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Dynamical Systems on Homogeneous Spaces

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American Mathematical Society

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Dynamical Systems on Homogeneous Spaces

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ДИНАМИЧЕСКИЕ СИСТЕМЫ НА ОДНОРОДНЫХ ПРОСТРАНСТВАХ

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ABSTRACT. The book provides an extensive survey of the theory of dynamical systems on homogeneous spaces (the so-called homogeneous flows). In particular, the author discusses ergodicity and mixing of homogeneous flows, the theory of unipotent flows (including recent proofs of the Raghunathan and Dani conjectures), and the dynamics of nonunipotent flows. The final chapter covers applications of homogeneous flows to number theory, mainly to the theory of Diophantine approximations. In particular, the author describes in detail the famous proof of the Oppenheim-Davenport conjecture using ergodic properties of homogeneous flows.

The book will be useful to graduate students and research mathematicians working in dynamical systems, ergodic theory, and number theory.

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Dynamical Systems on Homogeneous Spaces

Introduction

In 1980, while a third year student at Moscow State University, the author was given[1] a rather simply formulated problem to solve. Take a connected Lie group G and a discrete subgroup Γ with the manifold G/Γ compact (such a subgroup Γ is called a *uniform lattice* in G). Now multiply Γ on the left by a one-parameter subgroup $g_{\mathbb{R}} \subset G$ and take the closure $\overline{g_{\mathbb{R}}\Gamma} \subset G/\Gamma$. The question: Is this closure a submanifold in G/Γ? If one lifts everything to the group G, the problem seems purely algebraic (and easy to solve if G is commutative or compact).

One can somewhat extend the problem and consider all *orbits* $g_{\mathbb{R}}x$, $x \in G/\Gamma$, of the action $(t, x) \to g_t x$, called a *homogeneous flow*. A more cautious formulation is the following: Is it true that the closure $\overline{g_{\mathbb{R}}x} \subset G/\Gamma$ is a manifold for a generic orbit (that is, for almost all $x \in G/\Gamma$)? This question is related to ergodic theory (an area about which the author then had only a rather vague idea). Experts advised him in this connection to read the book of Auslander, Green and Hahn [**AGH**] "Flows on homogeneous spaces", written in the early sixties. This is how the author found himself in the area that combines ergodic theory and Lie groups. As became clear later, the problem of orbit closures was already at the focus of attention of experts due to applications in number theory, but the main events were yet to follow.

Having no idea about the story, the author first decided to study the case of nilpotent Lie groups G, the next in complexity after the commutative case. Fortunately, the book [**AGH**] contained all the information necessary to solve the problem: Green's ergodicity criterion and a result due to Auslander saying that any ergodic nilflow $(G/\Gamma, g_{\mathbb{R}})$ is *minimal*, that is, all orbits are dense in G/Γ. An easy induction shows that in the nonergodic case, the closures $\overline{g_{\mathbb{R}}x}$, as for commutative groups G, are not only submanifolds but even *homogeneous subspaces* (that is, for every point $x \in G/\Gamma$ there exists a subgroup $F \subset G$ such that $\overline{g_{\mathbb{R}}x} = Fx$). The next class – compact extensions of nilpotent groups – revealed that all orbit closures are still smooth but not always algebraic.

To report the results obtained, the author came to (and became a constant participant in) the seminar on dynamical systems headed by D. V. Anosov and A. M. Stepin. It became clear at once that in the general case one cannot hope that all orbits are "good". As an example one takes either the geodesic flow on a compact surface of constant curvature -1 (here $G = \mathrm{SL}(2, \mathbb{R})$), or the suspension over the automorphism $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$ of the torus $\mathbb{T}^2$ (this gives rise to a three-dimensional solvable group of exponential type). In both cases one obtains *Anosov flows*, the theory of which was developed by Anosov, Sinai, Smale, et al. Orbits of such flows were studied in detail by means of symbolic dynamics (see, for instance, [**Al**] or

[1] In connection with integration of Hamiltonian systems [**N, KK, MF**].

[**Bo79**]). Moreover, long ago, back in the twenties, Morse had constructed an orbit of the geodesic flow whose closure is nowhere locally connected (and locally looks like the product of a Cantor set and an interval)! (Note that the algebraic origin of the geodesic flow was first explored only in the fifties by Gelfand and Fomin [**GF**], and this led to the modern theory of dynamical systems on homogeneous spaces.)

