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Compressible Navier–Stokes Equations

Theory and Shape Optimization

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Preface

This book is a result of scientific collaboration, for more than ten years, between three countries, France, Russia and Poland. The main support for the joint research comes from Université de Lorraine, Institut Élie Cartan and from Centre Nationale de la Recherche Scientifique (CNRS) for the visits of Pavel I. Plotnikov in France. Also the support of the Systems Research Institute of the Polish Academy of Sciences for the visits in Poland is acknowledged. The research of Pavel I. Plotnikov was partially supported by Contract 02.740.11.0617 with the Ministry of Education and Science of the Russian Federation and by grant 10-01-00447-a from the Russian Foundation of Basic Research, while the research of Jan Sokołowski was partially supported by the project ANR-09-BLAN-0037 *Geometric analysis of optimal shapes (GAOS)* financed by the French Agence Nationale de la Recherche (ANR) and by grant N51402132/3135 of the Polish Ministerstwo Nauki i Szkolnictwa Wyższego "Optymalizacja z wykorzystaniem pochodnej topologicznej dla przepływów w ośrodkach ściśliwych" in the Systems Research Institute of the Polish Academy of Sciences. The monograph is published in the Polish mathematical series by Birkhäuser Basel. The authors are greatly indebted to Jerzy Trzeciak for his careful work on editing the monograph, and his patience.

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Introduction

This monograph is devoted to the study of boundary value problems for equations of viscous gas dynamics, named compressible Navier-Stokes equations. The mathematical theory of Navier -Stokes equations is very interesting in its own right, but its principal significance lies in the central role Navier-Stokes equations now play in fluid dynamics. Most of this book concentrates on those aspects of the theory that have proven useful in applications.

The mathematical study of compressible Navier-Stokes equations dates back to the late 1950s. It seems that Serrin [121] and Nash [92] were the first to consider the mathematical questions of compressible viscous fluid dynamics. An intensive treatment of compressible Navier-Stokes equations starts with pioneering papers by Itaya [60], Matsumura & Nishida [87], Kazhikhov & Solonnikov [63], and Hoff [56] on the local theory for nonstationary problems, and by Beirão da Veiga [11, 12], Padula [103], and Novotný & Padula [98, 99] on the theory of stationary problems for small data. A global theory of weak solutions to compressible Navier-Stokes equations was developed by P.-L. Lions in 1998. These results were essentially improved, sharpened and generalized by E. Feireisl. We refer the reader to the books by Lions [80], Feireisl [34], Novotný & Straškraba [101], and Feireisl & Novotný [37] for the state of the art in the domain.

Although the theory is satisfactory in what concerns local time behavior and small data, many issues of global behavior of solutions for large data are far from being understood. There are a vast range of unsolved problems concerning questions such as regularity of solutions to compressible Navier-Stokes equations, the theory of weak solutions for small adiabatic exponents, existence theory for heat conducting fluids. However, these problems will not be our primary concern here. We are mainly interested in three problems that we describe briefly below.

Existence theory. This issue is important since no progress in the mathematical theory of Navier-Stokes equations can be made without answering the basic questions on their well-posedness. We focus on existence results for the inhomogeneous in/out flow problem, in particular the problem of the flow around a body placed in a finite domain. Notice that the majority of known results are related to viscous gas flows in domains bounded by impermeable walls. In/out flow problems are still poorly investigated. We refer to the paper by Novo [93], where an existence theorem was proved for constant boundary data, and recent work by Girinon [52], where the existence of a weak solution was established for convex flow domains with inlet independent of the time variable. We give an existence result in the general nonstationary case without imposing restrictions on the geometry of the flow domain and the behavior of boundary data. In contrast, the question of existence of global weak solutions to the stationary in/out flow problem remains essentially unsolved. Local strong solutions close to the uniform flow have been studied by Farwig [29] and Kweon & Kellogg [69, 72, 73]. With applications to shape optimization theory in mind, we consider the problem of the flow around a body placed in a bounded domain for small Mach and Reynolds numbers.

