ELECTRICAL AND ELECTRONIC ENGINEERING SERIES

Introduction to
ELECTROMAGNETIC FIELDS

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PREFACE

There has been a growing awareness that the undergraduate student in electrical engineering requires a broad and deep understanding of the principles of electric and magnetic fields as a background for the rapidly developing areas referred to as electromagnetic engineering by some and as energy conversion by others. Thus, in addition to the general aspects of electromagnetic fields, a discussion of the sources and points of development of forces and torques in such fields, and the general features of energy storage, energy flow, energy transfer, and energy conversion require added emphasis. Since many of these requirements are different from those of the past, it is not surprising that most previous texts on electric and magnetic fields do not quite satisfy the needs, and new books in this classical field are justified. This book has been written to meet the introductory phase of the needs of the electrical engineering student.

As a background to the text, it is assumed that the reader is familiar with the calculus and that he has an understanding of the important concepts of classical mechanics. Where any special concepts are required, these are introduced within the context of the development of the book.

A number of special features have been adopted in the development of the text material. These include pedagogical ideas as well as a changed emphasis from the technical content of the past. Among these are the following:

Since the presentation of electric and magnetic fields has long been a challenge to educators owing to the mathematical complexity inherent in such a study and the rather elusive character of fields, this text presents the material in order of increasing complexity of field concepts. It begins with a consideration of scalar fields and then leads into vector fields. The standard treatment of electric and magnetic fields has been abandoned in favor of that suggested by K. Küpfmüller's book, "Einfuhrung in die theoretische Elektrotechnik." This order of presentation has been classroom tested with very satisfactory results.

To emphasize the fact that the lumped circuit parameters, resistance, capacitance, and inductance, are quantities defined from static field considerations, these and related concepts proceed from the systematic exploitation of static fields before dynamic fields are introduced.

Considerable use has been made of the principles of virtual displacement and virtual work in order to explore the general ideas of energy storage in fields and the forces at boundaries (dielectric boundaries in electric fields, and magnetic boundaries in magnetic fields). In dynamic problems, the seat of the forces and torques and the flow and storage of energy are carefully considered.

The development of the Maxwell field equations, the relationships of the static field equations to these, and the relationship of the fundamental mathematical problems to the solution of the field equations are emphasized. Frequent opportunity is taken to show that the mathematical problem is substantially the same in various phases of the work, that Poisson's or Laplace's equation is the controlling equation for most static field problems.

The use of vector notation and vector mathematics has been introduced early as an integral part of the text development.

An extensive number of illustrative examples have been included throughout the text to relate the results to practical problems.

The text material divides itself into five major areas: current flow fields, static electric fields, static magnetic fields, dynamic fields, Maxwell's equations, and electromagnetic phenomena. Every effort is made, however, to show the interrelationship among these areas, rather than to emphasize the differences. Such an interrelationship is readily possible since the electric charge, whether stationary, in uniform motion, or in accelerated motion, is fundamental to a discussion of most of these phenomena.

The author wishes to acknowledge his indebtedness to the many fine books covering the broad aspects of electromagnetic theory which have been written during the past seventy-five years. He owes a particular debt of gratitude to the late Prof. W. W. Hansen of Stanford University who, during the war, did so much to make field theory a living and vital study. It is a pleasure also to acknowledge the help of former colleagues, Dr. Edward Erdelyi and Dr. Richard E. Gildersleeve, in the early stages of the book's development, and to thank Dr. Robert Plonsey for his assistance in proofreading the entire text.

Samuel Seely

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CHAPTER 1

THE ELECTRIC FIELD IN MULTIDIMENSIONAL CONDUCTORS

This chapter will introduce the basic field concepts through a study of the experimental results which may be obtained with the aid of an electrolytic tank, or equivalently through the use of a flux plotter provided with Teledeltos paper. The field plots so obtained for any specified field configuration will be examined physically, and from this physical field distribution, which will be described in mathematical terms, many of the important field quantities will be developed.

