Deformation of Elastic Solids

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Preface

Most aerospace, mechanical, and civil engineering graduate programs in major U.S. universities require a series of courses in the general area of solid mechanics. The details of these courses vary from school to school and program to program. At one end of the spectrum, some give preference to courses that emphasize *unified* and *exact* treatment of solid and fluid mechanics, often referred to as *modern continuum mechanics*. At the other end, a more traditional approach is taken, whereby an approximate, linear treatment of *engineering problems* is emphasized.

It has become apparent over the years that neither of the two extremes is appropriate for modern engineering programs. Most engineering students find the continuum mechanics approach too "heavy" for their taste and training, while the classical linear approach denies them the basic knowledge and insight that are essential for facing the challenges of the future. It would certainly be best if both approaches could be taught in two complete sets of courses. However, this can be done only under a Ph.D. degree program in mechanics or under a major field of specialization in solid mechanics, where the number of other required courses can be adjusted. In recent years, such specialized programs have been abandoned in favor of more traditional, discipline oriented programs—e.g., aerospace engineering, mechanical engineering, and civil engineering. The major field of specialization in mechanics of solids is also becoming somewhat rare, resulting in the need for a new approach to the teaching of solid mechanics, whereby the student is given the opportunity to learn the essential elements of both approaches in a time frame that is reasonable under his or her program of study.

As a response to these changes, graduate courses in solid mechanics have been redesigned in many universities in recent years. At the University of California at Los Angeles (UCLA), a sequence of courses has been created. The first course deals with the fundamental principles of continuum mechanics; the treatment is exact but restricted to elastic solids only. The approximate linear theory is also introduced in this course as a special case. The second course is devoted entirely to the solution of relevant linear problems, with emphasis on analytical techniques. These are followed by a series of specialized courses on plasticity, wave propagation, fracture mechanics, etc.

This book is based on the first two of these courses. Both authors have drawn upon their experience in teaching earlier versions of similar courses for over 15 years. The material has been used by a fairly large number of students and scrutinized by a number of faculty members with interest in solid mechanics. We hope that the book will be received enthusiastically by students, faculty, and others with a similar interest.

Chapter 1 contains a review of the theory of elasticity as it is normally taught in undergraduate courses. The discussions are quite terse, since more detailed and advanced treatments of the various topics reviewed here are given in later chapters. The main purpose of including the review is to provide the motivation and setting for the more advanced treatment that follows.

A full appreciation of the theoretical treatment of solid mechanics requires a good knowledge of mathematics. Since the mathematics preparation of the readers of this book is likely to be quite varied, it was deemed necessary to make the book as self-contained as possible by including most of the relevant intermediate-level mathematics. This is done in Chapter 2. The topics include *Cartesian tensors* and certain *integral theorems*. The treatment is rigorous but not exhaustive.

Chapter 3 deals with *kinematics of deformation*. A number of exact measures of deformation are introduced, and their geometrical interpretations are given with illustrative examples. Linear approximations leading to the classical *infinitesimal strains* are carefully discussed.

The *balance laws* of continuum mechanics are presented in Chapter 4. The concept of *stress* and its various measures are also introduced in this chapter. The *equations of motion* and the *energy equation* are derived in their exact forms in the presence or absence of thermal effects. The linearized forms of these equations are obtained as special cases.

Chapter 5 deals with *constitutive equations* for *elastic solids*. The general rules of constitutive theory are stated in simple terms, and the exact forms of these equations for isotropic solids are derived. The linear equations for *isotropic* as well as *anisotropic* solids are obtained as special cases. The linear constitutive equations for the overall behavior of *fiber-reinforced composites* are also presented in this chapter.

Exact solutions of a number of nonlinear and linear elastic problems is presented in Chapter 6. For the nonlinear case, consideration is restricted to the simplest possible problems that bring out the essential features of nonlinearity. For the linear case, some general theorems are discussed first, followed by the exact solutions of a number of simple but useful problems.

