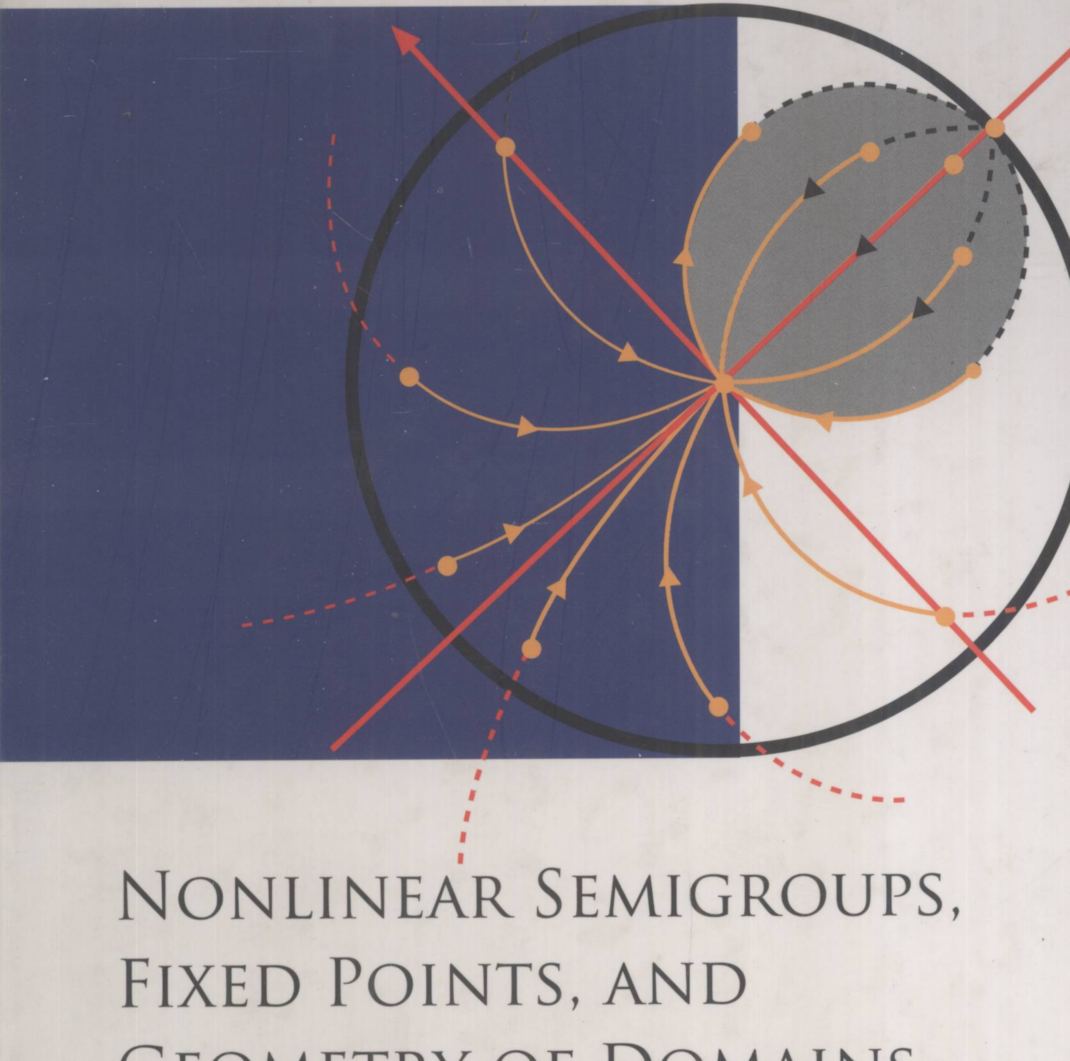
Simeon Reich • David Shoikhet



NONLINEAR SEMIGROUPS, FIXED POINTS, AND GEOMETRY OF DOMAINS IN BANACH SPACES

Imperial College Press

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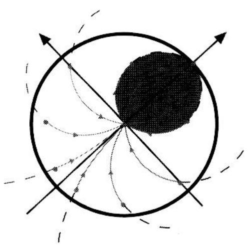

Nonlinear Semigroups, Fixed Points, and Geometry of Domains in Banach Spaces

Preface

Nonlinear semigroup theory is not only of intrinsic interest, but is also important in the study of evolution problems. In recent years many developments have occurred, in particular, in the area of nonexpansive semigroups in Banach spaces. As a rule, such semigroups are generated by accretive operators and can be viewed as nonlinear analogs of the classical linear contraction semigroups.

