Mathematics and Its Applications

Nikolay Sidorov,
Boris Loginov,
Aleksandr Sinitsyn and
Michail Falaleev

Lyapunov-Schmidt Methods in Nonlinear Analysis and Applications

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by

Nikolay Sidorov

Irkutsk State University, Irkutsk, Russia

Boris Loginov

Ulyanovsk State Technical University, Ulyanovsk, Russia

Aleksandr Sinitsyn

Universidad Nacional de Colombia, Departamento de Matematicas, Bogota, Colombia

and

Michail Falaleev

Irkutsk State University, Irkutsk, Russia

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This monograph is published
to mark the 70-th anniversary
of Professor Vladilen A. Trenogin

The Authors

Preface

Preface

Constructing nonlinear parameter-dependent mathematical models is essential in modeling in many scientific research fields. The investigation of branching (bifurcating) solutions of such equations is one of the most important aspects in the analysis of such models. The foundations of the theory of bifurcations for the functional equations were laid in the well known publications by A.M. Lyapunov (1906) [1, vol. 4] (on equilibrium forms of rotating liquids) and E. Schmidt (1908) [1]. The approach proposed by them has been throughly developed and is presently known as the Lyapunov–Schmidt method (see M.M. Vainberg and V.A. Trenogin [1, 2]). A valuable part in the foundations of the bifurcation theory belongs to A. Poincarés ideas [1].

Later, to the end of proving the theorems on existence of bifurcation points, infinite-dimensional generalizations of topological and variational methods were proposed by M.A. Krasnoselsky [1], M.M. Vainberg [1] and others. A great contribution to the development and applications of the bifurcation theory has been made by a number of famous 20th century pure and applied mathematicians (for example, see the bibliography in E. Zeidler [1]).

Well known are applications of the bifurcation theory in mechanics (convection; wave theory; oscillations; aero-hydro-elasticity; bending of bars, membranes and shells) described in profound investigations of A.I. Nekrasov, T. Levi-Chivit, N. Kochin, D. Stroyk, M.A. Lavrentyev, K.O. Fridrichs, J. Stoker, D. Ioseph, J. Keller, J. Toland, A.M. Ter-Krikorov, V.A. Trenogin, I.I. Vorovitch, A.S. Wolmir, M. Berger, Ya.I. Sekerzh-Zen'kovitch, V.I. Yudovitch, B.V. Loginov, L.S. Srubschik, V.V. Pukhnachov, V.V. Bolotin and many others.

The sphere of applications of both the Lyapunov–Schmidt method and the theory of bifurcations has been extending since the time of their advent. Currently it has embraced many new areas of natural science, economics and engineering. Specific nonlinear problems of phase transition and plasma physics

(A.A. Vlasov, V.P. Maslov, V.V. Vedeniapin, J. Batt, H. Neunzert, G. Rein, L. Arkeryd, P.L. Lions, C. Bardos, P. Degond, R. Glassey, Y. Guo, J. Dorning, M. Hesse and K. Schindler and others) are to be noted in this connection. Some of such applications are considered in our monograph.

In recent years the Lyapunov–Shmidt method has been applied in the bifurcation theory not alone but in combination with methods of the theory of representation and group analysis, finite-dimensional topological and variational methods, methods of the theory of perturbations as well as the theory of regularization. Such combined approaches have given us the possibility to prove more general theorems of existence of branching solutions, conduct their algorithmic and qualitative analysis, and develop asymptotical and iterative methods. It is no accident that this has allowed mathematicians to solve new challenging problems of theoretical and applied mathematics.

Our monograph presents some results obtained in the abovementioned area by the group of authors — Russian mathematicians — during the recent 25 years. The corresponding general theory of operator and differential-operator equations in Banach spaces is constructed. Its use is illustrated by a number of natural science examples of application to boundary value problems and to integral and integro-differential equations.

The limited possible size of the monograph has allowed the authors to include only some part of the total set of results in the area. Many other interesting results of several authors (global existence theorems, cosymmetry by Yudovich, projective-iterative techniques, etc.) have remained beyond its scope.