Stability of solutions with respect to nonsmooth data and domain perturbations. Propagation of rapid oscillations in compressible fluids. In compressible viscous flows, any irregularities in the initial and boundary data are transferred inside the flow domain along fluid particle trajectories. The transport of singularities in viscous compressible flows was studied by Hoff [56, 57]. In this book we discuss the propagation of rapid oscillations of the density, which can be regarded as acoustic waves. The main idea is that any rapidly oscillating sequence is associated with some stochastic field named the Young measure (see Tartar [128] and Perthame [106] for basic ideas). We establish that the distribution function of this stochastic field satisfies a kinetic equation of a special form, which leads to a rigorous model for propagation of nonlinear acoustic waves. Notice that oscillations can be induced not only by oscillations of initial and boundary data, but also by irregularities of the boundary of the flow domain.

Domain dependence of solutions to compressible Navier-Stokes equations. This issue is important because of applications to shape optimization theory. The latter is a branch of the general calculus of variations which deals with the shapes of geometric and physical objects instead of parameters and functions. The classic examples of shape optimization problems are the isoperimetric problem and Newton's problem of the body of minimal resistance. We refer to [126], [18], [21], [46], [54], [62], [91] for a general account of the theory and the relevant references. The first global result on domain dependence of solutions to compressible Navier-Stokes equations is due to Feireisl [33], who proved that the set of solutions to compressible Navier-Stokes equations is compact provided the set of flow domains is compact in the Kuratowski-Mosco topology and their boundaries have "uniformly small" volumes. We prove that the compactness result holds true if the set of flow domains is compact in the Kuratowski-Mosco topology, and also that some cost functionals, such as the drag and the work of hydrodynamical forces, are continuous in this topology. With applications to shape optimization in mind, we consider the shape differentiability of strong solutions and give formulae for the shape derivative of the drag functional. Let us also mention that in the incompressible case the shape differentiability of the drag functional was considered in [14], [15], [123]. Finally, we refer to Mohammadi & Pironneau [90] for the relevant references in applied shape optimization for fluids.

We also discuss the mathematical questions which are not related directly to Navier-Stokes equations. Among them are the theory of boundary value problems for transport equations and the problem of vanishing viscosity for diffusion equations with convective terms.

The basic idea of the book is to give as more details as possible and to avoid using complicated mathematical tools. In particular, we do not use compensated compactness results, the Bogovski lemma, or semigroup theory. Only the undegraduate background in mathematical analysis and elementary facts from functional analysis are assumed of the reader.

The material is organized in twelve chapters and an appendix. We start in Chapter 1 with a review of standard topics from real and functional analysis. The chapter includes, mainly without proofs, basic facts on measure and integral, functional analysis, elliptic and parabolic equations. We focus on measure theory, since the notion of Young measure is widely used throughout the book. Most of the material will be familiar to the reader and can be omitted. Possible exceptions are Section 1.3.1 containing a general formula for integration by parts in the Lebesgue-Stieltjes integral, Section 1.4 where the notion of Young measure is introduced, and Sections 1.1.2 and 1.5 devoted to interpolation theory and Sobolev spaces.

In Chapter 2 we collect the basic physical facts concerning compressible Navier-Stokes equations including the formulation of equations in a moving coordinate frame and formulae for the hydrodynamical forces and the work of these forces. In Chapter 3 we give the mathematical formulation of the main boundary value problem for compressible Navier-Stokes equations and discuss the notions of weak and renormalized solutions.

Chapters 4–11 can be considered as the core of the book. Our considerations are based on the approach developed by P.-L. Lions and E. Feireisl. The main ingredients of their method are Lions's result on weak continuity of the effective viscous flux, L^p-estimates of the pressure function, and the theory of the oscillation defect measure developed by E. Feireisl. Some of these results are of general character and hold true for any system of mass and impulse-momentum laws.