In the last forty years the theory of monotone and accretive operators has been intensively developed by many mathematicians (see, for example, [Brézis (1973)] and [Barbu (1976)]) with many applications to nonlinear analysis and optimization. This theory is closely connected with the generation theory of nonlinear one-parameter semigroups of nonexpansive mappings and with nonlinear evolution problems.

In a parallel development (and even earlier) the generation theory of one-parameter semigroups of holomorphic mappings in $\mathbb{C}^n$ has been an object of interest in the theory of Markov stochastic processes and, in particular, in the theory of branching processes (see, for example, [Harris (1963)] and [Sevastyanov (1971)]). The central problem in the study of such processes is to locate the extinction probability which can be defined as the smallest common fixed point of a semigroup of holomorphic mappings or, equivalently, as the smallest null point of its generator.

Later such semigroups appeared in other fields: one-dimensional complex analysis [Löwner (1923); Kufarev (1943); Kufarev (1947); Lebedev (1975); Aleksandrov (1976)], finite-dimensional manifolds [Kaup and Vigué (1990); Abate (1992)], the geometry of complex Banach spaces [Arazy (1987); Isidro and Vigué (1984); Kaup (1983); Dineen (1989)], control theory and optimization [Helmke and Moore (1994)], and Krein spaces [Vesentini (1987a)–(1987b); (1991)]. For the finite-dimensional case, M. Abate proved in [Abate (1992)] that each continuous semigroup of holomorphic mappings is everywhere differentiable with respect to its parameter, i. e., is generated by a holomorphic mapping. In addition, he established a criterion for a holomorphic mapping to be a generator of a one-parameter semigroup. (Such a problem is equivalent to the global solvability of a complex dynamical system.) Earlier, for the one-dimensional case, similar facts were presented by E. Berkson and H. Porta in their study [Berkson and Porta (1978)] of linear continuous semigroups of composition operators in Hardy spaces. It seems that the first deep study of semigroups of holomorphic mappings in the infinite-dimensional case is due to E. Vesentini. In [Vesentini (1987a)] he investigates semigroups of those fractional-linear transformations on the open unit Hilbert ball $\mathbb{B}$ which are isometries with respect to the hyperbolic metric on $\mathbb{B}$. The approach used there is based on the correspondence between such nonlinear semigroups and the strongly continuous semigroups of linear operators which leave invariant the indefinite metric on a Pontryagin space of defect 1. In [Vesentini (1987b)] and [Vesentini (1991)] this approach has been developed for general Pontryagin spaces and also for Krein spaces. Note that, generally speaking, such semigroups are not everywhere differentiable, and the generator of the corresponding linear semigroup is only densely defined. As a matter of fact, it turns out that the everywhere differentiability of a semigroup of holomorphic mappings on a bounded domain is equivalent to its continuity in the topology of local uniform convergence. Since, in the finite-dimensional case, this topology is equivalent to the compact open topology, the study of complex dynamical systems generated by holomorphic mappings includes in this case the study of semigroups of holomorphic mappings which are pointwise continuous. On the other hand, holomorphic self-mappings of a domain D in a complex Banach space are nonexpansive with respect to any pseudometric ρ assigned to D by a Schwarz-Pick system [Harris (1979)]. Therefore it is natural to inquire whether a theory analogous to the theory of monotone and accretive operators can be developed in the setting of those mappings which are nonexpansive with respect to such pseudometrics. We note in passing that the class of ρ-nonexpansive mappings properly contains the class of holomorphic mappings.

It seems that the need to investigate holomorphic mappings in infinite-dimensional spaces arose for the first time in connection with the study of nonlinear integral equations with an analytic nonlinear part at the end of the 19th and the beginning of the 20th centuries by A. Liapunov, E. Schmidt, A. Nekrasov and others.

Later in the 20th century the interest in analytic methods diminished temporarily due to the rapid development of degree theory by J. Leray, J. Schauder, G. Birkhoff, M. Krasnoselskii, P. Zabreiko, Y. Borisovich and others; see the references in [Krasnoselskii and Zabreiko (1984)].

