Classical Complex Analysis

A Geometric Approach — Vol. 2

I-Hsiung Lin

World Scientific

Classical Complex Analysis

A Geometric Approach — Vol. 2

I-Hsiung Lin

National Taiwan Normal University, Taiwan

World Scientific

NEW JERSEY • LONDON • SINGAPORE • BEIJING • SHANGHAI • HONG KONG • TAIPEI • CHENNAI

Published by

World Scientific Publishing Co. Pte. Ltd.
5 Toh Tuck Link, Singapore 596224
USA office: 27 Warren Street, Suite 401-402, Hackensack, NJ 07601
UK office: 57 Shelton Street, Covent Garden, London WC2H 9HE

British Library Cataloguing-in-Publication Data
A catalogue record for this book is available from the British Library.

CLASSICAL COMPLEX ANALYSIS — Volume 2
A Geometric Approach

ISBN-13 978-981-4271-28-8
ISBN-13 978-981-4271-29-5 (pbk)

Typeset by Stallion Press
Email: enquiries@stallionpress.com

Printed in Singapore by B & Jo Enterprise Pte Ltd

To my wife Hsiou-o and my children
Zing, Ting, Ying and Fei

Preface

Complex analysis, or roughly equivalently the theory of analytic functions of one complex variable, budded in the early ages of Gauss, d'Alembert, and Euler as a main branch of mathematical analysis. In the 19th century, Cauchy, Riemann, and Weierstrass laid a rigorous mathematical foundation for it (see Ref. [47]). Nourished by the joint effort of generations of brilliant mathematicians, it grows up into one of the remarkable branches of exact science, and serves as a prototype or model of other theories concerned with generalizations of analytic functions such as Riemann surfaces, analytic functions of several complex variables, quasiconformal and quasiregular mappings, complex dynamics, etc. Its methods and theory are widely used in many branches of mathematics, ranging from analytic number theory to fluid mechanics, elasticity theory, electrodynamics, string theory, etc.

Elementary complex analysis stands as a discipline to the whole mathematical training. This book is designed for beginners in this direction, especially for upper level undergraduate and graduate mathematics majors, and to those physics (or engineering) students who are interested in more theoretically oriented introduction to the subject rather than only in computational skills. The content is thus selective and its level of difficulty should be then adequately arranged.

Beside its strong intuitive flavor, it is the geometric (mapping) properties, derived from or characterized the analytic properties, that makes the theory of analytic functions differ so vehemently from that of real analysis and so special yet restrictive in applications. This is the reason why I favor a geometric approach to the basics. The degree of difficulty, as a whole, is not higher than that of L.V. Ahlfors's classic *Complex Analysis* [1]. But I try my best to give detailed and clear explanations to the theory as much as possible. I hope that the presentation will be less arduous in order to be more available to not-so-well-prepared or not-so-gifted students and be easier for self-study. Neither the greatest possible generality nor the most up-to-date terminologies is our purpose. Please refer to Berenstein and Gay

[9] for those purposes. I would consider my purpose fulfilled if the readers are able to acquire elementary yet solid fundamental classical results and techniques concerned.

Knowledge of elementary analysis, such as a standard calculus course including some linear algebra, is assumed. In many situations, mathematical maturity seems more urgent than purely mathematical prerequisites. Apart from these the work is self-contained except some difficult theorems to which references have been indicated. Yet for clearer and thorough understanding where one stands for the present in the whole mathematical realm and for the ability to compare with real analysis, I suggest readers get familiar with the theory of functions of two real variables.

Sketch of the Contents

If one takes a quick look at the Contents or read over Sketch of the Content at the beginning of each chapter and then s/he will have an overall idea about the book.

A complex number is not just a plane vector but also carries by itself the composite motion of a one-way stretch and rotation, and hence, is a two-dimensional "number". They constitute a field but cannot be ordered. Mathematics based on them is the one about similarity in global geometric sense and is the one about conformality in local infinitesimal sense. *Chapter 1* lays the algebraic, geometric and point-set foundations barely needed in later chapters.