Still, one could hope that the closure of a generic orbit is a manifold. This is supported by the easy observation that almost all orbits of an ergodic flow are everywhere dense. Moreover, it is natural to extend the class of homogeneous spaces under question by considering all spaces G/D of *finite volume*, that is, those carrying a smooth finite G-invariant measure (such spaces need not be compact, as is seen from the example $\mathrm{SL}(2,\mathbb{R})/\mathrm{SL}(2,\mathbb{Z})$). Thus, one has to study *nonergodic* flows on spaces of finite volume. It turned out that the ergodicity criterion for homogeneous flows was essentially known by that time. More precisely, in the sixties Moore [**Mo66**] examined the semisimple case, and Auslander [**Au**] the solvable case. Spectral invariants of ergodic flows in these cases were found by Stepin [**Ste69,73**] and Safonov [**Saf**]. The key step in reducing the general case to the semisimple and solvable cases was made by Dani [**Da77**]. This stage of the theory of homogeneous flows was summarized in the paper of Brezin and Moore [**BM**], where they found the ergodicity criterion and calculated the spectrum of homogeneous flows on the so-called admissible spaces of finite volume.[2] It remained to construct explicitly the *ergodic decomposition* of homogeneous flows. The fact that the space G/Γ breaks into closed invariant (not necessarily homogeneous) submanifolds such that the flow on each of them is ergodic relative to a smooth invariant measure was not difficult to prove [**St83**]. But even this implies that the closure of a generic orbit is smooth (more specifically, is equal to the corresponding ergodic submanifold). More surprising was that every ergodic manifold can be finitely covered (together with the flow thereon) by a homogeneous space of finite volume (see [**St86b,89**]). Therefore the theory of homogeneous flows was reduced to the ergodic case.

These results are expounded in a fairly detailed way in Chapter 1 of the present book. It is worth noting that the ergodicity criterion is based on the Mautner phenomenon [**Mo80**] for unitary representations; to establish the latter we apply a nice argument suggested by Margulis [**Ma91b**] which does not involve spectral theory. While studying homogeneous spaces we apply the methods of algebraic groups.

As for *individual orbits*, it became clear after the example with the geodesic flow that everything depends on the subgroup $g_\mathbb{R} \subset G$. On one hand, it may be the case that the action of $g_\mathbb{R}$ by left translations on the group G has at most a polynomial rate of divergence of close orbits relative to the right-invariant metric (then $g_\mathbb{R}$ is called a *quasi-unipotent* subgroup). Considering the adjoint action of g_t on the Lie algebra of the group G shows that such a subgroup must have all eigenvalues of absolute value 1 (if all eigenvalues are equal to 1, the subgroup $g_\mathbb{R}$ is called *unipotent*). On the other hand, the rate of divergence may be exponential (then $g_\mathbb{R}$ is said to be *partially-hyperbolic*). In this case the flow on G/Γ induced

[2] Still, the author was there in time to make "one or two lucky misimprovements" [**St86a,87a**]. In particular, he proved that every space of finite volume is admissible (see also [**Wu, Wi87, Z87**]). Thus, this subject was closed.

by $g_{\mathbb{R}}$ is *uniformly partially-hyperbolic.*[3] Such a flow possesses the *contracting and expanding* smooth foliations formed by orbits of the associated *horospherical* subgroups. For homogeneous actions, the techniques of partially-hyperbolic flows were already applied in the sixties by Auslander, Moore and Stepin; the general theory of smooth uniformly partially-hyperbolic flows was built in the late seventies by Pesin and Brin. An impression emerged that all orbit closures of a quasi-unipotent flow are always smooth (this was supported by examining the class of solvable groups G), whereas a partially-hyperbolic flow must have a "bad" orbit (see [**St87b**]).