The traditional methods for solving nonlinear operator equations have been based on either the Banach fixed point principle for contractive maps or the Leray–Schauder principle for compact operators.

However, the application of these principles is not always possible, or else if the operator depends on a parameter, these methods (as well as the classical Lyapunov–Nekrasov method) give only local results.

Parallel with the achievements mentioned above, the first results regarding holomorphic mappings on infinite-dimensional spaces appeared in the works of H. Cartan, R. Phillips, L. Nachbin, L. Harris, T. Suffridge, M. Hervé, E. Vesentini, K. Goebel, T. Kuczumov, A. Stachura, S. Reich, J.-P. Vigué, P. Mazet and many others (see the references in [Franzoni and Vesentini (1980); Goebel and Reich (1984); Hervé (1989); Dineen (1989); Chae (1985)]). A bridge between nonlinear equations with noncompact analytic operators and the theory of holomorphic mappings has been built in the book [Khatskevich and Shoikhet (1994a)].

In the one-dimensional case, the classical Denjoy–Wolff theorem provides information on both the location of fixed points and the behavior of the iterates of a holomorphic self-mapping. Over the last twenty years this result has been developed in at least three directions. The first one concerns increasing the dimension of the underlying space. Finite-dimensional extensions can be found, for instance, in the papers by Kubota [Kubota (1983)], MacCluer [MacCluer (1983)], Chen [Chen (1984)], Abate [Abate (1989); (1998)], and Mercer [Mercer (1991)–(1993); (1997); (1999)]. Infinite-dimensional generalizations are due, for example, to Fan [Fan (1978); (1979); (1982); (1983); (1986); (1988)], Wlodarczyk [Wlodarczyk (1985)–(1987); (1995)], Goebel [Goebel (1981); (1982)], Vesentini [Vesentini (1983); (1985)], Sine [Sine (1989)] and Mellon [Mellon (1996)]. These authors used a variety of approaches and assumed diverse conditions on the mappings and the domains. The second direction is concerned with analogues of the Denjoy–Wolff theorem for continuous semigroups. This approach has been used by several authors to study the asymptotic behavior of solutions to Cauchy problems. The third direction yields extensions of this theorem to the wider class of those self-mappings which are nonexpansive with respect to Schwarz–Pick pseudometrics.

It turns out that the asymptotic behavior of solutions to evolution equations is applicable to the study of the geometry of certain domains in complex spaces. For example, it is a well known result, due to R. Nevanlinna (1921), that if f is holomorphic in $|z| < 1$ and satisfies $f(0) = 0, \ f'(0) \neq 0$, then f is univalent and maps the unit disk onto a starlike domain (with respect to 0) if and only if $Re[zf'(z)/f(z)] > 0$ everywhere. This result, as well as most of the work on starlike functions on the unit disk, can be obtained from the identity

$$\frac{\partial}{\partial\theta} \arg f(re^{i\theta}) = \operatorname{Re}\left\{\frac{re^{i\theta} f'(re^{i\theta})}{f(re^{i\theta})}\right\}.$$

This idea does not extend readily to a higher-dimensional space. Moreover, such an approach is crucially connected with the initial condition $f(0) = 0$. Much later, Wald [Wald (1978)] gave a characterization of those functions which are starlike with respect to another center. Observe that although the classes of starlike, spirallike and convex functions were studied very extensively, little was known about functions that are holomorphic on the unit disk Δ and starlike with respect to a boundary point. In fact, only in 1981 Robertson [Robertson (1981)] introduced two relevant classes of univalent functions and conjectured that they coincide. In 1984 his conjecture was proved by Lyzzaik [Lyzzaik (1984)]. Finally, in 1990 Silverman and Silvia [Silverman and Silvia (1990)], using a similar method, gave a full description of the class of univalent functions on Δ, the image of which is starlike with respect to a boundary point. However, the arguments used in their work have a crucially one-dimensional character (because of the Riemann mapping theorem, the de Branges theorem, and Carathéodory's theorem on kernel convergence). In addition, the conditions given by Robertson and by Silverman and Silvia, characterizing starlikeness with respect to a boundary point, essentially differ from Wald's and Nevanlinna's conditions of starlikeness with respect to an interior point. Hence, it is difficult to trace the connections between these two closely related geometric objects. Therefore, even in the one-dimensional case the following problem arises: to find a unified condition of starlikeness (and spirallikeness) with respect to an interior or a boundary point. It seems that the idea to use a dynamical approach was first suggested by Robertson [Robertson (1936)] and developed by Brickman [Brickman (1973)], who introduced the concept of Φ-like functions as a generalization of starlike and spirallike functions (with respect to the origin) of a single complex variable. Suffridge [Suffridge (1977); (1970); (1973)], Pfaltzgraff [Pfaltzgraff (1974); (1975)] and Gurganus [Gurganus (1975)] developed a similar approach in order to characterize starlike, spirallike (with respect to the origin), convex and close-to-convex mappings in higher dimensional cases. Since 1970 the list of papers on these subjects has become quite long. Nevertheless, it seems that there has been no extension of Wald's as well as Silvia and Silverman's results to higher dimensions.

