant,*
(3) $\mathbf{Gcr}\,\mathcal{S}$ *can be axiomatized by standard rules and* $\vdash \boldsymbol{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S}$
(4) $\mathbf{Gcr}\,\mathcal{S}$ *forms a Gentzen system on* $\mathbf{Fm}_{\mathcal{L}}$,
(5) $\boldsymbol{R}_{\mathcal{L}}\mathrm{Th}\,\mathcal{S} = \mathrm{Thm}\,(\mathbf{Gcr}\,\mathcal{S})$.

Closure relations were introduced in [9] as a framework for studying metatheoretical properties of Hilbert systems. The fact that $\mathbf{Gcr}\,S$ form a Gentzen system was first observed also in [9]. The Gentzen system $\mathbf{Gcr}\,S$ formalizes a metalogic over the Hilbert system $\mathcal{S}$. This metalogic is quite weak and equivalent in expressive power to the strict universal Horn logic without equality [4]. Although $\mathbf{Gcr}\,S$ is almost trivial, since can be axiomatized by only taking all proper sequents of $\mathrm{Th}\,\mathcal{S}$ and standard Gentzen rules, by Proposition 3.3(3), it is proved to be useful as a framework for working with other kinds of closure relations like full or axiomatic [1].

We make a distinction between an element a and a vector $\langle a \rangle$ of length one with this element as its only component. This approach requires the following set of technical operators, that would allow us smooth transitions from formulas to sequents and back, and also between the theories of Hilbert and Gentzen systems. Define for every $X \subseteq \mathrm{Fm}_{\mathcal{L}}$ and every $\mathcal{A} \subseteq \mathrm{Fm}^{+}_{\mathcal{L}}$

$$\begin{aligned} \rhd X &:= \{\rhd \alpha \mid \alpha \in X\}, \\ \mathrm{Thm}\,\mathcal{A} &:= \{\alpha \in \mathrm{Fm}_{\mathcal{L}} \mid \rhd \alpha \in \mathcal{A}\}, \\ \boldsymbol{\Theta}\,\mathcal{A} &:= \{\rhd \alpha \in \mathrm{Fm}_{\mathcal{L}} \mid \rhd \alpha \in \mathcal{A}\}. \end{aligned}$$

Thus we obtain operators

$$\begin{aligned} &(\rhd) : \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}) \to \mathcal{P}(\mathrm{Fm}^{1}_{\mathcal{L}}), \\ &\mathrm{Thm} : \mathcal{P}(\mathrm{Fm}^{+}_{\mathcal{L}}) \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}}), \\ &\boldsymbol{\Theta} : \mathcal{P}(\mathrm{Fm}^{+}_{\mathcal{L}}) \to \mathcal{P}(\mathrm{Fm}^{1}_{\mathcal{L}}). \end{aligned}$$

Mnemonically, the Greek letter $\boldsymbol{\Theta}$ above stands for "Theorems".

Reminder. In the following proofs we rely heavily on, so called, "exponential" notation for closures of sets. Namely, if $\mathcal{C}$ is a closure system on some set X, then for all $Y \subseteq X$:

$$Y^{\mathcal{C}} = (Y)^{\mathcal{C}} := \textstyle\bigcap_{Y \subseteq F \in \mathcal{C}} F.$$

Definition 3.4. For a Hilbert system $\mathcal{S}$, define the set of *axiomatic closure relations of* $\mathcal{S}$ as follows:

$$\mathbf{Acr}\,\mathcal{S} := \{(\rhd T)^{\mathbf{Gcr}\,\mathcal{S}} \mid T \in \mathrm{Th}\,\mathcal{S}\}.$$

An element of $\mathbf{Acr}\,\mathcal{S}$ is called *an axiomatic closure relation for* $\mathcal{S}$. □

Note, that, in the definition above, the set $\rhd T$ contains sequents of the form $\rhd\alpha$, $\alpha \in T$, where $T \subseteq \mathrm{Fm}_{\mathcal{L}}$, and we take the closure of $\rhd T$ in the family of Gentzen theories, each of them is a set of sequents itself.

Proposition 3.5. *For every Hilbert system* $\mathcal{S}$ *of type* $\mathcal{L}$

(1) $\mathbf{Acr}\,\mathcal{S} \subseteq \mathbf{Gcr}\,\mathcal{S}$,
(2) $\mathcal{A} \in \mathbf{Acr}\,\mathcal{S} \implies \mathcal{A} = (\boldsymbol{\Theta}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}}$,
(3) $\mathbf{Acr}\,\mathcal{S} = \{(\rhd X)^{\mathbf{Gcr}\,\mathcal{S}} \mid X \subseteq \mathrm{Fm}_{\mathcal{L}}\}$,
(4) $\mathbf{Acr}\,\mathcal{S} = \{\mathbf{R}_{\mathcal{L}}[T)_{\mathrm{Th}\,\mathcal{S}} \mid T \in \mathrm{Th}\,\mathcal{S}\}$.