Chapter 7 is devoted entirely to *solution techniques* for *linear elastic* problems. Typical two- and three-dimensional boundary-value problems are considered in detail.

A brief exposition of *linear elastodynamics* is presented in Chapter 8. The main features of elastic waves are discussed through simple illustrative examples.

A review of *matrix algebra* is given in the appendix. Results that are needed for the discussions in the main body of the book are presented without proof.

We have made a strong effort to present an account of the exact theory of deformation in a form that can be easily understood by most first-year graduate students as well as by others with a baccalaureate degree from a U.S. university or its equivalent. The treatment is reasonably self-contained in that no significant prior knowledge of solid mechanics or of tensor analysis is needed, although some familiarity with elementary strength of materials can be helpful in appreciating the finer points of the exact theory. Complex variables are used in Chapter 7 to solve a few problems; readers who are not comfortable with this technique can omit this small section without any significant loss, since it is shown that these problems can be solved by other methods.

We have selected the material of the book with great care in order to keep its size relatively small without sacrificing the essential features of the subject. The problem sets have been deliberately kept small, with the hope that the students will have sufficient time to work them all thoughtfully. We strongly recommend that they be supplemented by additional problems of interest to the instructor and students.

There are a number of weaknesses; most of these turned out to be unavoidable. First, the notations for tensors and other mathematical quantities did not come out to be as uncluttered and consistent as we had originally hoped them to be. This is due primarily to the fact that the theoretical treatment of solid mechanics requires the introduction of an enormous number of variables, and one eventually tends to run out of symbols! There is also the need to conform to traditions and customs in the use of symbols. In spite of these difficulties, the notations used in the book should be easy to follow and should not cause any confusion.

A second weakness is the fact only a few problems of nonlinear elasticity have been included in Chapter 6. Certainly, the inclusion of a number of other relevant problems would have been possible. However, this would require a fuller exposition of the exact theory than was felt appropriate for this book. As explained earlier, the objective here is to introduce the students to the most essential features of the nonlinear theory.

Finally, several important topics involving inelastic effects (e.g., viscoelasticity, plasticity) have not been included. Students interested in these special topics should be able either to take such courses or to read up on their own after they have mastered the material included in the book.

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Valuable criticism of an earlier version of the manuscript was provided by Dr. Keith Walton and Dr. Gareth Parry of the University of Bath, England. Professors Russell Westmann of UCLA, Subhendu Datta of the University of Colorado, R. L. Brown of Montana State University, Marijan Dravinski of the University of Southern California, Arthur H. England and W. A. (Tony) Green of the University of Nottingham, and E. Rhian Green of Leicester University read the manuscript and suggested improvements. The input of Dr. M. R. Karim and many graduate students of UCLA, especially Shyh-Shiuh Lih, Ruey-Bin Yang, Ching-Chung Yin, and Russel Lund, was extremely helpful. We express our gratitude to these individuals.

The subject matter of the book is one of the oldest in the history of science and engineering. A large number of excellent books have appeared over the years. Our presentation has been heavily influenced by the books listed in the general bibliography.

Finally, the book could not have been completed without the patient understanding and encouragement of Rosita Mal and Harmohinder Singh.

Ajit K. Mal
Sarva Jit Singh

Contents

Chapter 5

CONSTITUTIVE EQUATIONS *123*

Chapter 6

ELASTOSTATICS *156*

Chapter 7

SOLUTION OF LINEAR ELASTOSTATIC PROBLEMS BY SPECIAL TECHNIQUES *191*

Chapter 8
LINEAR ELASTODYNAMICS *265*

Appendix
A REVIEW OF MATRICES *321*

GENERAL BIBLIOGRAPHY *330*

AUTHOR INDEX *333*

SUBJECT INDEX *335*

1

A Review of the Elementary Theory of Elasticity

In this chapter we present a brief exposition of the linear theory of elasticity as it can be found in most undergraduate texts. This will set the stage for more advanced treatment, which is the primary objective of this book.