The first chapter of this book is an introductory chapter which sets the stage for the remainder of the book by giving basic notions in functional analysis and operator theory on metric and normed spaces.

The second chapter defines differentiable and holomorphic (analytic) mappings and presents a generalization of classical function theory to Banach spaces.

The third chapter contains material that is not usually covered in basic graduate courses, but is needed in the study of fixed point theory in metric spaces and semigroups of nonexpansive mappings with respect to the so-called hyperbolic metric.

Chapter 4 contains some classical and modern fixed point principles while Chapter 5 demonstrates a special approach to fixed point theory of holomorphic mappings, which is based on the development of the classical Denjoy–Wolff Theorem in various settings.

Chapters 6–9 are devoted to nonlinear semigroup theory of those mappings which are nonexpansive with respect to some special metrics on domains in Banach spaces. The description is most complete in the case of nonlinear semigroups of holomorphic self-mappings of a convex domain (which are nonexpansive with respect to the hyperbolic metric).

The last chapter consists of some material devoted to less developed geometric function theory in infinite dimensional spaces. It demonstrates a dynamical approach to this theory which is based on the asymptotic behavior of semigroups of holomorphic mappings.

The latter topic is itself of intrinsic interest and is considered in more detail in Chapter 9.

Summing up, we hope that this book may be considered a first step in establishing bridges between nonlinear semigroup theory, fixed points, and the geometry of domains.

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Simeon Reich and David Shoikhet

Contents

Chapter 1

Mappings in Metric and Normed Spaces

1.1 Topological Spaces

1.1.1 *Topology*

Let X be a set. *A **topology** on X is a collection τ of subsets of the set X, satisfying three conditions:*

(a) the intersection of any two elements of τ is an element of τ;
(b) the union of the elements of any subfamily of the family τ belongs to τ;
(c) the set X and the empty set belong to τ.

The set X is called the space of the topology τ and the pair (X, τ) is called a **topological space**. When no confusion arises, we simply write "*X is a topological space*". The elements of the topology τ are called **τ-open** *(or simply open)* subsets.

Let τ_1 and τ_2 be two topologies on X; τ_1 is *weaker* (smaller, rougher) than τ_2, or τ_2 is *stronger* (greater, finer) than τ_1, if $\tau_1 \subseteq \tau_2$. It is possible that, for two given topologies τ_1 and τ_2 on X neither τ_1 is stronger than τ_2, nor τ_2 is stronger than τ_1; in this case τ_1 and τ_2 are said to be not comparable.

1.1.2 *Neighborhoods*

*A **neighborhood** (τ-neighborhood) of a point x in a topological space (X, τ) is any subset of this space which contains an element U of the topology τ with the property $x \in U$.* For example, in the case of the space $\mathbb{C}$ of complex numbers with the usual topology generated by the collection of open disks in $\mathbb{C}$, a neighborhood of a point is any subset of $\mathbb{C}$ containing an open disk which contains the point in question.

A set $\mathcal{D} \subseteq X$ is **open** *if and only if for every $x \in \mathcal{D}$, $\mathcal{D}$ is a neighborhood of x.*

The complement of an open set is a **closed** set. A set that is both closed and open is called a *clopen* set.

A set may be both open and closed, or it may be neither. In particular, both $\emptyset$ and X are both open and closed. The family of closed sets has the following properties, which are dual to the properties of the open sets.