1-1. The Current-flow Field. Our study will begin with an experiment. This experiment may be performed by the student with equipment which is available commercially or which may be constructed locally. The equipment is generally referred to as *flux-plotting equipment.* In one form, it consists of a sheet of conducting Teledeltos paper. This paper may be cut to any desired shape, to represent a specified configuration. Electrodes of prescribed shape may be attached to the paper in specified positions. By applying a known source of potential between the electrodes, a current, known as a *conduction* current, will flow between the electrodes. By means of a high-impedance voltmeter, it is possible to explore the potential distribution over the surface of the conducting paper. Of course, the equipment may consist of a sheet of iron on an insulated support. It might also consist of a nonconducting shallow tank which contains a uniform layer of conducting liquid. A typical flux-plotting equipment is illustrated in Fig. 1-1.

Suppose that such a typical flux-plotting equipment consists of a large rectangular conducting surface, with small circular electrodes which have been attached to it. If the conducting sheet is sufficiently large, then the system approximates a pair of conducting electrodes immersed in an infinite conducting plane. This system is closely analogous to a pair of wires immersed in an infinite conducting medium, a cross section of which is being explored. The situation is substantially that illustrated in Fig. 1-2. The electrode terminals are connected to the source of power, and the voltmeter will be used to explore the field. That a potential variation should be expected follows from the fact that current will flow between the terminals attached to the conducting sheet. It is this current-flow field that is to be carefully investigated.

Suppose that it is possible to adjust the potential of the source so that the indicating voltmeter reads 100 scale divisions when connected between

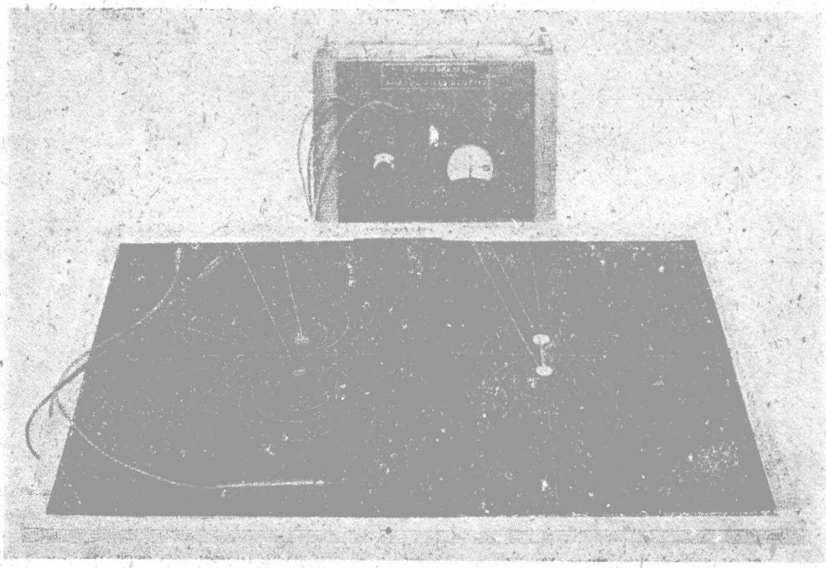

FIG. 1-1. Photograph of a commercial flux-plotting equipment. (*Courtesy Sunshine Scientific Instrument.*)

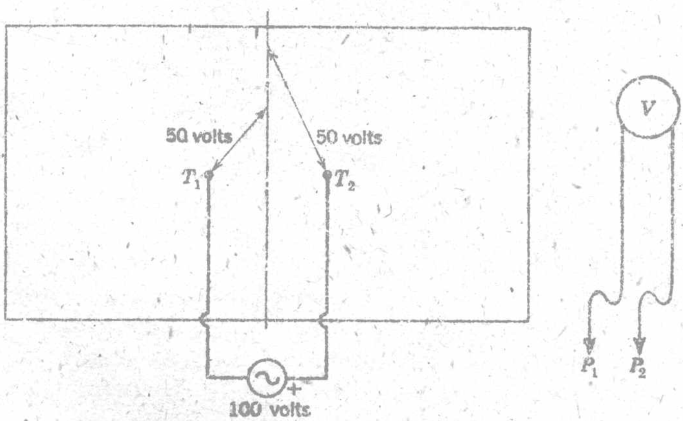

FIG. 1-2. Flux-plotting equipment for the system under survey.