1.1 STRAIN

To begin with, let us consider a deformation that is independent of one Cartesian coordinate, say z, and parallel to the xy-plane. This type of deformation is known as *plane strain*. We shall return to the problem of three-dimensional deformation after we have completed our study of deformation in a plane.

Let a point P with coordinates (X, Y) in the undeformed state be displaced to the point P' with coordinates (x, y) due to the deformation of the body (Fig. 1-1). The vector *displacement* of the point P has Cartesian components (U, V), where

$$U = x - X, \qquad V = y - Y \tag{1.1}$$

It will be assumed that the displacement components are continuous and twice differentiable functions of X, Y or x, y. Consider a small rectangular element $PQRS$ in the undeformed state, with sides (dX, dY) parallel to the coor-

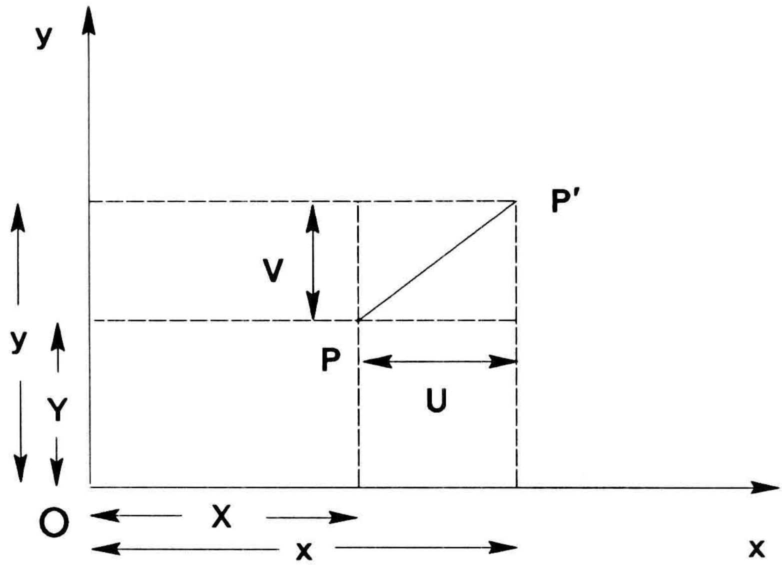

Figure 1-1 The displacement vector.

dinate axes (Fig. 1-2). Let the points P, Q, R, and S move, respectively, to P', Q', R', and S' after deformation. The coordinates of Q relative to P are $(dX, 0)$ and the coordinates of Q' relative to P' are (dx, dy), where

$$dx = dX + dU = dX + \frac{\partial U}{\partial X}\, dX \tag{1.2a}$$

$$dy = dV = \frac{\partial V}{\partial X}\, dX \tag{1.2b}$$

We thus have

$$P'Q' = \left[\left(dX + \frac{\partial U}{\partial X}\, dX\right)^2 + \left(\frac{\partial V}{\partial X}\, dX\right)^2\right]^{1/2}$$

$$\approx dX + \frac{\partial U}{\partial X}\, dX$$

where we have neglected the squares and higher powers of $(\partial U/\partial X)$ and $(\partial V/\partial X)$. Therefore, the increase in length per unit length of the line PQ, denoted by e_{xx}, is given by