The monograph includes 7 chapters. Chapter 1 outlines linear problems. Some results needed for further analysis — generalized Jordan sets of linear operators; abstract techniques for construction of regularization algorithms (R.A.) in the sense of Tikhonov–Lavrentyev needed to work with linear equations; iterative methods of computing both isolated Fredholm points and elements of generalized Jordan sets of operator functions — are considered here. All these concepts and constructions are used throughout the monograph.

Chapter 2 describes an elementary approach to proving existence theorems and computing asymptotics for the branches of real-valued solutions of nonlinear operator equations of the form

$$F(x, \lambda) = 0 \tag{1}$$

in the neighbourhood of the branching point λ_0. This approach, which can be traced back to works of A.M. Lyapunov, A. Poincaré, E. Shmidt and L. Kronecker, extensively employs the analysis of branching equations with the aid of V.A. Trenogin's Jordan type of chains, Kronecker's and Morse–Conley's index theories, as well as methods of finite-dimensional topology and simple variational techniques.

Chapters 3 and 4 discuss the problem of obtaining branching solutions for equation (1) by iterative techniques. The equation is assumed to be given approximately (by an approximation to equation (1)), and errors are assumed possible in the process of computations (i.e., the computational process itself can contain errors). Chapter 3 describes some techniques of constructing regularizing equations whose solutions uniformly approximate the branches of the exact solution and can be obtained approximately, for example, by the Newton–Kantorovich method. Accumulation of errors in the process of constructing asymptotics, and also the technique of parameter continuation are considered. Chapter 4 describes the employment of power geometry methods proposed by A.D. Bruno [1] in order to make uniform the solution branches, on the basis of which the N-step method of sequential approximations in the neighbourhood of branching points is constructed. The theory of branching for interlaced equations (see also chapter 5) is proposed here. This theory gives us the possibility to investigate different branches of solutions dependent on free parameters. The Lyapunov–Kantorovich method of convex majorants, which is introduced here, is employed for estimating the domain of existence and possible extension of solution branches.

In the multi-dimensional branching theory, nonlinear equations often have families of small solutions depending on one or several parameters. As a rule these parameters have a group sense — the nonlinear equation turns out to be invariant ($\equiv$ equivariant) with respect to some group of transformations. In physics the case in which there are non-group parameters is understood as random degeneracy. For the boundary value problems group symmetry is usually stipulated by symmetry of the domain. In computing families of branching solutions and asymptotics, group invariance simplifies constructing and investigation of the branching equation (BEq), which is equivalent to the nonlinear problem.

The very first results concerned with application of group symmetry in the theory of branching belong to V.I. Yuodovich [3, 4], who had considered "one case of branching in the presence of a multiple spectrum" as well as the applications to computation of secondary stationary fluid flows between one-sided rotating cylinders.

The subsequent development of the bifurcation theory under group invariance conditions was continued by B.V. Loginov and V.A. Trenogin [1]. In [1] the group stratification method for constructing a reduced BEq was proposed (see wide bibliography of the monograph [10] (B.V. Loginov) — the survey of results up to 1980). Particularly, in [2] (B.V. Loginov, V.A. Trenogin) a theorem on inheritance of the initial nonlinear problem's group symmetry by the corresponding BEq has been proved.

Since the mid 1970s symmetry methods in the branching theory have been elaborated independently by western and soviet mathematicians. The theorem of inheritance was later proved and applied to the Benard problem by D.H. Sattinger [2, 3]. Some results concerned with pattern formation in branching problems were also obtained by B.V. Loginov and applied to the statistical theory of crystals.

The most general result — existence of a bifurcation near the odd-multiplicity eigenvalue of the analytical operator function of a spectral parameter — was proved by N.A. Sidorov and V.A. Trenogin (1971, see chapter 2) who had applied the theory of mapping degree directly to the BEq. In the equivariant branching theory, this result allows one to obtain (see Chapter 5) existence theorems for the solutions which are invariant with respect to the subgroups, in particular, with respect to normal divisors — the most general result of A. Vanderbauwhede's "equivariant branching lemma" [3].