the two electrodes. For convenience, suppose that each scale division is 1 volt, so that the maximum scale reading is 100 volts. Suppose, now, that one of the voltmeter probes, say, *P*1, is attached to terminal *T*1 and it is desired to search for all points on the sheet which will indicate 50 volts on the voltmeter V. By actually performing this experiment, it will be found that the 50-volt level of potential occurs on the perpendicular bisector of the line between the two terminals $T1$ and $T2$. Such a result would be expected from the symmetry of the arrangement. Clearly, if probe $P1$ is placed anywhere along this line and probe $P2$ is placed on terminal $T2$, a 50-volt reading will also result.

Now a systematic exploration of the field is to be undertaken. Connect probe $P1$ to terminal $T1$, and search the field for the curves of 10 volts potential difference by moving probe $P2$ over the field. When this has been completed and the curve of the 10-volt potential difference

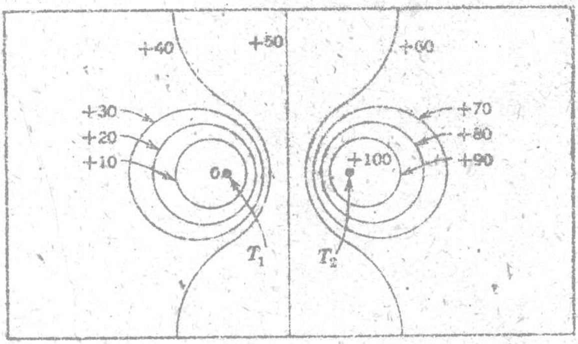

FIG. 1-3. Equipotential lines of the current field.

has been drawn, the process will be repeated for the 20-volt curve, then the 30-volt curve, etc., until the entire field has been systematically explored. The curves of equal potential difference are referred to as *equipotential lines*, or *curves*. A system of curves will result, as illustrated in Fig. 1-3. Clearly, if both probes are applied to two points of the same equipotential, no deflection will be noted on the voltmeter. This is, in fact, the real meaning of the term *equipotential*.

If the conducting material were to be of substantial thickness, then points of equal potential difference could be found within the material. These points constitute surfaces whose base lines are the equipotentials on the surface of the sheet. The surfaces are referred to as *equipotential surfaces*. As an extension of the above, there is no potential difference between any two points of the same equipotential surface.

1-2. Potential and Potential Difference. Each equipotential curve of Fig. 1-3 has been labeled with a number which specifies the potential difference relative to $T1$, which has been assigned the value zero and is being chosen as the reference, or datum, of potential. Also, the + sign applied to the numerical value of potential difference has been chosen to be consistent with the designated reference positive potential. Similarly, the reference current direction is here being chosen to coincide with the requirement that the direction of conventional current be from a region of higher to a region of lower potential. That is, the current in the conducting sheet will be from $T2$ to $T1$ when the applied potential has the reference polarity. Clearly, if the terminal connections are interchanged, the direction of the current will reverse although the same distribution of equipotentials will result. Now, however, the equipotentials will carry $-$ signs in order to denote that they are the negatives of the previous data.

The reference point, or datum, of the potential is entirely arbitrary, as all effects depend only on differences of potential. That is, $T1$ may be chosen at any desired reference level without in any way affecting the equipotentials in the field. All that happens is that the equipotentials in the field will be either increased or decreased, depending upon the specified reference level of $T1$. Conversely, any point of the field may be chosen as the reference point, or datum, of potential. For example, suppose that a point on the bisector (shown as the $+50$ in Fig. 1-3) is chosen as the zero reference level of potential. In this case, all potential values of Fig. 1-3 will be reduced by the constant value of 50 volts. $T1$ will now have associated with it the designation, -50 volts, and $T2$ will be designated as $+50$ volts. All other values will be correspondingly changed. The potential difference between any two points in the field is independent of the choice of reference point of potential. Clearly, therefore, it follows that the potential at any point of the field is equal to the potential difference between this point and a reference point. Conversely, it follows that the potential difference between two points of the field is equal to the difference of potential between these two points.