(5) *For every* $X \subseteq \mathrm{Fm}_{\mathcal{L}}$,

$$\bar{\alpha} \rhd \alpha \in (\rhd X)^{\mathbf{Gcr}\,\mathcal{S}} \iff \alpha \in \{\bar{\alpha}\}^{\mathcal{S}} \vee X^{\mathcal{S}} \iff X, \bar{\alpha} \vdash_{\mathcal{S}} \alpha.$$

Proof. (1) By definition.

(2) If $\mathcal{A} \in \mathbf{Acr}\,\mathcal{S}$, then $\mathcal{A} = (\rhd T)^{\mathbf{Gcr}\,\mathcal{S}}$ for some $T \in \mathrm{Th}\,\mathcal{S}$. Then

$$\rhd T \subseteq \mathbf{\Theta}\,\mathcal{A} \subseteq \mathcal{A} \implies \mathcal{A} = (\rhd T)^{\mathbf{Gcr}\,\mathcal{S}} \subseteq (\mathbf{\Theta}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}} \subseteq \mathcal{A}^{\mathbf{Gcr}\,\mathcal{S}} = \mathcal{A}$$

$$\implies \mathcal{A} = (\mathbf{\Theta}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}}.$$

(3) If $\mathcal{A} \in \mathbf{Acr}\,\mathcal{S}$, then $\mathcal{A} = (\mathbf{\Theta}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}} = (\rhd \mathrm{Thm}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}}$. For the other direction, suppose $\mathcal{A} = (\rhd X)^{\mathbf{Gcr}\,\mathcal{S}}$, for some $X \subseteq \mathrm{Fm}_{\mathcal{L}}$. Then $\mathcal{A} = (\mathbf{\Theta}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}} = (\rhd \mathrm{Thm}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}}$, because

$$(\supseteq)\ \mathbf{\Theta}\,\mathcal{A} \subseteq \mathcal{A} \implies (\mathbf{\Theta}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}} \subseteq \mathcal{A}^{\mathbf{Gcr}\,\mathcal{S}} = \mathcal{A},$$

$$(\subseteq)\ \mathcal{A} = (\rhd X)^{\mathbf{Gcr}\,\mathcal{S}} \implies \rhd X \subseteq \mathcal{A} \implies \rhd X \subseteq \mathbf{\Theta}\,\mathcal{A}$$

$$\implies \mathcal{A} = (\rhd X)^{\mathbf{Gcr}\,\mathcal{S}} \subseteq (\mathbf{\Theta}\,\mathcal{A})^{\mathbf{Gcr}\,\mathcal{S}}.$$

(4) Suppose $\mathcal{A} \in \mathbf{Acr}\,\mathcal{S}$. Then, by (3), $\mathcal{A} = (\rhd T)^{\mathbf{Gcr}\,\mathcal{S}}$, where $T = \mathrm{Thm}\,\mathcal{A} \in \mathrm{Th}\,\mathcal{S}$. Let $\mathcal{C} = [T)_{\mathrm{Th}\,\mathcal{S}}$. Being a general closure relation for $\mathcal{S}$, $\mathcal{A} = \mathbf{R}_{\mathcal{L}}\,\mathcal{D}$, for some algebraic closure system $\mathcal{D} \subseteq \mathrm{Th}\,\mathcal{S}$. Then $\mathcal{A} = \mathbf{R}_{\mathcal{L}}\,\mathcal{C}$, because

$$(\supseteq)\ T \overset{(3)}{=} \bigcap \mathcal{D} \implies \mathcal{D} \subseteq [T)_{\mathrm{Th}\,\mathcal{S}} = \mathcal{C} \implies \mathbf{R}_{\mathcal{L}}\,\mathcal{C} \subseteq \mathbf{R}_{\mathcal{L}}\,\mathcal{D} = \mathcal{A},$$

$$(\subseteq)\ \mathbf{\Theta}\,\mathbf{R}_{\mathcal{L}}\,\mathcal{D} = \rhd(\bigcap \mathcal{D}) = \rhd T = \rhd(\bigcap \mathcal{C}) = \mathbf{\Theta}\,\mathbf{R}_{\mathcal{L}}\,\mathcal{C} \implies \rhd T \subseteq \mathbf{R}_{\mathcal{L}}\,\mathcal{C}$$