$$e_{xx} = \frac{P'Q' - PQ}{PQ} = \frac{\partial U}{\partial X} \tag{1.3a}$$

Similarly, the increase in length per unit length of the line PS is

$$e_{yy} = \frac{\partial V}{\partial Y} \tag{1.3b}$$

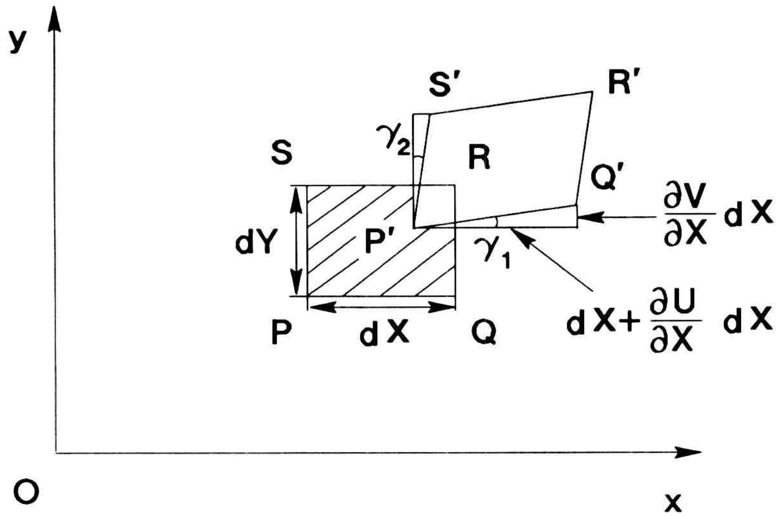

Figure 1-2 Deformation of line elements dX and dY.

The quantities e_{xx} and e_{yy} are known as *normal strains*.

If γ_1 denotes the angle that $P'Q'$ makes with the x-axis, we have (from Fig. 1-2)

$$\tan \gamma_1 = \frac{(\partial V/\partial X)}{1 + (\partial U/\partial X)}$$

Assuming that the angle γ_1 (measured in radians) is small and neglecting small quantities of the second and higher orders in $(\partial U/\partial X)$ and $(\partial V/\partial X)$, we find

$$\gamma_1 = \frac{\partial V}{\partial X} \tag{1.4a}$$

Similarly, the angle γ_2 of Fig. 1-2 is

$$\gamma_2 = \frac{\partial U}{\partial Y} \tag{1.4b}$$

Let $2e_{xy}$ denote the decrease in the angle between the two lines PQ and PS, which are parallel to the x- and y-axes, respectively, before deformation. Equations (1.4) then yield

$$2e_{xy} = \gamma_1 + \gamma_2 = \frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X} \tag{1.5}$$

The quantity e_{xy} is the *shearing strain*. The symbol $\gamma_{xy} = 2e_{xy}$ is often used to denote the shearing strain in engineering applications.

Relations (1.3) and (1.5) are obtained under the assumption that the displacement derivatives are small compared to unity. The theory of elasticity in which the products and squares of derivatives of the displacement components with respect to the space coordinates are neglected in comparison with the derivatives themselves is known as infinitesimal or *linear theory*. In this chapter we shall confine ourselves to the linear theory only.

We have seen that in the two-dimensional problem of plane strain, there are two normal strains, e_{xx} and e_{yy}, and one shearing strain, e_{xy}. In the general case of three-dimensional deformation, there are three normal strains, e_{xx}, e_{yy}, e_{zz}, and three shearing strains, e_{yz}, e_{zx}, and e_{xy}. They are related to the displacement components U, V, and W through the equations

$$e_{xx} = \frac{\partial U}{\partial X}, \qquad e_{yy} = \frac{\partial V}{\partial Y}, \qquad e_{zz} = \frac{\partial W}{\partial Z} \tag{1.6a}$$

$$e_{yz} = \frac{1}{2}\left(\frac{\partial V}{\partial Z} + \frac{\partial W}{\partial Y}\right) = \frac{1}{2}\gamma_{yz}, \qquad e_{xz} = \frac{1}{2}\left(\frac{\partial W}{\partial X} + \frac{\partial U}{\partial Z}\right) = \frac{1}{2}\gamma_{xz} \tag{1.6b}$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial U}{\partial Y} + \frac{\partial V}{\partial X}\right) = \frac{1}{2}\gamma_{xy} \tag{1.6c}$$

1.2 STRESS

A solid may be acted upon by two types of external forces: *body forces* and *surface forces*. Body forces act upon every volume element of the solid. Surface forces, in contrast, are forces that act upon every element of the surface of the body. In addition to the external forces, there are *internal forces*, which arise from the mutual interaction between various parts of the body.