Consider, therefore, that the potential of point a of the field is given by the symbol ϕ_a relative to some arbitrary point as reference. Correspondingly, the potential of some other point b of the field relative to the same arbitrary reference point is given as ϕ_b. The potential difference between these two points is

$$V_{ab} = \phi_a - \phi_b \tag{1-1}$$

The numerical value given by Eq. (1-1) may be positive or negative, depending upon whether ϕ_a is greater or less than ϕ_b. To avoid confusion, in what follows V_{ab} will denote the *potential drop* from point a to point b. If, therefore, V_{ab} is negative, this merely means that the potential of point a is less than the potential of point b, relative to the same reference points.

1-3. Vectors and Two Vector Operations. To continue with the description of current-flow fields, it will be found desirable to do so in terms of vectors and certain vector operations. Consequently, this section will serve as a mathematical digression to introduce these concepts.

Physical quantities may be of two general classes. Those quantities which can be described by a single number, as, for example, temperature and humidity, are known as *scalar* quantities. Those quantities, such as force and velocity, which have a direction in space as well as a magnitude are known as *vector* quantities. Vector quantities are represented geometrically by means of straight lines with arrowheads, the arrow pointing in the direction of the vector, the length being proportional to its magnitude. They are represented symbolically by boldface type. Attention is specifically directed to the difference between the complex number (phasor, sinor) of a-c circuit theory and the space vector here being considered.

The sum of two vectors **A** and **B** is the vector **C**, or

$$\mathbf{A} + \mathbf{B} = \mathbf{C} \tag{1-2}$$

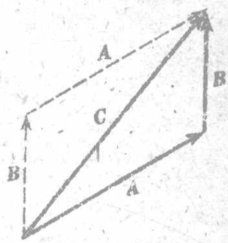

FIG. 1-4. Addition of vectors.

Observe carefully that, since **A** and **B** each possesses a magnitude and direction in space, **C** will also possess both magnitude and direction. Geometrically, the mathematical process is that illustrated in Fig. 1-4. Thus, the vector **C** is obtained from **A** and **B** by a parallelogram process, in which the origin of **B** is made to coincide with the terminus of **A**, and with **C** having the origin of **A** and the terminus **B** as its origin and terminus, respectively. Observe from the diagram that interchanging **A** and **B** does not affect the resultant **C**, whence

$$\mathbf{A} + \mathbf{B} = \mathbf{B} + \mathbf{A} \tag{1-3}$$

It follows from this that addition follows the commutative law. The addition of more than two vectors is readily accomplished geometrically by continuing the process defined in Eq. (1-2), adding successive vectors to the result of the previous operations. Thus, it readily follows that such an addition follows the associate law, namely,

$$(\mathbf{A} + \mathbf{B}) + \mathbf{C} = \mathbf{A} + (\mathbf{B} + \mathbf{C}) \tag{1-4}$$

It is noted that vectors, and the operation of vector arithmetic, have been defined without reference to any system of coordinates. In fact, it will be found that all the subsequent vector operations will be expressed in general terms. Often, however, it is convenient to refer the vector and its operations to a particular system of coordinates, but this will be the secondary and not the primary approach to our study of vector analysis. In fact, in this text, we shall limit consideration to three different orthogonal systems, the rectangular, cylindrical, and spherical systems of axes, with primary consideration given to the rectangular system of axes.