In the spring of 1985, G. A. Margulis gave a talk at the seminar about a certain application of homogeneous flows to number theory, and clarified the situation with the orbit closures conjecture. First, any (not necessarily homogeneous) uniformly partially-hyperbolic flow on a compact manifold has an orbit which does not come back to a neighborhood of the original point (and hence its closure is not a manifold). Second, as followed from the talk, the situation with (quasi)unipotent flows is much more delicate. More precisely, in the seventies Raghunathan noticed a connection between unipotent orbits on $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$ and the long-standing Oppenheim–Davenport conjecture on values of quadratic forms at integral points: the latter would be settled if one managed to prove that closures of all bounded orbits of the unipotent subgroup

$$u_t = \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}$$

are algebraic. The solution of this problem was precisely the subject of the talk (see [**Ma87**]). There is a more general conjecture (let us call it the *Raghunathan topological conjecture*, though it was formulated by Dani in [**Da81**]) stating that all orbits of a unipotent flow must have algebraic closures (that is, they must be homogeneous subspaces of finite volume).[4] Moreover, all ergodic measures of such a flow must have an algebraic origin (the *Dani measure conjecture*).

By that time, the case of a solvable group G was completely understood (see [**St84a**]). However, the most important affirmations of these conjectures come from the study of the *horocycle flow* induced by the subgroup $h_t = \left(\begin{smallmatrix} 1 & t \\ 0 & 1 \end{smallmatrix}\right) \subset G = \mathrm{SL}(2,\mathbb{R})$. Back in the thirties, Hedlund (by purely geometric methods) had proved that h_t has only periodic and everywhere dense orbits provided that the space G/Γ is of finite volume; moreover, the flow h_t is minimal if G/Γ is a compact space! The corresponding results on ergodic measures were obtained by Furstenberg [**Fu72**] and by Dani and Smillie [**DS**]. The conjectures were also supported by results of [**Bo76, V75, V77, EP**] on actions of horospherical subgroups on compact homogeneous spaces. Every horospherical subgroup is unipotent, but not vice versa: for instance, the main difficulty in the study of the flow $(\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z}), u_{\mathbb{R}})$ is that the subgroup $u_{\mathbb{R}}$ is not horospherical. In the general case, as was mentioned by Margulis, the Raghunathan–Dani conjectures seem to be very difficult to prove (especially the measure part).

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[3] Experts in smooth dynamical systems will notice at once that the quasi-unipotent and partially-hyperbolic cases correspond to whether or not the entropy of the flow is trivial.

[4] Thanks to the above remark of Margulis, the Raghunathan conjecture easily implies the criterion for all orbit closures to be smooth [**St90**].

The difficulties in studying unipotent actions were already clear from an early result of Margulis [**Ma71**] saying that no unipotent trajectory in $\mathrm{SL}(n,\mathbb{R})/\mathrm{SL}(n,\mathbb{Z})$ exits to infinity (this was utilized in one of the proofs of his celebrated arithmeticity theorem). The proof was highly nontrivial (now it is more accessible – see [**DM90b, KM98**]), but the result did not even ensure that such a trajectory is recurrent (that is, comes back to any neighborhood of the original point)! Nevertheless, it proved to be one of the most important tools for studying dynamics of unipotent flows – especially after Dani [**Da84,86b**] established a general result of this kind stating that any unipotent trajectory on a space of finite volume visits a compact subset with positive density of times.

In the eighties and nineties the study of unipotent actions took the central position in the theory. Partial results (again for horospherical subgroups) were obtained in [**Da86a**] and [**St91**], and for unipotent subgroups of $\mathrm{SL}(3,\mathbb{R})$ in [**DM89,90a,b**]. On the other hand, in the early eighties Ratner discovered surprising rigidity properties of the horocycle flow [**R82a,b,83**]. For instance, a measure-theoretic isomorphism of two horocycle flows turned out to have an algebraic origin.[5] The proof utilized an H-property of the horocycle flow which made the polynomial nature of divergence more precise and, like the Margulis homecoming theorem, was based on certain properties of polynomials (subsequently Witte [**Wi85**], by exploring Ratner's idea, established the measure-theoretic rigidity of all ergodic unipotent flows). It was Ratner who by applying the so-called R-property of arbitrary unipotent flows (a modification of the H-property of the horocycle flow) managed to prove the measure conjecture in a series of papers [**R90a,b,c,91a**] whose length totals more than 100 pages. It is important to emphasize that the algebraicity of finite ergodic measures was proved in a much more general situation than it was originally conjectured: for *arbitrary* discrete subgroups $\Gamma \subset G$. Quite soon, by applying the measure conjecture and the Dani–Margulis results on homecoming, she managed to settle the topological conjecture as well [**R91b**] (simultaneously, Shah [**Sh91**], making use of [**R91a**], proved the conjecture for regular unipotent subgroups of semisimple groups of rank 1). In a sense, these two theorems of Ratner topped off the present stage of the theory of homogeneous flows. Margulis and Tomanov [**MT94,96**] (by using some ideas of Ratner as well as some from earlier papers of Dani and Margulis) suggested a more accessible proof of Ratner's measure theorem. Earlier Dani and Margulis [**DM93**] gave an alternative proof of the topological theorem.