- Both $\emptyset$ and X are closed.
- A finite union of closed sets is closed.
- An arbitrary intersection of closed sets is closed.

1.1.3 *Examples of topologies*

The following examples illustrate the variety of topological spaces:

Example 1.1 The trivial topology or the indiscrete topology on a set X consists of only X and $\emptyset$. These are also the only closed sets.

Example 1.2 The discrete topology on a set X consists of all subsets of X. Thus every set is both open and closed.

Example 1.3 The open intervals on the real line $\mathbb{R} = (-\infty, \infty)$ generate a topology on $X = \mathbb{R}$. The extended real line $\mathbb{R}^* = [-\infty, \infty] = \mathbb{R} \cup \{-\infty, \infty\}$ has a natural topology too. It consists of all subsets U such that for each $x \in U$:

(a) If $x \in \mathbb{R}$, then there exists some $\varepsilon > 0$ with $(x - \varepsilon,\, x + \varepsilon) \subset U$;
(b) If $x = \infty$, then there exists some $y \in \mathbb{R}$ with $(y, \infty] \subset U$; and
(c) If $x = -\infty$, then there exists some $y \in \mathbb{R}$ such that $[-\infty, y) \subset U$.

Example 1.4 A different, and admittedly contrived, topology on $\mathbb{R}$ consists of all sets A such that for each x in A, there is a set of the form $U \backslash C \subset A$, where U is open in the usual topology, C is countable, and $x \in U \backslash C$.

Example 1.5 Let $N = \{1, 2 \ldots\}$. The collection of sets consisting of the empty set and all sets containing 1 is a topology on N. The closed sets are N and all sets not containing 1.

1.1.4 *Interiors and closures. Limit points*

Let (X, τ) be a topological space, and let A be any subset of X. The topology τ defines two sets intimately related to A.

The ***interior*** *of A, denoted by A°, is the largest (with respect to inclusion) open set included in A.* (It is the union of all open subsets of A.) The interior of a nonempty set may be empty.

The ***closure*** *of A, denoted by $\bar{A}$, is the smallest closed set including A; it is the intersection of all closed sets including A.*

It is not hard to verify that $A \subset B$ implies $A^\circ \subset B^\circ$ and $\bar{A} \subset \bar{B}$. Also, it is obvious that a set A is open if and only if $A = A^\circ$, and a set B is closed if and only if $B = \bar{B}$. Consequently, for any set $A, \overline{(\bar{A})} = \bar{A}$ and $(A^\circ)^\circ = A^\circ$. Thus, a neighborhood of a point x is any set V containing x in its interior.

The collection of all neighborhoods of a point x, called the ***neighborhood base****, or the neighborhood system, at x, is denoted by N_x.*

It is easy to verify that N_x has the following properties.

(a) $X \in N_x$.
(b) For each $V \in N_x$, we have $x \in V$ (so $\emptyset \notin N_x$).
(c) If $V, U \in N_x$, then $V \cap U \in N_x$.
(d) If $V \in N_x$ and $V \subset W$, then $W \in N_x$.

A topology on X is called ***Hausdorff*** *(or* ***separated****) if any two distinct points can be separated by disjoint neighborhoods of the points. That is, for each pair $x, y \in X$ with $x \neq y$ there exist neighborhoods $U \in N_x$ and $V \in N_y$ such that $U \cap V = \emptyset$.*

A point x is a point of closure or ***closure point*** *of the set A if every neighborhood of x meets A.* Note that $\bar{A}$ coincides with the set of all closure points of A.

A point x is an **accumulation point** (or a **limit point**, or a **cluster point**) of A if for each neighborhood V of x we have $(V \backslash \{x\}) \cap A \neq \emptyset$.

To see the difference between closure points and limit points, let $A = [0, 1) \cup \{2\}$, a subset of $\mathbb{R}$. Then 2 is a closure point of A in $\mathbb{R}$, but not a limit point. The point 1 is both a closure point and a limit point of A.

Let A be any subset of a topological space X, and let A^c be its complement, *i.e.*, $A^c = X \backslash A$.

A point x is a ***boundary point*** *of A if each neighborhood V of x satisfies both $V \cap A \neq \emptyset$ and $V \cap A^c \neq \emptyset$.* Clearly, accumulation and boundary points of A belong to its closure $\bar{A}$. Let A' denote the set of all accumulation points