Just as one experienced in calculus, we need to know some standard *elementary* complex-valued *functions* of a complex variable before complex differentiation and integration are formally introduced. It is the *isolated-zero principle* (see (3) and (4) in (3.4.2.9)) that makes many of their *algebraic properties* or algebraic identities similar to their real counterparts. Owing to the complex plane $\mathbf{C}$ having the same topological structure as the Euclidean plane $\mathbf{R}^2$, their *point-set properties* (such as continuity and convergence) are the same as the real ones, too. It is the *geometric mapping properties* owned by these elementary functions that distinguish them from the real ones and make one feel that complex analysis is not just a copy of the latter. In particular, the local and global single-valued continuous branches of $\arg z$ are deliberately studied, and then, prototypes of Riemann surfaces are introduced. *Chapter 2* tries to figure out, though loosely and vaguely organized, the common analytic and geometric properties owned by these individual elementary functions and then, to foresee what properties a general analytic function might have.

A complex valued function $f(z)$, defined in a domain (or an open set) Ω, is called *analytic* if any one of the following equivalent conditions is satisfied:

1. $f(z)$ is differentiable everywhere in Ω (Chap. 3).
2. $f(z)$ is continuous in Ω and $\int_{\partial\Delta} f(z)dz = 0$ for any triangle Δ contained in Ω (Chaps. 3 and 4).
3. For each fixed point $z_0 \in \Omega$, $f(z)$ can be expressed as a convergent power series $\sum_{n=0}^{\infty} a_n(z-z_0)^n$ in a neighborhood of z_0 (Chap. 5).

An analytic function $f(z)$ infinitesimally, via the Cauchy–Riemann equations, appears as a conformal mapping in case $f'(z) \neq 0$ and $\overline{f(z)}$ can be interpreted as the velocity field of a solenoidal, irrotational flow (see (3.2.4.3)). *Chapter 3* develops the most fundamental and important analytic and geometric properties, both locally and globally, which an analytic function might possess. The most subtle one, among all, is that a function $f(z)$ analytic at z_0 can always be written as $f(z) = f(z_0) + (z - z_0)\varphi(z)$ where $\varphi(z)$ is another function analytic at z_0. From this, many properties, such as the isolated-zero principle, maximum–minimum modulus principle and the open mapping property, inverse and implicit function theorems, can be either directly or indirectly deduced. This chapter also studies some global theorems such as Schwarz's Lemma, symmetry principle, argument principle and Rouché's theorem and their illustrative examples.

After proving homotopic and homologous forms of Cauchy integral theorem, most part of Chapter 4 is devoted to the residue theorem and its various applications in evaluating integrals, summation of series, and the Fourier and Laplace transforms.

Chapter 5 starts with various local power series representations of an analytic function and analytic continuation of power series which eventually lead to the monodromy theorem. Besides power series representation, an entire function can also be factorized as an infinite product of its zeros such as polynomials do, and meromorphic function can be expanded into partial fractions via its poles as rational functions. The most remarkable example is Euler's gamma function $\Gamma(z)$ and its colorful properties. Riemann zeta function is only sketched. The essential limit process in the whole chapter is the method of *local uniform convergence*. Weierstrass's theorem and Montel's normality criterion are two of the most fundamental results in this direction. Both are used to prove Picard's theorems via the elliptic modular function. These classical theorems can also be obtained by Schwarz–Ahlfor's Lemma, a geometric theorem.

Riemann mapping theorem initiated the study of global geometric mapping properties of univalent analytic functions, simply called univalent mappings. Schwarz–Christoffel formulas provide fruitful examples for the theorem. As a consequence, *Chapter 6* solves Dirichlet's problem for an open disk, a Jordan domain and hence, a class of general domains via Perron's method. This, in turn, is adopted to determine the canonical mappings and canonical domains for finitely connected domains.

Based on our intuitive and descriptive knowledge about Riemann surfaces of particularly chosen multiply-valued functions, scattered from Chaps. 2–6, *Chapter* 7 tries to give a formal, rigorous yet concise introduction to *abstract* Riemann surfaces. We will cover the fundamental group, covering surfaces and covering transformations, and finally highlight the proof of the uniformization theorem of Riemann surfaces via available classical methods, even though most recently it admits a purely differential geometric one-page proof [17].

Almost all sections end up with *Exercises A* for getting familiar with the basic techniques and applications; most of them also have *Exercises B* for practice of combining techniques and deeper thinking or applications; few of them have *Exercises C* for extra readings of a paper or *Appendixes A–C*.

As far as starred sections are concerned, see "How to use the book" below.