We assemble all such results in Chapter 4, where we consider the system of balance laws which is formulated as follows. Assume that a medium occupies a domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$. We want to find a velocity field $\mathbf{u} : \Omega \times (0, T) \to \mathbb{R}^d$ and a density function $\varrho : \Omega \times (0, T) \to \mathbb{R}^+$ satisfying the momentum and mass balance equations

$$\begin{aligned} &\partial_t(\varrho\mathbf{u}) + \operatorname{div}(\varrho\,\mathbf{u}\otimes\mathbf{u}) = \operatorname{div}\mathbb{T} + \varrho\mathbf{f} \quad \text{in } \Omega\times(0,T), \\ &\partial_t\varrho + \operatorname{div}(\varrho\mathbf{u} - \mathbf{g}) = 0 \quad \text{in } \Omega\times(0,T), \end{aligned} \tag{0.0.1}$$

where $\mathbf{f}$, $\mathbf{g}$ are given vector fields, and $\mathbb{T}$ is the stress tensor. The class of such systems includes compressible Navier-Stokes equations and their numerous modifications. At this stage we do not specify the form of the stress tensor and do not impose boundary and initial conditions on the density and the velocity. Instead we assume that they satisfy some integrability conditions. In Chapter 4 we consider the basic properties of solutions to equations (0.0.1). The results we present include elementary facts on integrability of functions with finite energy (Section 4.2), and the standard material concerning weak compactness properties of the impulse momentum $\varrho\mathbf{u}$ and the kinetic energy tensor $\varrho\mathbf{u}\otimes\mathbf{u}$ (Section 4.4). Section 4.6 is more important: we prove there the general form of P.-L. Lions's result on weak continuity of the viscous flux, which is the most important result of the mathematical theory of viscous compressible flows.

Chapter 5 deals with existence theory for the nonstationary in/out flow boundary value problem for compressible Navier-Stokes equations, which are a particular case of system (0.0.1) with the stress tensor of the form

$$\mathbb{T} = \nabla\mathbf{u} + (\nabla\mathbf{u})^\top + (\lambda - 1)\,\mathrm{div}\,\mathbf{u}\,\mathbb{I} - p(\varrho)\,\mathbb{I}, \tag{0.0.2}$$

where λ is some constant, and $p(\varrho)$ is a monotone function such that $p(\varrho)\sim\varrho^\gamma$ at infinity. The in/out flow problem for equations (0.0.1)–(0.0.2) can be formulated as follows. Let a vector field $\mathbf{U} : \Omega\times(0,T)\to\mathbb{R}^d$ and a nonnegative function $\varrho_\infty : \Omega\times(0,T)\to\mathbb{R}$ be given. We want to find a solution of (0.0.1)–(0.0.2) satisfying the initial and boundary conditions

$$\begin{aligned}&\mathbf{u}=\mathbf{U},\quad \varrho=\varrho_\infty \quad \text{on } \partial\Omega\times\{t=0\},\\ &\mathbf{u}=\mathbf{U} \quad \text{on } \partial\Omega\times(0,T),\quad \varrho=\varrho_\infty \quad \text{on } \Sigma_{\text{in}},\end{aligned} \tag{0.0.3}$$

where the inlet Σ_{in} is the open subset of $\partial\Omega\times(0,T)$ which consists of all points (x,t) such that the vector $\mathbf{U}(x,t)$ points to the inside of $\Omega\times(0,T)$. The peculiarity of this problem is that we deal with the boundary value problem for the mass balance equations. In Chapter 5 we prove that for the adiabatic exponent $\gamma>2d$ and smooth initial and boundary data satisfying the compatibility conditions, the problem has a renormalized solution. We follow the multilevel regularization scheme proposed by E. Feireisl, but with a different regularization technique. The main ingredient of our method is the estimates of the normal derivatives of solutions to singularly perturbed transport equations (Section 5.3.8). These estimates are nontrivial and their derivation is based on Aronson-type inequalities for the heat kernels of diffusion equations with convective terms. Another essential ingredient of our method is the systematic use of Young measure theory.

Chapter 6 is of technical character. There we prove that for the solution to problem (0.0.1)–(0.0.3) constructed in Chapter 5, the pressure $p(\varrho)$ is locally integrable with some exponent greater than 1.