Refer to Fig. 1-5, which shows a vector **A** oriented with respect to a cartesian system of coordinates. The unit vectors **i**, **j**, **k** (vectors of unit magnitude) are chosen in the X, Y, Z directions, respectively, as illustrated. But, based on Eq. (1-2), it is possible to define a vector in terms of a number of other vectors. It is convenient, therefore, to express the vector **A** in terms of the sum of three vectors parallel to the rectangular (orthogonal) axes, thus,

$$\mathbf{A} = A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k} \tag{1-5}$$

In this expression, there appears the product of a vector and a scalar. Such a product is defined as a vector having a magnitude equal to the

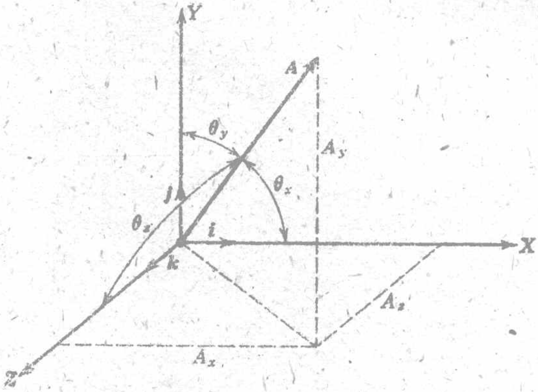

FIG. 1-5. Rectangular components of vector **A**.

product of the scalar and the magnitude of the vector (in this case, unity), and having a direction that is the direction of the original vector. The scalars A_x, A_y, A_z are the components of **A** and are given (see Fig. 1-5) by

$$\begin{aligned} A_x &= A \cos \theta_x \\ A_y &= A \cos \theta_y \\ A_z &= A \cos \theta_z \end{aligned} \tag{1-6}$$

where θ_x, θ_y, θ_z are the angles between **A** and the positive directions of the axes and where A is the magnitude of **A**.

To find the vector sum of two vectors **A** and **B** requires that each vector be referred to the same set of axes. Thus, for two vectors having the three components,

$$\begin{aligned} \mathbf{A} + \mathbf{B} &= (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) + (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k}) \\ \text{or} \quad \mathbf{A} + \mathbf{B} &= (A_x + B_x)\mathbf{i} + (A_y + B_y)\mathbf{j} + (A_z + B_z)\mathbf{k} \end{aligned} \tag{1-7}$$

This equation follows directly from the associative property of vector addition, as in Eq. (1-4).

Consider now the product of two vectors. There are two types of product, the *scalar* product and the *vector* product. These names serve to indicate that the result of the respective multiplication in the first case is a scalar and in the second case is a vector. Since our present needs involve only the scalar product of two vectors, we shall limit our discussion to these.

By definition, the *scalar*, or *dot*, product of two vectors **A** and **B** is a scalar quantity and is the product of the magnitudes of the vectors and the cosine of the angle between them, thus,

$$\mathbf{A} \cdot \mathbf{B} = AB \cos (\mathbf{A},\mathbf{B}) \tag{1-8}$$

Note that the dot between the vectors is essential to the expression. Clearly, from this definition,

$$\mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A} \tag{1-9}$$

and the scalar product obeys the commutative law. Also, the scalar product of a vector with itself is simply

$$\mathbf{A} \cdot \mathbf{A} = A^2 \tag{1-10}$$

since the angle between the two vectors is zero. Of course, the scalar product of two vectors which are perpendicular to each other will be zero, owing to the presence of the cos 90° that would appear in the expression for the dot product.

The scalar product of **A** and **B**, given by Eq. (1-8), may be expressed in terms of their rectangular components, thus,

$$\mathbf{A} \cdot \mathbf{B} = (A_x\mathbf{i} + A_y\mathbf{j} + A_z\mathbf{k}) \cdot (B_x\mathbf{i} + B_y\mathbf{j} + B_z\mathbf{k})$$

Note, however, that

$$\mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$

and

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0$$

Therefore, it follows that

$$\mathbf{A} \cdot \mathbf{B} = A_xB_x + A_yB_y + A_zB_z \tag{1-11}$$

This shows that the scalar product of two vectors is equal to the sum of the products of the corresponding components.