$$\implies \mathcal{A} = (\rhd T)^{\mathbf{Gcr}\,\mathcal{S}} \subseteq \mathbf{R}_{\mathcal{L}}\,\mathcal{C}.$$

$$(5)\ \ \bar{\alpha} \rhd \alpha \in (\rhd T)^{\mathbf{Gcr}\,\mathcal{S}} \overset{(3)}{=} \mathbf{R}\,[T)_{\mathrm{Th}\,\mathcal{S}}$$

$$\iff \alpha \in \{\bar{\alpha}\}^{[T)_{\mathrm{Th}\,\mathcal{S}}} = (T \cup \{\bar{\alpha}\})^{\mathcal{S}} = T \vee \{\bar{\alpha}\}^{\mathcal{S}} \iff T, \bar{\alpha} \vdash_{\mathcal{S}} \alpha. \ \square$$

Lemma 3.6. $\mathbf{Acr}\,\mathcal{S}$ *is a closure system iff for all families* $\{\mathcal{A}_i\}_{i \in I} \subseteq \mathbf{Acr}\,\mathcal{S}$

$$\textstyle\bigcap_{i \in I} \mathcal{A}_i = (\bigcap_{i \in I} \mathbf{\Theta}\mathcal{A}_i)^{\mathbf{Gcr}\,\mathcal{S}}.$$

Proof. It follows directly from the implications

$$(\Rightarrow)\ \mathbf{\Theta}(\textstyle\bigcap_{i \in I} \mathcal{A}_i) = \bigcap_{i \in I} \mathbf{\Theta}\mathcal{A}_i$$

$$\implies \textstyle\bigcap_{i \in I} \mathcal{A}_i \overset{3.5(2)}{=} (\mathbf{\Theta}(\bigcap_{i \in I} \mathcal{A}_i))^{\mathbf{Gcr}\,\mathcal{S}} = (\bigcap_{i \in I} \mathbf{\Theta}\mathcal{A}_i)^{\mathbf{Gcr}\,\mathcal{S}}.$$

$$(\Leftarrow)\ \textstyle\bigcap_{i \in I} \mathcal{A}_i = (\bigcap_{i \in I} \mathbf{\Theta}\mathcal{A}_i)^{\mathbf{Gcr}\,\mathcal{S}} \overset{3.5(3)}{\in} \mathbf{Acr}\,\mathcal{S}.$$

$\square$

4. The Deduction-Detachment Theorem

The following is a standard definition:

Definition 4.1. A Hilbert system $\mathcal{S}$ admits a *multiterm deduction-detachment theorem* (DDT_Δ) with respect to a finite (may be empty) set $\Delta(x,y)$ of formulas of two variables if the following holds

(1) $x, \Delta(x,y) \vdash_{\mathcal{S}} y$ / *Δ-detachment*,

(2) $\dfrac{\Gamma, \alpha \vdash_{\mathcal{S}} \beta}{\Gamma \vdash_{\mathcal{S}} \Delta(\alpha,\beta)}$, for all $\alpha, \beta \in \mathrm{Fm}_{\mathcal{L}}$, / *$\Delta$-deduction.* □

Lemma 4.2. *Suppose* $\mathbf{Acr}\,\mathcal{S}$ *for some Hilbert system* $\mathcal{S}$ *is an invariant closure system, and let* $\Delta(x,y)$ *be a nonempty set of formulas of two variables. Then* $\mathcal{S}$ *admits* DDT_Δ *iff*

$$\{x \rhd y\}^{\mathbf{Acr}\,\mathcal{S}} = (\rhd \Delta(x,y))^{\mathbf{Acr}\,\mathcal{S}}. \qquad (*)$$

Proof. ($\Rightarrow$) The proof is straightforward.
($\Leftarrow$) The statement follows from the implications:

$$x \rhd y \in (\rhd \Delta(x,y))^{\mathbf{Acr}\,\mathcal{S}} \overset{3.5}{=} (\rhd \Delta(x,y))^{\mathbf{Gcr}\,\mathcal{S}}$$

$$\overset{3.5(5)}{\Longrightarrow} y \in \{x\}^{\mathcal{S}} \vee (\Delta(x,y))^{\mathcal{S}} \Longrightarrow x, \Delta(x,y) \vdash_{\mathcal{S}} y. \quad / \ \Delta\text{-detachment}$$