In Chapter 2 we try to give an idea of what is involved in studying unipotent flows. The exposition starts with a detailed treatment of the geodesic and horocycle flows. Unfortunately, in the general case the proof of the measure conjecture is too complicated to reproduce within the framework of this book. To give its flavour, we examine the proof on the examples $G = \mathrm{SL}(2,\mathbb{R})$ and $G = \mathrm{SL}(2,\mathbb{C})$, following [**R92**] and [**MT96**] respectively. The proof of Margulis' homecoming theorem for unipotent trajectories is also demonstrated on the simplest example $G = \mathrm{SL}(2,\mathbb{R})$. The topological conjecture is proved in full scope along the lines of [**DM93**].

For one-parameter unipotent flows there is a stronger result: every trajectory not only has its closure homogeneous, but is uniformly distributed therein [**R91b**]. We expound various generalizations of the assertion which are due to Dani and Margulis [**DM93**], Ratner [**R94**], Eskin, Mozes and Shah [**MS**], [**Sh94**], [**EMS96,97**].

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[5] For nilflows, the corresponding results were earlier proved by Parry [**P71,73**].

Essentially, all of them are concerned with the convergence in the weak* topology on the space of probability measures on G/Γ.

Of course, the proof of the Raghunathan–Dani conjectures gave a new stimulus to research in this area and served as a base for numerous applications. For instance, in the class of arbitrary one-parameter homogeneous flows one managed to settle Rokhlin's problem on multiple mixing and to better understand their minimal sets (Starkov [**St93,95a**]). There has been great progress in studying orbits and ergodic measures for actions of arbitrary subgroups $F \subset G$ (Ratner [**R94**], Mozes [**Moz95a**], Shah and Weiss [**Sh96, W, SW**], Katok and Spatzier [**KS96**], Margulis and Tomanov [**MT96**]).

Also, in Chapter 3 we discuss the following topics. After Dani [**Da85,86c**] found the Hausdorff dimension of the family of bounded orbits in a special case, the full solution of the problem was given by Kleinbock and Margulis [**KM96**]. Nice results on topological rigidity are due to Markus [**Mar83**], Benardete [**Ben**] and Witte [**Wi90**]. The structure of time changes for homogeneous flows is not well understood: we only mention results of Livshitz [**Liv**] and Katok and Spatzier [**KS94**] for Anosov actions, a classical theorem of Kolmogorov [**Ko**] for rectilinear flows on the 2-torus (subsequently developed by many authors) and a result of Ratner [**R78,79**] for horocycle flows. In recent times, the phenomenon of exponential mixing of certain homogeneous K-flows became one of the most powerful tools (Moore [**Mo87**], Ratner [**R87a**], Katok and Spatzier [**KS94**], Kleinbock and Margulis [**KM96,99**]).

It is worth emphasizing that, due to newly found *applications in number theory*, the interest in homogeneous flows rose considerably. For instance, at two recent International Congresses of Mathematicians three lectures were devoted to the theory of homogeneous flows, and two of them (those of Margulis at ICM-90 and of Ratner at ICM-94) were plenary ones. One of the first applications of homogeneous flows in number theory was given in [**AGH**], where nilflows were utilized to prove Weyl's theorem on uniform distribution of the fractional parts of values of polynomials. A classical result due to Kolmogorov [**Ko**] on time changes for rectilinear flows on the 2-torus reveals a connection with Diophantine properties of the rotation number. Dani [**Da85**] showed how Diophantine properties of reals affect the behavior of corresponding orbits of the geodesic flow. There exist numerous refinements of this connection; in particular, a well-known conjecture of Littlewood reduces to a certain (not yet established) statement on the structure of orbits of the Cartan diagonal subgroup on $\mathrm{SL}(3,\mathbb{R})/\mathrm{SL}(3,\mathbb{Z})$. After Margulis settled the Oppenheim–Davenport conjecture, Dani and Margulis [**DM93**], and also Eskin, Mozes and Margulis [**EMM**], found the exact asymptotics of the number of Diophantine solutions of the inequality $a < Q(x) < b$ for indefinite quadratic forms.