In the 1980s monographs by A. Vanderbauwhede [3], M. Golubitsky, I. Stewart and D. Schaeffer [1, 2] describing various applications were published. They suggest detailed surveys of the results obtained by mathematicians of western countries in the theory of equivariant branching. The main tool of the investigations [1, 2] is the singular theory of smooth mappings. However, in our opinion the Newton type of polytope methods developed by A.D. Bruno [1] provide more insight since these assume investigation of BEqs of any order n of degeneration of the linearized operator (see Chapter 4).

The theorem on inheritance of group symmetry had given a new approach in the theory of equivariant branching — application of methods of group analysis of differential equations (L.V. Ovsyannikov [1, 2]). These methods allow one to solve the problem of constructing a general form of BEq at the expence of inherited group symmetry in cases of both stationary and non-stationary bifurcations.

Chapter 5 is devoted to applications of ideas of symmetry in the theory of bifurcations. The principal objective implied consideration of applications to problems of mathematical physics as well as suggestion of illustrative examples. In the first two sections of this chapter the authors investigate properties of hereditance of symmetry by the branching equation. The theory of resolving systems has been applied in section 2 for proving the Grobman–Hartman theorem for differential equations in Banach spaces with a degenerate operator at the derivative (see also Chapter 6). This result may be considered as an introduction to center manifold methods for such equations.

Necessary and sufficient conditions for simultanious reduction with respect to both the unknowns and the equations (truncation reduction) are obtained. They serve as the basis for the possibility of applying the iteration procedure in obtaining families of multi-parameter solutions (see sections 1 and 3). Application to some problems of mathematical physics are given as illustrative examples. Section 3 discusses constructing general form of BEq assuming group symmetry. The case of potential BEqs is considered separately (see also Chapters 2 and 4).

The comparison of various approaches shows that more efficient for solving this problem are S. Lie–L.V. Ovsyannikov invariant manifolds methods — so called group analysis methods. They are applied in section 4, 5, 6 and 7 of Chapter 5 to some problems of mathematical physics bound up with the Helmholz equation with a nonlinear perturbation, the theory of capillary–gravity surface waves in hydrodynamics and problems of phase transitions in the statistical theory of crystals. These problems can be interconnected and considered as problems of symmetry violation considered in the branching theory enlighten in sections 3 and 4 from the general viewpoint. Section 8 includes applications of methods of group analysis in construction and investigation of Lyapunov–Schmidt BEqs in the case of the Andronov–Hopf bifurcation (cycle birth bifurcation). Section 9, the final one, discusses the questions of stability of branching solutions.

Chapter 6 describes applications of Lyapunov–Schmidt's ideas in the theory of *differential operator equations* (DOE)

$$B(t)\dot{u} = F(t, u) \tag{2}$$

with the irreversible operator $B(0)$ in the main part (with a singularity — briefly, singular I-DOE). A number of 'initial value and boundary value' problems, which model real dynamic processes of filtering, thermal convection, deformation of mechanical systems, electrical engineering (models of Barrenblatt–Zheltova, Kochina, Oskolkov, Hoff, V. Dolexal and others), can be reduced to such equations.

Singular differential operator equations have been investigated in the works of S.G. Krein, N.A. Sidorov, B.V. Loginov, G.A. Sviridyuk, I.V. Melnikhova, A.I. Kozhanov, R.E. Schowalter, M.V. Falaleev and others. Extended bibliographies can be found in monographs by N.A. Sidorov [20], R.W. Cassol and R.E. Schowalter [1], and in the survey by G.A. Sviridyuk [1].

The problem of applying Lyapunov–Schmidt's ideas to singular differential operator equations having Fredholm operators in the main part had been stated already by L.A. Lusternik in the course of work of his symposia held at Moscow State University in the mid 1950s. It appeared obvious that the analog of the classical branching equation for such equations (see chapters 2 and 3) is a system of differential equations of an infinite order (see Sidorov [1]). In view of substantial difficulties which arise in the investigation of this system, the theory of singular differential operator equations is presently far from being completed, moreover, there are few results for the nonlinear case.