In Chapter 7 the results obtained are extended to the range of adiabatic exponents $(3/2,\infty)$ common for homogeneous boundary value problems. In this chapter we propose a new approach to the boundary value problems with fast oscillating boundary data and develop a theory of such problems based on the kinetic formulation of the governing equation. We deal with the sequence of solutions $(\mathbf{u}_\epsilon,\varrho_\epsilon)$, $\epsilon>0$, to problem (0.0.1)–(0.0.3) with regularized pressure functions of the form $p_\epsilon = p(\varrho)+\varepsilon\varrho^n$, and the initial and boundary data ϱ_∞^ϵ. We assume that the sequence ϱ_∞^ϵ is only bounded, but need not converge to any limit in the strong sense. In particular this class of data includes rapidly oscillating functions of the form

$$\varrho_\infty^\epsilon = F\Big(x, t, \frac{x}{\epsilon}, \frac{t}{\epsilon}\Big),$$

where $F(x,t,y,\tau)$ is a bounded function, periodic in y and τ. Under these assumptions, the sequence $(\varrho_\epsilon, \mathbf{u}_\epsilon, p(\varrho_\epsilon))$ converges only weakly to some limit $(\overline{\varrho}, \mathbf{u}, \overline{p})$ as $\epsilon \to 0$. Following [128] we conclude that this limit admits a representation

$$\overline{\varrho}(x,t) = \int_{\mathbb{R}} s\, d\mu_{x,t}(s), \quad \overline{p}(x,t) = \int_{\mathbb{R}} p(s)\, d\mu_{x,t}(s), \tag{0.0.4}$$

where $\mu_{x,t}$ is a probability measure on the real line named the Young measure. It is completely characterized by the distribution function $f(x,t,s) = \mu_{x,t}(-\infty, s]$. Notice that the sequence ϱ_ϵ converges strongly if and only if the distribution function is deterministic, i.e., $f(1-f) = 0$. The basic idea underlying the method of kinetic equations (see [106]) is that the distribution function satisfies a differential relation named a kinetic equation. Usually the kinetic equation contains some undefined terms and cannot be considered as an equation in the strict sense of this word. A remarkable property of compressible Navier-Stokes equations is that in this particular case the kinetic equation can be written in closed form as

$$\partial_t f + \operatorname{div}(f\mathbf{u}) - \partial_s\Big(sf \operatorname{div}\mathbf{u} + \frac{s}{\lambda+1}\int_{(-\infty,s]} (p(\tau) - \overline{p})\, d_\tau f(x,t,\tau)\Big) = 0.$$

In Chapter 7 we derive the kinetic equation and show that, when combined with relations (0.0.4) and the momentum balance equations, it gives a closed system of integro-differential equations which describes the propagation of rapid oscillations in a compressible viscous flow. We also prove that if the data are deterministic and the function f satisfies some integrability condition, then any solution to the kinetic equation satisfying some integrability conditions is deterministic. This fact is a general property of the kinetic equation and has no connection with the theory of Navier-Stokes equations. It follows that if ϱ_∞^ϵ converges strongly, then so does ϱ_ϵ.

In the next chapters we apply the kinetic equation method to the analysis of the domain dependence of solutions to compressible Navier-Stokes equations. We restrict our considerations to the problem of the flow around an obstacle placed in a fixed domain. In this problem $\Omega = B \setminus S$ is a condenser type domain, B is a fixed *hold-all domain* and S is a compact obstacle. It is assumed that $\mathbf{U}$ vanishes on $S \times (0,T)$.

In Chapter 8 we collect the basic facts concerning domain convergence and related questions from capacity theory. The most important is Hedberg's theorem (Theorem 8.2.22) on approximation of Sobolev functions. In this chapter we also introduce the notion of $\mathcal{S}$-convergence, which plays a key role in the next chapters. Denote by $C_S^\infty(B)$ the set of all smooth functions defined in B and vanishing on $S \Subset B$. Let $W_S^{1,2}(B)$ be the closure of $C_S^\infty(B)$ in the $W^{1,2}(B)$-norm. It is clear that $W^{1,2}_S(B)$ is a closed subspace of $W^{1,2}(B)$. A sequence of compact sets $S_n \Subset B$ is said to $\mathcal{S}$-converge to S if

- there is a compact set $B' \Subset B$ such that $S_n, S \subset B'$ and S_n converges to S in the standard Hausdorff metric;
- for any sequence $u_n \rightharpoonup u$ weakly convergent in $W^{1,2}(B)$ with $u_n \in W^{1,2}_{S_n}(B)$, the limit element u belongs to $W^{1,2}_S(B)$;
- whenever $u \in W^{1,2}_S(B)$, there is a sequence $u_n \in W^{1,2}_{S_n}(B)$ with $u_n \to u$ strongly in $W^{1,2}(B)$.