Features of the Book

(1) *Style of Writing.* As a textbook for beginners, I try to introduce the concepts clearly and the whole theory gradually, by giving definite explanations and accompanying their geometric interpretations whenever possible. Geometric points of view are emphasized. There are about 546 figures and many of them are particularly valid or meaningful only for complex variable but not for reals. Most definitions come out naturally in the middle of discussions, while main results obtained after a discussion are summarized and are numbered along with important formulas.

(2) *Balance between Theory and Examples.* As an introductory text or reference book to beginners, how to grasp and consolidate the basic theory and techniques seems more important and practical than to go immediately to deeper theories concerned. Therefore there are sufficient

amount of examples to practice main ideas or results. Exercises are usually divided into parts A and B; the former is designed to familiarize the readers with the established results, while the latter contains challenging exercises for mature and minded readers. Both examples and exercises are classic and are benefited very much from Refs. [31, 52, 58, 60, 80]. What should be mentioned is that many exercise problems in Ref. [1] have been adopted as illustrative examples in this text.

(3) *Careful Treatment of Multiple-Valued Functions.* Owing to historical and pedagogical reasons, complex analysis is conventionally carried out in the (one-layer) classical complex plane. Later development shows that the most natural place to do so is multiple-layer planes, the so-called Riemann surfaces or one-dimensional complex manifolds in its modern terminology. Multiple-valuedness is a difficult subject to most beginners and most introductory books just avoid or sketch it by focusing on $\sqrt[n]{z}$ and $\log z$ only. To provide intuitive feeling toward abstract Riemann surfaces in Chap. 7, Chaps. 2–6 take no hesitation to treat multiple-valued analytic functions whenever possible in the theory and in the illustrative examples. Once the trouble-maker $\arg z$, the origin of multiple-valuedness, is tamed (see Sec. 2.7.1), what is left is much easier to handle with. Also we construct many (purely descriptive and nonrigorous) Riemann surfaces or their line complexes of specified multiple-valued functions, merely for purposes of clearer illustration, wherever we feel worthy to do so.

(4) *Emphasis on the Difference between Real and Complex Analyses.* The origin of all these differences comes from the very character of what a complex number is (see the second paragraph inside the title Sketch of the Contents). This fact reflects, upon differentiating process, in the aspects of algebra, analysis as well as in geometry (see (3.2.2) for short; Secs. 3.2.1–3.2.3 for details).

(5) *Paving the Way to Advanced study.* The contents chosen are so arranged that they will provide solid background knowledge to further study in fields mentioned in the first paragraph of this Preface. Besides, the book contains more materials than what is required in a Ph.D. qualifying examination for complex analysis.

How to Use the Book (a Suggestion to the Readers)?

The book is rich in contents, examples, and exercises when comparing to other books on complex analysis of the same level. It is designed for a variety of usages and motivations for advanced studies concerned.

The whole content is divided into two volumes: Volume 1 contains Chaps. 1–4 and Appendix A, while Volume 2 contains Chaps. 5–7 and Appendixes B and C.

May I have the following suggestions for different proposes:

Chapters 1 and 2 are preparatory. Except those basic concepts such as limits and functions needed, topics in these two chapters could be selective, up to one's taste.

(1) As an introductory text for undergraduates
Sections 2.5.2, 2.5.4, 2.6, 2.7.2, 2.7.3 (sketch only), 2.9 (sketch only); 3.2.2, 3.3.1 (only basic examples and $\sqrt[n]{z}$, $\log z$), 3.3.2, 3.4.1–3.4.4, 3.5.1–3.5.5, 3.5.7 (sketch only), 4.8, 4.9, 4.10 (sketch only), 4.11 (sketch only), 4.12.1–4.12.3, 5.3.1, 5.4.1, 5.5.2 (optional and sketch).
As a whole, examples and Exercises A should be selective. Minded readers should try more, both examples and exercises, and pay attentions to more elementary multiply-values functions and their Riemann surfaces, if possible.
In a class, the role played by a lecture to select topics is crucial.

(2) As a beginning graduate text
With a solid understanding of materials in (1), the following topics are added: Secs. 2.8, 3.4.5, 3.4.6, 4.1–4.7, 4.12.3A–4.15 (selective), 5.1.3, 5.2, 5.3.2, 5.5–5.6 (selective), 5.8.1–5.8.3; Chap. 6 except 6.6.4.
Examples and Exercises A (even Exercise B) should be emphasized. Of course, the adding or deleting some topics are still possible.