Let a deformable solid be in its unstrained state with no forces acting on it and let a system of forces be then applied to it. On account of the application of these forces, the solid becomes deformed, and a system of internal forces is set up within it to oppose this deformation. These internal forces give rise to what is known as stress within the solid.

Let us consider *a part* of the solid occupying a region V enclosed by the surface S in the *deformed state*. The boundary S is acted upon by surface forces caused by the action of the material exterior to V on that within V. It will be assumed that these surface forces are continuously distributed over S. A suitable measure of such forces is their intensity, i.e., the amount of force per unit area of the surface on which they act.

To specify the stress acting on a small area δS at a point P on S, we assume that the forces acting across this elementary area, due to the action of the material outside V, can be reduced to a single force $\delta \mathbf{p}$ (Fig. 1-3) and that the limit

$$\lim_{\delta S \to 0} \frac{\delta \mathbf{p}}{\delta S} \tag{1.7}$$

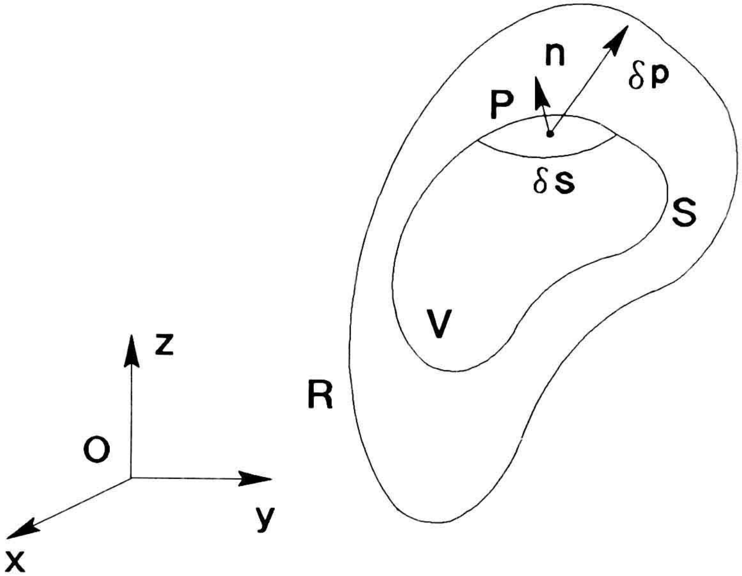

Figure 1-3 The surface force.

exists. This limit is known as the *traction*, or the *stress vector*, on δS at the point P.

In general, the traction is inclined to the area δS on which it acts, and we can resolve it into two components: a *normal stress* perpendicular to the area and a *shearing stress* acting in the plane of the area δS. Alternatively, we can resolve the traction into its Cartesian components. Let the outward drawn unit normal to δS be **n** and the x-, y-, and z-components of the traction acting on δS be denoted by $t_x^{(n)}$, $t_y^{(n)}$, and $t_z^{(n)}$, respectively.

Consider a small cubic element with sides parallel to the coordinate axes (Fig. 1-4). The components of the traction acting on the face with normal in the positive x-direction are $t_x^{(1)}$, $t_y^{(1)}$, $t_z^{(1)}$. We use the notations

$$t_x^{(1)} = \sigma_{xx}, \qquad t_y^{(1)} = \sigma_{xy}, \qquad t_z^{(1)} = \sigma_{xz} \tag{1.8}$$

Similarly, the components of the traction acting on the face of the cube with normal in the positive y-direction are σ_{yx}, σ_{yy}, σ_{yz}, and the components of the traction acting on the face of the cube with normal in the positive z-direction are σ_{zx}, σ_{zy}, σ_{zz}. Thus the first suffix indicates the direction of the normal to the face and the second suffix indicates the direction of the traction component. In all, we have nine components, σ_{xx}, σ_{xy}, $\cdots$, σ_{zz}, which are known as the *components of stress*. We can display these components in the form of the matrix:

$$\begin{bmatrix} \sigma_{xx} & \sigma_{xy} & \sigma_{xz} \\ \sigma_{yx} & \sigma_{yy} & \sigma_{yz} \\ \sigma_{zx} & \sigma_{zy} & \sigma_{zz} \end{bmatrix} \tag{1.9}$$