We investigate in great detail the properties of $\mathcal{S}$-convergence and give examples of classes of obstacles which are compact with respect to this convergence.

In Chapter 9 we prove the central result on the domain stability of solutions to compressible Navier-Stokes equations. We show that if a sequence S_n of compact obstacles $\mathcal{S}$-converges to a compact obstacle S then the sequence of corresponding solutions to the in/out flow problem contains a subsequence which converges to a solution to the in/out flow problem in the limiting domain. In Chapter 10 we sharpen this result by proving that the typical cost functionals, such as the work of hydrodynamical forces, are continuous with respect to $\mathcal{S}$-convergence. As a conclusion we establish the solvability of the problem of minimization of the work of hydrodynamical forces in the class of obstacles with a given fixed volume.

Chapter 11 is devoted to the shape sensitivity analysis of the stationary boundary problem for compressible Navier-Stokes equations. Here we prove the local existence and uniqueness results for the in/out flow problem for compressible Navier-Stokes equations under the assumption that the Reynolds and Mach numbers are sufficiently small. We show the weak differentiability of solutions with respect to the shape of the flow domain and derive formulae for the derivatives and corresponding system of adjoint equations, which are of a practical interest. The results obtained are based on the theory of strong solutions to boundary value problems for transport equations which is presented in Chapter 12.

Contents

Chapter 1

Preliminaries

We use the following notation throughout the monograph.

A vector in n-dimensional Euclidean space is denoted by $\mathbf{v} = (v_1, \ldots, v_n)$, whether it is a column vector or a row vector. For a matrix $\mathbf{A} = (A_{ij})$, $i, j = 1, \ldots, n$, with real entries A_{ij}, i is the row index and j the column index, and $\mathbf{A}^\top = (A_{ji})$ stands for the transposed matrix. The product of a matrix by a column vector is denoted by $\mathbf{Av}$, and of a row vector by a matrix by $\mathbf{vA}$. The product of two matrices $\mathbf{C} = \mathbf{AB}$ is a matrix with the entries $C_{ij} = A_{ik}B_{kj}$ with the summation convention over repeated indices.

In particular, the scalar product of two vectors is $\mathbf{v} \cdot \mathbf{u} = v_i u_i$, and $(\mathbf{Av})_i = A_{ij}v_j$, while

$$\mathbf{vA} = \mathbf{A}^\top \mathbf{v}$$

with $(\mathbf{vA})_i = (\mathbf{A}^\top \mathbf{v})_i = A_{ji}v_j$.

The tensor product of two vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ is the matrix $\mathbf{A} := \mathbf{u} \otimes \mathbf{v}$ with the entries $A_{ij} := u_i v_j$, $i, j = 1, \ldots, n$. For the product of this matrix with a vector we have

$$(\mathbf{u} \otimes \mathbf{v})\mathbf{w} = \mathbf{u}(\mathbf{v} \cdot \mathbf{w}) \quad \text{and} \quad \mathbf{w}(\mathbf{u} \otimes \mathbf{v}) = (\mathbf{w} \cdot \mathbf{u})\mathbf{v}.$$