Kleinbock and Margulis [**KM98**] obtained remarkable results proving a general conjecture of Sprindzhuk [**Sp80**] on Diophantine approximation on manifolds. Skriganov [**Skr98**] used homogeneous flows to obtain asymptotic estimates of the number of lattice points inside polyhedra (the first results in the two-dimensional case go back to the twenties and are due to Hardy and Littlewood [**HL**] and Khintchine [**Kh23**]). Eskin, Mozes and Shah [**EMS96**] sharpened results of [**DRS**] and [**EM**] on the asymptotics of the number of lattice points on homogeneous manifolds in $\mathbb{R}^n$. All of these questions are discussed in Chapter 4. Before that, we give preliminaries from the theory of Diophantine approximation.

A considerable number of recently found applications show that the theory of homogeneous flows is far from being completed. The reader especially interested in number theory is referred to surveys of Borel [**B**] and Margulis [**Ma97**].

For various reasons, we do not touch upon certain subjects. These are Bernoulli properties of homogeneous flows, Markov partitions and symbolic dynamics (see [**Si68**], [**Kat**], [**Bo79**], [**OW**]); the Bernoullicity criterion was studied by Dani [**Da76a**] but in the general case it is not yet proved. All Lie groups throughout the book are real; hence we are not concerned with dynamical systems on homogeneous spaces over other local fields; we only note that the Raghunathan–Dani conjectures are settled in this setup as well (see [**R93,95a,98**] and [**MT94,96**]). Entropy theory will also play a minor role. We do not touch upon the theory of flows on spaces of infinite volume. Whereas the solvable case is completely understood [**St87b**], the case $G = \mathrm{SO}(1,n)$ presents an independent and rather delicate object of research in the theory of Fuchsian and Kleinian groups. Here the results are not definitive; see the book [**Ni**] and the survey [**St95b**].

Other subjects we try to present in as much detail as possible. In addition to [**AGH**], we intensively used the surveys [**Da96,99**], [**Gh**], [**Ma91b,97**], [**R84,95b**], [**SSS**], [**St97**]. The reader may consult the list of open problems given in [**Ma99**] to find directions for future research.

The author is very grateful to A. S. Mishchenko, whose question introduced him into the topic. Since then, the author had a happy opportunity to interact with many remarkable experts in ergodic theory and Lie groups: D. V. Anosov, S. Dani, A. B. Katok, G. A. Margulis, M. E. Ratner, A. M. Stepin, E. B. Vinberg. The author is also happy to acknowledge the great impact of conversations with A. Eskin, D. Kleinbock, S. Mozes, V. V. Ryzhikov, N. Shah, G. Tomanov, D. Witte, and many others.