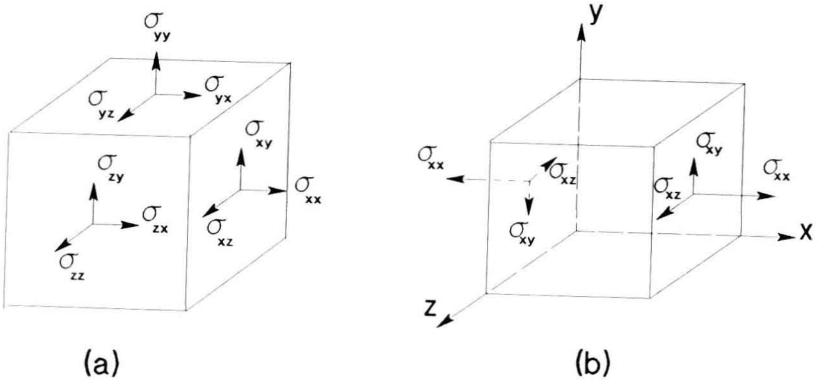

Figure 1-4 Stress components.

Obviously the components σ_{xx}, σ_{yy}, σ_{zz} represent normal stresses and the components σ_{xy}, σ_{yx}, σ_{xz}, σ_{zx}, σ_{yz}, σ_{zy} represent shearing stresses.

We use the convention that the normal stress on a surface is positive when it produces tension and negative when it produces compression of the material within the element. The positive direction of a component of the shearing stress on any face of the cubic element is taken in the positive (negative) direction of the coordinate axis if a tensile stress on the same face is in the positive (negative) direction of the corresponding axis. This rule is illustrated in Fig. 1-4 by indicating the positive directions of the components σ_{xx}, σ_{xy}, and σ_{xz} for the two faces of the cubic element with normals in the positive x-direction and the negative x-direction, respectively.

1.3 EQUATIONS OF EQUILIBRIUM

Consider the equilibrium of a small rectangular parallelepiped with its center at $P(x, y, z)$ and edges δx, δy, δz parallel to the coordinate axes (Fig. 1-5). The centers of the six faces of the parallelepiped are at the points

$$(x \pm \tfrac{1}{2}\delta x, y, z), \qquad (x, y \pm \tfrac{1}{2}\delta y, z), \qquad (x, y, z \pm \tfrac{1}{2}\delta z)$$

If the components of stress at P are $\sigma_{xx}, \sigma_{xy}, \cdots, \sigma_{zz}$, then the components of the traction acting on the face with its center at $(x + \delta x/2, y, z)$ are

$$\sigma_{xx} + \frac{1}{2}\frac{\partial \sigma_{xx}}{\partial x}\delta x, \qquad \sigma_{xy} + \frac{1}{2}\frac{\partial \sigma_{xy}}{\partial x}\delta x, \qquad \sigma_{xz} + \frac{1}{2}\frac{\partial \sigma_{xz}}{\partial x}\delta x$$

since the outward normal to this face is in the positive x-direction. The components of the traction acting on the face with its center at $(x - \delta x/2, y, z)$ are

$$-\left(\sigma_{xx} - \frac{1}{2}\frac{\partial \sigma_{xx}}{\partial x}\delta x\right), \qquad -\left(\sigma_{xy} - \frac{1}{2}\frac{\partial \sigma_{xy}}{\partial x}\delta x\right), \qquad -\left(\sigma_{xz} - \frac{1}{2}\frac{\partial \sigma_{xz}}{\partial x}\delta x\right)$$