$$\Gamma, \alpha \vdash_{\mathcal{S}} \beta \overset{3.5(5)}{\Longrightarrow} \alpha \rhd \beta \in (\rhd \Gamma)^{\mathbf{Gcr}\,\mathcal{S}} = (\rhd \Gamma)^{\mathbf{Acr}\,\mathcal{S}}$$

$$\Longrightarrow (\rhd \Delta(\alpha,\beta))^{\mathbf{Gcr}\,\mathcal{S}} = (\rhd \Delta(\alpha,\beta))^{\mathbf{Acr}\,\mathcal{S}} \overset{(*)}{=} \{\alpha \rhd \beta\}^{\mathbf{Acr}\,\mathcal{S}} \subseteq (\rhd \Gamma)^{\mathbf{Gcr}\,\mathcal{S}}$$

$$\Longrightarrow \Delta(\alpha,\beta) \subseteq \Gamma^{\mathcal{S}} \overset{3.5(5)}{\Longrightarrow} \Gamma \vdash_{\mathcal{S}} \Delta(\alpha,\beta). \quad / \ \Delta\text{-deduction} \qquad \square$$

Examples. 1) Consider the normal modal logic $S4$. Let $\mathrm{Th}\,S4_{\vdash}$ be the family of sets of modal formulas that are closed under modus ponens and each contains all theorems of $S4$. Then, by deduction theorem for $S4$,

$$\{x \rhd y\}^{\mathbf{Acr}\,S4_{\vdash}} = \{\Box x \to y\}^{\mathbf{Gcr}\,S4_{\vdash}} = \{\Box x \to y\}^{\mathbf{Acr}\,S4_{\vdash}}.$$

2) The *inconsistent* Hilbert system $\mathcal{S} = \langle \mathbf{Fm}_{\mathcal{L}}, \{\mathrm{Fm}_{\mathcal{L}}\}\rangle$ over the language $\mathcal{L}$ admits DDT_Δ with respect to any finite set $\Delta \subseteq \mathrm{Fm}_{\mathcal{L}}$ of formulas, because

$$(\rhd \Delta)^{\mathbf{Acr}\,\mathcal{S}} = (\rhd \Delta)^{\mathbf{Gcr}\,\mathcal{S}} = \mathrm{Fm}^{+}_{\mathcal{L}} = \{x \rhd y\}^{\mathbf{Acr}\,\mathcal{S}}.$$

3) We also define that the *almost inconsistent* Hilbert system $\mathcal{S} = \langle \mathbf{Fm}_{\mathcal{L}}, \{\emptyset, \mathrm{Fm}_{\mathcal{L}}\}\rangle$ over $\mathcal{L}$ admits $\mathrm{DDT}_\emptyset$, because

$$\{x \rhd y\}^{\mathbf{Acr}\,\mathcal{S}} = \mathbf{R}_{\mathcal{L}}\{\emptyset, \mathrm{Fm}_{\mathcal{L}}\} = \mathrm{Fm}^{+}_{\mathcal{L}} \setminus \mathrm{Fm}^{1}_{\mathcal{L}} = (\emptyset)^{\mathbf{Gcr}\,\mathcal{S}} = (\emptyset)^{\mathbf{Acr}\,\mathcal{S}}.$$

Theorem 4.3. *Let $\mathcal{S}$ be a Hilbert system with theorems. Then* $\mathbf{Acr}\,\mathcal{S}$ *forms a Gentzen system iff $\mathcal{S}$ admits a multiterm deduction-detachment theorem.*

Proof. In view of the remarks above, it suffices to prove the theorem for $\mathcal{S}$ that is not inconsistent.
($\Rightarrow$) Suppose $\mathbf{Acr}\,\mathcal{S}$ is a closure system, then there is a closure of the set $\{x \rhd y\}$ in $\mathbf{Acr}\,\mathcal{S}$. If $\{x \rhd y\}^{\mathbf{Acr}\,\mathcal{S}} = (\emptyset)^{\mathbf{Gcr}\,\mathcal{S}}$, then

$$\{x \rhd y\}^{\mathbf{Acr}\,\mathcal{S}} = (\emptyset)^{\mathbf{Gcr}\,\mathcal{S}} \Longrightarrow x \rhd y \in (\emptyset)^{\mathbf{Gcr}\,\mathcal{S}} \overset{3.5(5)}{\Longrightarrow} x \vdash_{\mathcal{S}} y,$$