(3) To readers whose are interested in Riemann surfaces
Pay more attention to multiply-valued functions and their descriptive Riemann surfaces such as Secs. 2.7, 3.3.3, 3.4.7, 3.5.6, 5.1, 5.2, and end up with the whole Chap. 7.

(4) Several complex variables
Sections 3.4.7, 3.5.6; Chap. 7.

(5) Quasiconformal mappings and complex dynamical systems
Section 3.2.3, Example 2 in Sec. 3.5.5; Secs. 5.3.4, 5.8; 6.6.4; Chap. 7, and Appendix C.

(6) As a general reference book supplement to other books on complex analysis.

Acknowledgments

The following students in Mathematics Department helped to type my hand-written manuscript:

Jing-ya Shui; Ya-ling Zhan; Yu-hua Weng; Ming-yang Kao; Wei-ming Su; Wen-jie Li; Shuen-hua Liang; Shi-wei Lin; D.C. Peter Hong; Hsuan-ya Yu; Yi-hsuan Lin; Ming-you Chin; Che-wei Wu; Cheng-han Yang; Kuo-han Tseng; Yi-ting Tsai; Yi-chai Li; Po-tsu Lin; Hsin-han Huang.

Yan-yu Chen graphed all the figures that appeared in this book. Yan-yu Chen, Aileen Lin, Wen-jie Li, and Ming-yang Kao helped to edit the final manuscript for printing. Here may I pay my sincerest thanks to all of them. Without their unselfish dedication, this book definitely could have not been published so soon.

Also, teaching assistant Jia-ming Ying helped to improve and correct partially my English writing. My colleagues Prof. Tian-yu Tsai proof-read the entire manuscript, and Prof. Yu-lin Chang proof-read Chaps. 5–7 and adopted parts of the content in his graduate course on complex analysis. Both of them pointed out many mistakes and gave me valuable suggestions. It would be my pleasure to express my gratitude toward them for their kindest help.

As usual, teaching assistant Ching-yu Yang did all the computer work for the several editions of the manuscript. Ms. Zhang Ji and Ms. Tan Rok Ting, editors in World Scientific, copy-edited the whole book with carefulness and expertise. Thank them so much.

I-hsiung Lin
2009/1/21
Taipei, Taiwan, China

Contents

Volume 2

CHAPTER 5

Fundamental Theory: Sequences, Series, and Infinite Products

Introduction

A (single-valued) function $f(z) : \Omega$ (an open set) $\to \mathbf{C}$ is said to be *analytic* in Ω if, for each point $z_0 \in \Omega$, $f(z)$ can be expressed locally by a power series $\sum_0^\infty a_n(z - z_0)^n$ in a neighborhood of z_0. The various power series representations of the same $f(z)$ at various points in Ω are connected by either direct or indirect *analytic continuation* (see (5.1.2.2), (5.1.3.5), (5.1.3.6), and Sec. 5.2). The starting points in this approach to the theory are the contents of Sec. 3.3.2; in particular, that of (3.3.2.4). Its main advantage over the differentiation (Chap. 3) and the integration (Chap. 4) methods lies on the fact that it can be easily applied to the study of the *local behaviors* of analytic functions (see, for instance, Secs. 5.1.2 and 3.5.1) and, hence, the introduction of the concept of *abstract Riemann surfaces* (see Section 3.3.3, Chap. 7, and, of course, the monumental book [81] by H. Weyl).

Sketch of the Content

Section 5.1 (and Sec. 3.3.2) investigates mainly the properties of an analytic function $f(z)$ defined by a *single* convergent power series $\sum_0^\infty a_n(z - z_0)^n$ in its open disk of convergence and on the boundary circle. In this restricted yet special case, it is easy to obtain important results such as the max–min modulus principle, and the Cauchy integral theorem and formula. The existence of at least one *singular point* on the circle of convergence presents unexpected complicated situations about the boundary behavior of a power series. And *Abel's limit theorem* in this direction is fundamental.