The derivatives of a scalar or a vector function with respect to the time variable are denoted by $\partial_t \mathbf{v} = \frac{\partial \mathbf{v}}{\partial t}$, and similarly for the spatial variables. There is a difference between the Jacobian of a vector function and its gradient: the Jacobian is denoted by

$$D\mathbf{v} = (\partial_{x_j} v_i) = \left[\frac{\partial \mathbf{v}}{\partial x_1}, \frac{\partial \mathbf{v}}{\partial x_2}, \frac{\partial \mathbf{v}}{\partial x_3}\right] = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_1}{\partial x_2} & \frac{\partial v_1}{\partial x_3} \\ \frac{\partial v_2}{\partial x_1} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_2}{\partial x_3} \\ \frac{\partial v_3}{\partial x_1} & \frac{\partial v_3}{\partial x_2} & \frac{\partial v_3}{\partial x_3} \end{bmatrix},$$

and the gradient is its transpose

$$\nabla \mathbf{v} = D\mathbf{v}^{\top} = (\partial_{x_i} v_j) = [\nabla v_1, \nabla v_2, \nabla v_3] = \begin{bmatrix} \frac{\partial v_1}{\partial x_1} & \frac{\partial v_2}{\partial x_1} & \frac{\partial v_3}{\partial x_1} \\ \frac{\partial v_1}{\partial x_2} & \frac{\partial v_2}{\partial x_2} & \frac{\partial v_3}{\partial x_2} \\ \frac{\partial v_1}{\partial x_3} & \frac{\partial v_2}{\partial x_3} & \frac{\partial v_3}{\partial x_3} \end{bmatrix}.$$

Therefore, the nonlinear term in the Navier-Stokes equations is a vector denoted by $\varrho\mathbf{v}\nabla\mathbf{v} = (\varrho(v_j\partial_{x_j} v_i)) = [\varrho\mathbf{v}\cdot\nabla v_1, \varrho\mathbf{v}\cdot\nabla v_2, \varrho\mathbf{v}\cdot\nabla v_3]$, where we sum over the repeated indices $j = 1, 2, 3$, and $\varrho\mathbf{v}\cdot\nabla v_1$, $\varrho\mathbf{v}\cdot\nabla v_2$, $\varrho\mathbf{v}\cdot\nabla v_3$ stand for column vectors according to our convention. In general, for a function $u : \mathbb{R}^d \to \mathbb{R}$ we denote by

$$\partial^{\alpha} u = \frac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \dots \partial x_d^{\alpha_d}}$$

the partial derivative of order $|\alpha| = \alpha_1 + \cdots + \alpha_d$ with the multiindex $\alpha = (\alpha_1, \dots, \alpha_d)$, for integers α_i, $i = 1, \dots, d$.

We also use the simplified notation e.g., $\partial_x^2 \varrho$ for the collection of all the second order derivatives $\partial^2_{x_i x_j}\varrho = \dfrac{\partial^2 \varrho}{\partial x_i \partial x_j}$, $i, j = 1, \dots, d$, of a scalar function $\mathbb{R}^d \ni x \mapsto \varrho(x) \in \mathbb{R}$, and $\partial_x^k \mathbf{u}$, $k = 1, 2$, for the collections of the first order and of the second order derivatives of a vector function $\mathbb{R}^d \ni x \mapsto \mathbf{u}(x) \in \mathbb{R}^d$. For simplicity we write, e.g., $\mathbf{u} \in L^p(\Omega)$ to mean that all components u_j of a vector function $\mathbf{u} = (u_1, \dots, u_d)$ belong to the space $L^p(\Omega)$, in other words $L^p(\Omega)$ stands here for $L^p(\Omega; \mathbb{R}^d)$. The same convention is used for other spaces, e.g., in our notation $(\mathbf{v}, \varphi, \zeta) \in C^{1+\gamma}(\Omega) \times C^{\gamma}(\Omega)^2$ means that all components of the vector function $\mathbf{v}$ belong to the space $C^{1+\gamma}(\Omega)$, and the scalar functions φ, ζ belong to $C^{\gamma}(\Omega)$.

The notation $\partial_x^2 \mathbf{u} \in L^p(\Omega)$ means that all the second order derivatives belong to $L^p(\Omega)$. In this way we avoid the notation with multiindices unless strictly necessary.

For a given symmetric tensor $\mathbb{S} = (S_{ij})$, its divergence is the vector denoted by $\operatorname{div}\mathbb{S}$ with the components $\operatorname{div}\mathbb{S}_i = \partial_{x_j} S_{ij}$, summed over $j = 1, 2, 3$. The product of two tensors is the scalar $\mathbb{A} : \mathbb{B} = A_{i,j} B_{i,j}$.