so $\mathcal{S}$ is either inconsistent or almost inconsistent, a contradiction with the assumption. Thus $\{x \rhd y\}^{\mathbf{Acr}\,\mathcal{S}} = (\rhd T)^{\mathbf{Gcr}\,\mathcal{S}}$, for some $T \in \mathrm{Th}\,\mathcal{S}$, such that $T \neq \mathrm{Thm}\,\mathcal{S}$, because $(\emptyset)^{\mathbf{Gcr}\,\mathcal{S}} = (\rhd \mathrm{Thm}\,\mathcal{S})^{\mathbf{Gcr}\,\mathcal{S}}$. Since $\{x \rhd y\}^{\mathbf{Acr}\,\mathcal{S}}$ is compact in $\mathbf{Acr}\,\mathcal{S}$, there is a finite subset $\mathcal{O} \subseteq T$, such that $\{x \rhd y\}^{\mathbf{Acr}\,\mathcal{S}} = \mathcal{O}^{\mathbf{Gcr}\,\mathcal{S}}$. Suppose σ is any substitution such that $\sigma\{x, y\} = \{x, y\}$ and $\sigma(\mathrm{Var} \setminus \{x, y\}) \subseteq \{x, y\}$ and let $\Delta(x, y) = \sigma\mathcal{O}$. Since $\mathbf{Acr}\,\mathcal{S}$ forms a Gentzen system, it is invariant under inverse substitutions, therefore

$$\{x \rhd y\}^{\mathbf{Acr}\,\mathcal{S}} = \{\sigma x \rhd \sigma y\}^{\mathbf{Acr}\,\mathcal{S}} = (\sigma\mathcal{O})^{\mathbf{Acr}\,\mathcal{S}} = (\rhd\Delta(x, y))^{\mathbf{Acr}\,\mathcal{S}}.$$

So, by Lemma 4.2, $\mathcal{S}$ admits DDT_Δ.
($\Leftarrow$) Suppose $\mathcal{S}$ has DDT_Δ, where $\Delta \neq \emptyset$. Δ can be viewed as a function $\Delta : \mathrm{Fm}^2_{\mathcal{L}} \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}})$. Furthermore it can be extended to a function from $\mathrm{Fm}^+_{\mathcal{L}}$ to $\mathcal{P}(\mathrm{Fm}_{\mathcal{L}})$ inductively as follows

$$\begin{aligned} &\Delta(\rhd\alpha) := \alpha, \\ &\Delta(\alpha_0, \ldots, \alpha_n \rhd \alpha) := \Delta(\alpha_0, \ldots, \alpha_{n-1} \rhd \Delta(\alpha_n, \alpha)) \\ &\qquad := \textstyle\bigcup_{\delta\in\Delta}\{\Delta(\alpha_0, \ldots, \alpha_{n-1} \rhd \delta(\alpha_n, \alpha))\}, \end{aligned}$$

and further, in the usual way, to a complex function $\Delta : \mathcal{P}(\mathrm{Fm}^+_{\mathcal{L}}) \to \mathcal{P}(\mathrm{Fm}_{\mathcal{L}})$. Thus, for every $\mathcal{A} \in \mathbf{Acr}\,\mathcal{S}$, the following holds

$$(1)\ \rhd\alpha \in \mathcal{A} \iff \Delta(\rhd\alpha) \overset{def}{=} \alpha \in \mathrm{Thm}\,\mathcal{A}$$

$$\begin{aligned} (2)\ &\bar{\alpha}, \alpha_{|\bar{\alpha}|} \rhd \alpha \in \mathcal{A} \overset{3.5(5)}{\iff} \mathrm{Thm}\,\mathcal{A}, \bar{\alpha}, \alpha_{|\bar{\alpha}|} \vdash_{\mathcal{S}} \alpha \overset{4.1}{\iff} \mathrm{Thm}\,\mathcal{A}, \bar{\alpha} \vdash_{\mathcal{S}} \Delta(\alpha_{|\bar{\alpha}|}, \alpha) \\ &\overset{3.5(5)}{\iff} \bar{\alpha} \rhd \Delta(\alpha_{|\bar{\alpha}|}, \alpha) \subseteq \mathcal{A} \iff \ldots \iff \rhd\Delta(\bar{\alpha}, \alpha_{|\bar{\alpha}|} \rhd \alpha) \subseteq \mathcal{A} \\ &\qquad \iff \Delta(\bar{\alpha}, \alpha_{|\bar{\alpha}|} \rhd \alpha) \subseteq \mathrm{Thm}\,\mathcal{A}. \end{aligned}$$

In other words: $\bar{\alpha} \rhd \alpha \in \mathcal{A} \iff \Delta(\bar{\alpha} \rhd \alpha) \in \mathrm{Thm}\,\mathcal{A}$. $\quad (*)$