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List of Notations

$\mathbb{Z}$ — the ring of integers
$\mathbb{N}$ — the semigroup of positive integers
$\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$ — the fields of rational, real and complex numbers
G, H, F, etc. — Lie groups
$\mathfrak{g}$, $\mathfrak{h}$, $\mathfrak{f}$, etc. — Lie algebras
$\Gamma \subset G$ — discrete subgroup of a Lie group G
G_s — the semisimple splitting of a Lie group G
Ad — the adjoint representation
$\mathrm{Aut}(G)$ — the automorphism group of a Lie group G
$D_0 \subset G$ — the identity component of a closed subgroup $D \subset G$
$\overline{D} \subset G$ — the closure of a subset $D \subset G$
$g_{\mathbb{R}} \subset G$ — one-parameter subgroup of a Lie group G
$g^{\mathbb{Z}} \subset G$ — the cyclic group generated by an element $g \in G$
G/D — the homogeneous space of a Lie group G by a closed subgroup $D \subset G$
$(G/D, g_{\mathbb{R}})$ — one-parameter homogeneous flow
$E(g_{\mathbb{R}})$, $\widetilde{E}(g_{\mathbb{R}})$ — the ergodic partition and decomposition of a homogeneous flow
$\mathrm{SL}(n, \mathbb{R})$ — the group of unimodular real $n \times n$ matrices
$A^{\intercal}$ — the transpose matrix
$\mathrm{diag}(t_1, \ldots, t_n)$ — the diagonal matrix with elements $t_1, \ldots, t_n$
det — the determinant of a matrix
card — the cardinality of a set
(X, μ) — measure space
$\mathcal{P}(X)$ — the space of Borel probability measures on a topological space
$C_c(X)$ — the space of continuous functions with compact support on a topological space
$\mathbb{H}$ — Hilbert space
$\mathrm{Im}(\phi)$ — the image of a homomoprhism ϕ
$\mathrm{Ker}(\phi)$ — the kernel of a homomorphism ϕ
$\mathbb{H}^n$ — the n-dimensional Lobachevskii space
$\mathcal{H}^n$ — the n-dimensional Hausdorff measure

Contents

Preliminaries

We recall some definitions from topology which are relevant to our purposes. All topological spaces X are assumed to be *metrizable*, *locally compact* (that is, every point $x \in X$ has a neighborhood with compact closure) and separable (that is, X contains a countable dense subset). In particular, X always possesses a *countable base* of open sets (that is, satisfies the *second countability axiom*).

Borel sets in X are sets obtained from open sets by at most countably many operations of union, intersection and complementation. The family of Borel sets forms a σ-algebra (in particular, a countable intersection and a countable union of Borel sets are Borel sets).

A subset $A \subset X$ is said to be *locally closed* if for every point $x \in A$ there exists a neighborhood $O(x) \subset X$ such that the intersection $A \cap O(x)$ is closed in $O(x)$. A subset $A \subset X$ is *locally connected* if for any point $x \in A$ and any neighborhood $O(x) \subset X$ there exists a neighborhood $O'(x) \subset O(x) \subset X$ such that the intersection $A \cap O'(X)$ is connected.

A. Ergodic theory

The information to follow is taken from [**KSF**].

1. Measure theory. A *measure* μ on a set X is a nonnegative σ-additive function $\mu : \Omega(X) \to \mathbb{R}^+ \cup \infty$ defined on a σ-algebra $\Omega(X)$ of subsets in X. Subsets in $\Omega(X)$ are called *measurable sets*. The measure μ is said to be *complete* if from $\mu(A) = 0$ and $A \in \Omega(X)$ it follows that $B \in \Omega(X)$ for every $B \subset A$.

A measure μ on X is called *absolutely continuous* relative to a measure μ' (we denote this by $\mu \leq \mu'$) if all μ'-measurable sets are μ-measurable and from $\mu'(A) = 0$ it follows that $\mu(A) = 0$. If $\mu \leq \mu'$ and $\mu' \leq \mu$, then the measures μ and μ' are called *equivalent* ($\mu \sim \mu'$). Measures μ and μ' are *mutually singular* if there exists a subset $A \subset X$ such that $\mu(A) = 0$ and $\mu'(X - A) = 0$.

A measure μ is said to be *finite* if $\mu(X) < \infty$, and *σ-finite* if $X = \bigcup_{i \in \mathbb{N}} X_i$, where $\mu(X_i) < \infty$. A finite measure μ normalized by the condition $\mu(X) = 1$ is called a *probability measure*. A pair (X, μ) (we omit $\Omega(X)$ to simplify notation) will be called a *measure space*. In what follows, we will always work with sufficiently good measures. To be precise, under rather natural conditions ($\Omega(X)$ has a countable basis (mod 0), the measure μ is complete and σ-finite, etc.; see [**KSF**] for more), the space (X, μ) is measure-theoretically isomorphic to the real line $\mathbb{R}$ equipped with a *Lebesgue–Stieltjes* measure (finite or infinite). If this is the case, the space (X, μ) will be called a *Lebesgue space*.[1] If in addition $\mu(X) = 1$, the pair (X, μ) is called a *probability space*.

[1] We emphasize that in contrast to the common definition we do not require the measure μ to be finite.