A convergent power series can be *analytically continued* through its *regular points* (if exist) on the circle of convergence to as far as possible in the complex plane. The resulted function is usually multiple-valued in its domain, and, indeed, will be *single-valued* if *the domain is simply-connected.* Section 5.2 highlights the proof of the *monodromy theorem.*

The important role played by the *local uniform convergence* of a sequence or a series of analytic functions is formally introduced and studied in Sec. 5.3. In addition to the basic *Weierstrass's theorem*, some other criteria, originated mainly from the maximum principles, for local uniform convergence are also derived; they include *Vitali's* and *Montel's theorems* (5.3.3.10). This kind of convergence *preserves* analyticity and the number of zeros (*Hurwitz's theorem*), and creates wonderful phenomena about the fixed points of an analytic function and its iterate functions.

Mittag–Leffler's partial fractions theorem extends the partial fraction expansion of a rational function, as shown in (2.5.3.5), to meromorphic functions defined on the entire plane. Section 5.4 adopts three methods to achieve this, including Cauchy's residue method and the $\overline{\partial}$-operator method (see Exercises B of Sec. 5.4.1).

Section 5.5 extends the factor product expression of a polynomial, as shown in (2.5.1.2), to entire functions, usually known as *Weierstrass's factorization theorem*, including the canonical form, genus and *Hadamard's order theorem.* The introduction of infinite products is sketched and preparatory.

The *gamma function* $\Gamma(z)$(see Sec. 5.6) and the *Riemann zeta function* $\zeta(z)$ (see Sec. 5.7) are two important illustrative examples of the materials in both Secs. 5.4 and 5.5. Various representations and characteristic properties, including *Stirling's formula*, of $\Gamma(z)$ are studied; while, that of $\zeta(z)$ is only sketched.

The concept of *normal* sequence (family) of analytic functions is the version for functions of Bolzano–Weierstrass's theorem for bounded infinite point set or sequence. Section 5.8 devotes to the study of the criteria for normality and its applications. The *elliptic modular function* is adopted to prove the important *Montel's normality criterion* which, in turn, is used to prove the famous *Picard's first and second theorems.* In Sec. 5.8.4, various types of *Schottky's theorem* are introduced and provide another proofs of Montel's and Picard's theorems. Also, L. V. Ahlfors (1938) extended Schwarz–Pick's lemma (see (3.4.5.2)) to *Schwarz–Ahlfor's lemma* (5.8.4.14), in the content of Poincaré's metric and curvature, which provides differential geometric proofs to some fundamental results such as Livioulle's theorem, Schottky's theorem, Montel's and Picard's theorems. To the end (see Sec. 5.8.5), we present some results in complex dynamical system as another meaningful application of the concept of normality.

5.1 Power Series

Before we start, it is supposed that the readers are familiar with the content of Sec. 3.3.2; in particular, (3.3.2.3) and (3.3.2.4). We will use these results directly.

5.1.1 *Algorithm of power series*

What we are going to do here is to provide a detailed account about (5) in (4.8.1) and the readers are urged to review examples presented there. Also, the readers might refer to Chap. 1 of Ref. [16] for a discussion of the algebra of *formal* power series.

Given two power series

$$f(z) = \sum_0^\infty a_n(z-z_0)^n, \quad |z-z_0| < r_1 \text{ (the radius of convergence)};$$

$$g(z) = \sum_0^\infty b_n(z-z_0)^n, \quad |z-z_0| < r_2 \text{ (the radius of convergence)}.$$

Three easier operations are as follows:

The identity operation:

$$\sum_0^\infty a_n(z-z_0)^n = \sum_0^\infty b_n(z-z_0)^n, \quad |z-z_0| \le \min(r_1, r_2)$$
$$\Leftrightarrow a_n = b_n, \quad \text{for } n \ge 0. \tag{5.1.1.1}$$

The addition and subtraction operations:

$$f(z) \pm g(z) = \sum_0^\infty (a_n \pm b_n)(z-z_0)^n, \quad |z-z_0| \le \min(r_1, r_2). \tag{5.1.1.2}$$

The Cauchy product operation: By absolute convergence of both series in $|z-z_0| \le \min(r_1, r_2)$, then

$$f(z)g(z) = \sum_{n=0}^\infty (a_n b_0 + a_{n-1}b_1 + \cdots + a_0 b_n)(z-z_0)^n,$$
$$|z-z_0| \le \min(r_1, r_2). \tag{5.1.1.3}$$

Proofs are left to the readers.