In Chapter 6, in explication of foundations of the theory of singular differential operator equations, the authors have employed the apparatus of generalized Jordan chains (developed in Chapter 1) and the fundamental operators of singular integro-differential expressions (constructed by M.V. Falaleev [1]), the theory of generalized functions, the Nekrasov–Nazarov's method of undetermined coefficients, which is combined with asymptotic methods of the theory of differential equations with singular points, topological methods, methods of semigroups and groups with kernels developed by G.A. Sviridyuk. Such a mixture of diverse methods has given the possibility of investigating a wide class of singular differential operator equations and partial differential operator equations with the Noether operator in the main part. In the linear case a number of classes of singular differential operator equations has been completely investigated.

Chapter 7 considers applied problems of mathematical physics. Here a system of Vlasov–Maxwell integro-differential equations, which describes the behaviour of multi-component plasma, is investigated. This system has a great importance for applications, and so it is intensively investigated by several schools of applied mathematics and theoretical physics (J. Batt and G. Rein, R. Glassey, J. Schaeffer and Y. Guo, P. Degond, etc.).

This chapter investigates the solutions for the system of Vlasov–Maxwell equations which correspond to the distribution functions introduced and employed in many works of Russian mathematicians (see the survey by G.A. Rudykh, N.A. Sidorov, A.V. Sinitsyn, Yu.A. Markov [1], and the paper by V.V. Vedenyapin [2], etc.) and in some works of mathematicians from western countries (J. Batt and Fabian [1], Braach [1], Glassey, Guo and Ragazzo [1]). Techniques of reduction to systems of elliptic equations and problems of existence and stability of solutions are considered. Classes of exact solutions are constructed and described for the case of concrete distribution functions. On the basis of results of Chapter 2, existence theorems for bifurcation points of solutions of the Vlasov–Maxwell system have been proved, and the asymptotics of the solutions have been computed. It is known that the Vlasov–Maxwell systems (classical and relativistic) make it possible to construct and investigate various models of magnetic insulation (for example, problems of magnetic insulation were investigated by Abdallah, Degond and Mehats [1]). The Degond model [1] has been proved to be efficient. In the appendix written by A. Sinitsyn [1] a brief derivation of that model is considered, existence theorems on solutions of the corresponding two-point boundary value problem are given and its dependence on physical parameters is described.

The chapters are devided into sections and subsections where appropriate. Mathematical relations are numbered autonomous in each section. In double numbering a first digit corresponds to the section number (or a subsection number in the chapter 4). The second digit numbers formulas inside a section (a subsection in the chapter 4).

The preface was written by B.V. Loginov and N.A. Sidorov, Chapters 1–4 by N.A. Sidorov, Chapter 5 by B.V. Loginov, Chapter 6 by M.V. Falaleev and

N.A. Sidorov, Chapter 7 by A.V. Sinitsyn and N.A. Sidorov, the Appendix by A.V. Sinitsyn.

Translation from Russian into English was made by A.V. Sinitsyn (Chapters 1–4, 7 and Appendix), B.V. Loginov (Chapter 5) and M.Yu. Chernyshov (Preface and Chapter 6).

We hope that this monograph will be helpful for specialists in both classical and applied mathematics, mechanics, theoretical physics, as well as for graduate and postgraduate students specializing in the areas indicated above. Any critical and improving, or complementing remarks will be accepted by the authors with gratitude.

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In addition N.A. Sidorov and M.V. Falaleev deliver regular courses of lectures on nonlinear and singular problems considered in our book at ISU (Irkutsk, Russia), B.V. Loginov in RIM (Taskent, Uzbekistan) and USTU (Uljanovsk, Russia). A.V. Sinitsyn held a number of seminars on the applications (Chapter 7 and Appendix) at the Univ. de Paule Sabatier (Toulouse, France) and at the Univ. Nacionale de Columbia (Bogota, Columbia).

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THE AUTHORS

Contents