We recall that a Lebesgue–Stieltjes measure m on $\mathbb{R}$ is given by a nondecreasing left continuous function $F : \mathbb{R} \to \mathbb{R}$. Here

$$m(a,b) = F(b) - F(a+0), \qquad m[a,b] = F(b+0) - F(a).$$

By a standard construction one obtains a σ-additive measure defined on the σ-algebra of Borel subsets in $\mathbb{R}$. The completion of this measure is called a Lebesgue–Stieltjes measure (note that the family of measurable sets depends on the function F). The usual Lebesgue measure l on $\mathbb{R}$ is given by the function $F(x) = x$.

If $m \preceq l$, the measure m is called *absolutely continuous.* For such a measure there exists a nonnegative measurable (relative to the Lebesgue measure) function $s : \mathbb{R} \to \mathbb{R}$ such that

$$\int_{\mathbb{R}} f dm = \int_{\mathbb{R}} f(x)s(x)dx$$

for any Lebesgue measurable function f. If the measure m is supported on a finite or countable[2] subset in $\mathbb{R}$, it is called a *discrete* measure. If every countable set is of zero m-measure and at the same time $m(\mathbb{R} - A) = 0$ for some set $A \subset \mathbb{R}$ of zero Lebesgue measure, then m is said to be a *singular* measure. It is known that every Lebesgue–Stieltjes measure is the sum of an absolutely continuous measure, a discrete measure, and a singular measure.

Let a map $f : X \to X'$ be given and let the space X be equipped with a measure μ. Then X' can be equipped with a measure $\mu' = f_*\mu$ by the rule

$$\mu'(A) = \mu(f^{-1}(A)) \text{ for any } A \subset X' \text{ with measurable inverse image } f^{-1}(A)$$

(note that $(X', f_*\mu)$ can fail to be a Lebesgue space).

Let (X,μ) and (X',μ') be two measure spaces. A map $f : X \to X'$ is called *measurable* if the inverse image $f^{-1}(A)$ of each measurable set $A \subset X'$ is measurable. A measurable map $f : X \to X'$ is called *measure-preserving* if $f_*\mu = \mu'$, that is, $\mu(f^{-1}(A)) = \mu'(A)$ for each measurable $A \subset X'$.

2. Actions on a measure space. An *automorphism* of a measure space (X,μ) is a measure-preserving invertible (mod 0) map $f : X \to X$. By $\mathrm{Aut}(X,\mu)$ we denote the group of all automorphisms of the space (X,μ) equipped with the following topology:

$$f_n \to f \text{ if } \mu(f_n(A) \cap B) \to \mu(f(A) \cap B) \text{ for all measurable sets } A, B \subset X.$$

Let F be a topological group (for instance, the group $\mathbb{R}$ of reals or the group $\mathbb{Z}$ of integers). In what follows, we assume that *F is locally compact and second countable.*[3] By a *continuous action* of F on a space (X,μ) we understand a continuous homomorphism $\phi : F \to \mathrm{Aut}(X,\mu)$. Throughout the book we consider only continuous actions, and instead of $\phi_g(x)$ we usually write gx, where $g \in F$, $x \in X$. In the case $F = \mathbb{R}$, a continuous action is called a *flow*, and in the case $F = \mathbb{Z}$ a *one-generator action.*

A measurable set $A \subset X$ is called *F-invariant* if $gA = A$ (mod 0) for all $g \in F$. Under our assumptions (the space (X,μ) is Lebesgue and F is second countable and locally compact) this is equivalent to the property $FA = A$ (mod 0) [**Ver**]. An action of F on (X,μ) is *ergodic* if for every F-invariant set $A \subset X$ either $\mu(A) = 0$ or $\mu(X - A) = 0$ (in this case μ is called an *F-ergodic measure*).

[2]Not necessarily discrete.

[3]Actually, all our groups will be finite-dimensional Lie groups.

3. Unitary representations. Let $L^2(X,\mu)$ be the Hilbert space of (equivalence classes of) complex-valued functions on X square integrable relative to μ. We equip the space with the scalar product

$$(f,f') = \int_X f(x)\overline{f'(x)}d\mu_x.$$

A norm in the space $L^2(X,\mu)$ is given as usual: $||f||^2 = (f,f)$. Since (X,μ) is always assumed to be a Lebesgue space, it follows that $L^2(X,\mu)$ is separable.