1-4. Potential Gradient and Line Integral. In addition to consideration of the potential at all points of the field, it is often convenient to consider the *gradient* of the potential at all points of the field. In order to understand the meaning of this term, consider two adjacent equipotential surfaces, one of which is denoted ϕ and the second denoted as $\phi + d\phi$, as illustrated in Fig. 1-6. Consider the point a on one equipotential and two points b and c on the adjacent equipotential. Point b is chosen to lie along the normal drawn to the equipotential at point a.

The distance between the adjacent equipotentials along the direction of the chosen normal, that is, the distance between points a and b, is designated as dn. Moreover, as illustrated in Fig. 1-6, the unit normal $\mathbf{n}$ is a space-vector quantity, and its direction would change, in general, if the point a were chosen elsewhere on the equipotential surface except when the surface is a plane. The magnitude of the normal $\mathbf{n}$ will be chosen by definition to be unity, and in the present case the positive direction of the normal is taken in the direction of increasing potential.

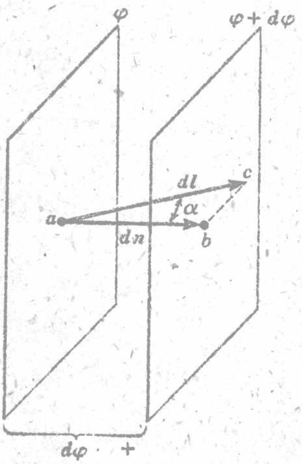

FIG. 1-6. Adjacent equipotentials in a potential field.

Consider now the quantity $d\phi/dn$, which specifies the greatest space rate of change of potential, and which is the slope, or gradient, of the potential field at the point a. Mathematically, the quantity discussed, when written with its associated direction $\mathbf{n}$, is called the *gradient of the potential*, or

$$\text{grad } \phi = \frac{d\phi}{dn}\mathbf{n} \tag{1-12}$$

Physically, therefore, the potential gradient, grad ϕ, is a measure of the maximum slope or the maximum rate of change of potential with distance in the direction of the increasing potential. For example, if one were to stand on the side of a hill, the gradient would be a measure of the slope in the direction of steepest ascent. In this case, the gradient is the tangent of the angle between the horizontal and the side of the hill. Often, in practice, the slope is specified as, say, $\frac{1}{100}$. This means that the ground rises 1 ft in each 100 of horizontal distance in the specified direction. Note particularly that, since the direction of steepest ascent is specified, the gradient has both magnitude and direction and is therefore a vector quantity. In vector analysis, it is often customary to write the symbol ∇ instead of the letters grad, where the symbol ∇ is called the *del* operator. Using this notation, Eq. (1-12) may be written as

$$\text{grad } \phi = \nabla\phi = \frac{d\phi}{dn}\mathbf{n} \tag{1-13}$$

Note from Fig. 1-6 that

$$dn = dl \cos \alpha$$

whence Eq. (1-13) becomes

$$\frac{d\phi}{dl} = |\text{grad } \phi| \cos \alpha$$

which is the *directional derivative* of the potential ϕ. This is the rate of change of ϕ in a particular direction and depends on the direction selected. If a rectangular system of axes were to be specified which was fixed at point a, then the directional derivative of ϕ in the direction of the X axis would be

$$\frac{\partial \phi}{\partial x} = (\text{grad } \phi) \cos \alpha_x = \mathbf{i} \cdot \text{grad } \phi$$

with corresponding terms for the Y and Z directions. Thus, for a general direction which possesses components along the three axes, it follows that

$$\text{grad } \phi = \nabla \phi = \mathbf{i} \frac{\partial \phi}{\partial x} + \mathbf{j} \frac{\partial \phi}{\partial y} + \mathbf{k} \frac{\partial \phi}{\partial z} \tag{1-14}$$

The corresponding forms for the gradient in cylindrical and spherical coordinates are written down, without proof.