On the other hand, points in $\mathbb{R}^d$ are denoted by x, y with coordinates $x = (x_i), y = (y_i)$; this is an exception from the vector notation.

1.1 Functional analysis

1.1.1 Banach spaces

We recall some well known facts; the main sources are [24] and [131]. A *normed space* A is a linear space over the field of real numbers equipped with a norm

$\|\cdot\|_A : A \to \mathbb{R}$ such that

$$\|u\|_A \geq 0, \quad \|u\|_A = 0 \quad \text{if and only if} \quad u = 0,$$
$$\|\lambda u\|_A = |\lambda|\,\|u\|_A \quad \text{for all } \lambda \in \mathbb{R} \text{ and } u \in A,$$
$$\|u+v\|_A \leq \|u\|_A + \|v\|_A \quad \text{for all } u, v \in A.$$

A sequence $u_n \in A$ converges (converges strongly) to $u \in A$ if $\|u_n - u\|_A \to 0$ as $n \to \infty$. In this case we write $u_n \to u$ or $\lim_{n\to\infty} u_n = u$. A normed A space is *complete* if $\|u_m - u_n\|_A \to 0$ as $m, n \to \infty$ implies the existence of $u \in A$ such that $u_n \to u$. Complete normed spaces are named *Banach spaces.*

A set $G \subset A$ is open if for any $a \in A$ there is $\varepsilon > 0$ such that the ball $\{x \in A : \|x - a\|_A < \varepsilon\}$ is contained in A. A set $F \subset A$ is closed if for any sequence $F \ni u_n \to u$ the limit u belongs to F. Obviously F is closed if and only if $A \setminus F$ is open. The closure of $D \subset A$ is denoted by $\operatorname{cl} D$ or $\overline{D}$. We say that a set $D \subset A$ is *dense* in a set $E \subset A$ if $E \subset \operatorname{cl} D$.

Embedding. We say that a Banach space A is *continuously embedded* in a Banach space B or that the *embedding* of A in B *is bounded* if $A \subset B$ and there exists $c > 0$ such that $\|u\|_B \leq c\|u\|_A$ for all $u \in A$. In this case we write $A \hookrightarrow B$.

Product, sum and intersection. The Cartesian product $A \times B$ of Banach spaces A, B consists of all pairs (u, v), where $u \in A$, $v \in B$, and is equipped with the norm $\|u\|_A + \|v\|_A$. Let A, B be Banach spaces, both subsets of an ambient Banach space Z. Then the intersection $A \cap B$ equipped with the norm $\|u\|_{A\cap B} = \|u\|_A + \|u\|_B$ and the algebraic sum $A + B := \{w = u + v : u \in A,\ v \in B\}$ equipped with the norm

$$\|w\|_{A+B} = \inf\{\|u\|_A + \|v\|_B : u + v = w\}$$

are Banach spaces.

Linear operators. Let A, B be Banach spaces. Linear mappings $T : A \to B$ are called *linear operators.* A linear operator is *bounded* if it has a finite norm

$$\|T\|_{\mathcal{L}(A,B)} := \sup_{\|u\|_A \leq 1} \|Tu\|_B = \inf\{c : \|Tu\|_B \leq c\|u\|_A \text{ for all } u \in A\}.$$

Equipped with this norm, the set $\mathcal{L}(A, B)$ of bounded linear operators becomes a Banach space.

Duality. The *dual space* A' of a Banach space A consists of all continuous linear functionals $u' : A \to \mathbb{R}$. The duality pairing between A' and A is defined by $\langle u', u\rangle := u'(u)$. Equipped with the norm $\|u'\|_{A'} := \sup_{\|u\|_A \leq 1} |\langle u', u\rangle|$, A' becomes a Banach space. We have (see [49, Thm. 5.13])

Theorem 1.1.1. *Let Banach spaces A and B be subsets of a Banach space Z and suppose $A \cap B$ is dense in Z. Then*

$$(A \cap B)' = A' + B', \quad (A + B)' = A' \cap B'.$$