Let a probability measure μ be invariant for an action (X,F). Then one defines a continuous unitary representation $\rho : F \to \mathbb{U}(L^2(X,\mu))$ by the rule

$$(\rho_g f)(x) = f(g^{-1}x), \qquad g \in F, \quad x \in X, \quad f \in L^2(X,\mu)$$

(one says that F is *unitary* if

$$(\rho_g f, \rho_g f') = (f,f'), \qquad g \in F, \quad f,f' \in L^2(X,\mu),$$

and this always holds whenever μ is F-invariant).

Since the action is continuous, it follows, in particular, that the *stabilizer*

$$\mathrm{Stab}(f) = \{g \in F : \ \rho_g f = f\}$$

for each element $f \in L^2(X,\mu)$ is a closed subgroup of F.

In the case $F = \mathbb{R}$, a continuous flow on (X,μ) will be denoted by $(X,\phi_{\mathbb{R}})$. An element $f \in L^2(X,\mu)$ is called an *eigenfunction* if there exists $\lambda \in \mathbb{C}$ such that $\rho_t f = \lambda^t f$, $t \in \mathbb{R}$ (here the equality means that the functions coincide almost everywhere on X). From the unitary property it follows that $|\lambda| = 1$; the case $\lambda = 1$ corresponds to an *invariant function.* In this setting the flow $(X,\phi_{\mathbb{R}})$ is ergodic if and only if apart from constants there are no $\phi_{\mathbb{R}}$-invariant functions in $L^2(X,\mu)$.

4. Ergodicity of rectilinear flows on the torus. Let X be the torus $\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n$ equipped with the usual $\mathbb{R}^n$-invariant Lebesque measure μ, and let $v \in \mathbb{R}^n$. The *rectilinear flow* on the torus $\mathbb{T}^n$ induced by the vector v is a one-parameter group of translations:

$$\phi_t(x+\mathbb{Z}^n) = (tv + x + \mathbb{Z}^n) \in \mathbb{R}^n/\mathbb{Z}^n, \qquad x \in \mathbb{R}^n.$$

One says that the coordinates $v_1, \ldots, v_n$ are *linearly independent over the field* $\mathbb{Q}$ of rationals if

$$(*) \qquad (\alpha_1 v_1 + \cdots + \alpha_n v_n = 0, \ \alpha_i \in \mathbb{Q}) \Rightarrow (\alpha_i = 0, \ i = 1, \ldots, n).$$

The equation $\alpha_1 x_1 + \cdots + \alpha_n x_n = 0$, $x \in \mathbb{R}^n$, defines a subspace $L \subset \mathbb{R}^n$. If all α_i are rational, then the subspace is called *rational* and, as is known, intersects the lattice $\mathbb{Z}^n \subset \mathbb{R}^n$ in a lattice (that is, $L/\mathbb{Z}^n \cap L$ is a torus). Therefore if the coordinates of v are not independent over $\mathbb{Q}$, the torus $\mathbb{T}^n$ fibers into a family of invariant subtori of equal dimensions. Clearly, in this case the rectilinear flow is not ergodic. On the contrary, if the coordinates of v are independent over $\mathbb{Q}$, then the flow is ergodic. In fact, $\mathbb{T}^n$ carries a transitive action of the group $\mathbb{R}^n$, and one can consider a continuous unitary representation $\rho : \mathbb{R}^n \to \mathbb{U}(L^2(\mathbb{T}^n,\mu))$. Assume that $f \in L^2(\mathbb{T}^n,\mu)$ and $\rho(tv)f = f$, $t \in \mathbb{R}$. The stabilizer $H = \mathrm{Stab}(f)$ is a closed subgroup of $\mathbb{R}^n$, and $\mathbb{R}v$, $\mathbb{Z}^n \subset H$. It suffices to prove that $H = (\overline{\mathbb{R}v\mathbb{Z}^n})_0 = \mathbb{R}^n$. Since the group $\mathbb{R}^n$ is commutative, it follows that the subgroup H is closed, contains $\mathbb{R}v$ and intersects $\mathbb{Z}^n$ in a lattice. If $\dim H < n$, we arrive