Cylindrical coordinates

$$\text{grad } \phi = \frac{\partial \phi}{\partial r} \mathbf{a}_r + \frac{\partial \phi}{r\, \partial \theta} \mathbf{a}_\theta + \frac{\partial \phi}{\partial z} \mathbf{a}_z \tag{1-15a}$$

Spherical coordinates

$$\text{grad } \phi = \frac{\partial \phi}{\partial r} \mathbf{a}_r + \frac{\partial \phi}{r\, \partial \theta} \mathbf{a}_\theta + \frac{1}{r \sin \theta} \frac{\partial \phi}{\partial \psi} \mathbf{a}_\psi \tag{1-15b}$$

where the $\mathbf{a}$'s are unit vectors in the direction specified by the subscript, with positive direction in the direction of increasing variable.

It is now possible to reverse the foregoing considerations and consider the question of the difference of potential between the two points a and b. By definition, the difference of potential between these two points is the potential difference that exists between these two points. According to Eq. (1-13), this potential difference is $d\phi$, so that

$$d\phi = (\text{grad } \phi)_n \, dn$$

where the subscript n denotes that the component of grad ϕ is chosen in the direction n. This is, of course, just grad ϕ itself. The total potential difference between the points a and b is thus given by the expression

$$V_{ba} = \phi_b - \phi_a = \int_a^b d\phi = \int_a^b (\text{grad } \phi)_n \, dn \tag{1-16}$$

Some simple mathematical manipulations of this expression lead to useful forms. Suppose, therefore, that the right-hand side of the expression is multiplied and divided by $\cos \alpha$. This yields

$$V_{ba} = \phi_b - \phi_a = \int_a^b (\text{grad } \phi)_n \cos \alpha \frac{dn}{\cos \alpha} \tag{1-17}$$

This equation may therefore be written in the form

$$V_{ba} = \phi_b - \phi_a = \int_a^b (\text{grad } \phi)_l \, dl \tag{1-18}$$

In the light of Eq. (1-8) for the scalar multiplication of two vectors, Eq. (1-18) may now be written in the following form:

$$V_{ba} = \phi_b - \phi_a = \int_a^b d\mathbf{l} \cdot \text{grad } \phi$$

or

$$V_{ba} = \phi_b - \phi_a = \int_a^b d\mathbf{l} \cdot \nabla\phi \tag{1-19}$$

In these expressions, $d\mathbf{l}$ has now been written as a vector element of path length and has the meaning indicated in Fig. 1-6.

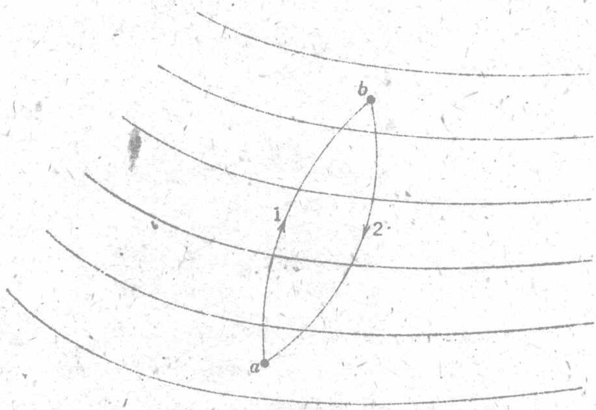

FIG. 1-7. Line integral of the potential gradient over a closed path.

Suppose that the two points a and b now denote any two points of the field, as illustrated in Fig. 1-7. The system of lines is to denote equipotentials of the field. The total potential difference between the points a and b along path 1 is simply the summation of the potential differences along all elements of this path. That is,

$$V_{ba} = \phi_b - \phi_a = \int_{\substack{a \\ \text{Path 1}}}^{b} d\phi$$

In a similar way, the total potential difference between the points b and a along path 2 is given by the expression

$$V_{ab} = \phi_a - \phi_b = \int_{\substack{b \\ \text{Path 2}}}^{a} d\phi$$

These two quantities are the negatives of each other, so that

$$V_{ba} = -V_{ab} \tag{1-20}$$

If in Eq. (1-18) (grad $\phi)_l$ is positive if its direction coincides with the