DEVELOPMENTAL PSYCHOLOGY SERIES

ACQUISITION OF MATHEMATICS CONCEPTS AND PROCESSES

Edited by

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Acquisition of Mathematics Concepts and Processes

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Acquisition of Mathematics Concepts and Processes

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Preface

This volume is a companion to Herbert Ginsburg's *Development of Mathematical Thinking* in the Developmental Psychology Series edited by Harry Beilin. The common goal of these two books is to present some of the most promising and productive areas of current research in mathematics learning and problem solving. Leading researchers in each area have been enlisted to characterize state-of-the-art developments in their fields of specialization, explain where their own work fits into this picture, describe one project or study as an exemplar of what is current, and identify some important directions for future research.

Because a community of scholars is beginning to emerge at the interface of several branches of psychology, mathematics, and mathematics education, the editors believe that it is timely to publish a set of volumes that might further this movement. Each volume was planned for readers from both the psychology and mathematics education research communities. Ginsburg's book was intended to focus on the work of researchers who approach mathematics learning and problem solving from a psychological perspective, whereas this volume was designed to represent a mathematics education point of view.

The distinctions between a psychological and a mathematics education perspective are discussed in the introduction (Chapter 1). One characteristic of the mathematics education research reported in this volume is an emphasis on mathematics as a highly structured content domain. Another is the important role of instruction, both as a goal of research and as a methodological tool in research settings.

Carpenter and Moser (Chapter 2) trace the development of whole-number arithmetic and describe their program of research on the procedures children use to obtain answers to addition and subtraction problems. The next three chapters provide information about three different, but in many ways complementary, approaches to research on the development of rational-number ideas. Karplus, Pulos, and Stage (Chapter 3) investigate the use of proportional reasoning patterns by early adolescents, seeking data on individual differences, sex differences, grade-level differences, and the relationship between student attitudes and success on proportional reasoning tasks. Behr, Lesh, Post, and Silver (Chapter 4) conducted rational-number research that included a large-scale testing component, a small-group instructional component, and the careful observation of the effects of theory-based instruction on rational-number understandings. Vergnaud (Chapter 5) describes three categories of multiplicative structures (isomorphism of measures, product of measures, and multiple proportion) and reports results of didactic experiments involving the topic of volume in seventh-grade classrooms.

Chapters 6 and 7 deal with research on space and geometry. Bishop (Chapter 6) reviews two major thrusts in this field: (*a*) the child's understanding of spatial and geometric concepts and (*b*) spatial abilities and visual processing. Hoffer (Chapter 7) provides an overview of the research and curricular developments in the Soviet Union and the United States that have been based on the Van Hiele levels of thought and phases of learning for geometry.

The last three chapters focus on mathematical problem-solving. Lester (Chapter 8) discusses the complex nature of mathematical problem-solving and reviews the necessarily slow progress that mathematics education researchers have made in developing a stable body of knowledge concerning its acquisition. He then describes past research and some recent studies conducted at Indiana University, and raises several key issues for future research, including a call for more open dialogue among researchers in various disciplines.

Lesh, Landau, and Hamilton (Chapter 9) define and illustrate a theoretical construct—a conceptual model—central to a current research project on applied mathematical problem-solving.

Schoenfeld (Chapter 10) characterizes the "managerial" aspects of expert and novice problem-solving behaviors among college students and describes the impact of managerial or "executive" actions on success or failure in problem solving.

The mathematics content treated in this volume ranges from early number-concepts in the primary grades to complex problem-solving at the college level. Thus the book should be of interest to teachers of mathematics and to trainers of mathematics teachers in grades K–13, as well as to mathematics education researchers and graduate students in mathematics education.

The editors are indebted to Christine Duffy for her help in preparing the manuscript.

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CHAPTER 1

Introduction

Richard Lesh and Marsha Landau

This book presents some of the most promising and productive current research in mathematics learning and problem solving to readers from both the psychology and mathematics education communities, and does so from a *mathematics education* point of view. What constitutes a distinctive mathematics education perspective? The answer includes an emphasis both on *mathematics* as a content domain, and on *education,* for example, the development of instructional materials. These dual emphases influence the nature of the research questions asked, the underlying assumptions made, the research procedures used, and the generalizations formulated.

In the past, mathematics education researchers borrowed most of their theoretical perspectives and research methodologies from other fields, largely within psychology (e.g., developmental psychology and information processing). However, the chapters in this book represent subareas of mathematics education research that have matured sufficiently so that "theory building" has replaced "theory borrowing." For example, Lester emphasizes in Chapter 8 the movement toward theory-based research in problem solving. A consequence has been that formerly useful, borrowed methodologies are frequently inconsistent with the purposes and assumptions underlying the newly emerging theoretical perspectives. Major mathematics education research projects, such as those discussed in this book, have had to engage in the development of research methodologies as well as in the generation of knowledge related to the improvement of mathematics instruction.

Many of the most promising techniques integrate some form of instructional intervention into the data gathering process. For example, Behr, Lesh, Post, and Silver's rational number research (Chapter 4) made extensive use of Soviet-style teaching experiment methodologies (Kantowski, 1978). The results of both

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Bishop's (Chapter 6) and Hoffer's (Chapter 7) geometry research were linked to the presence–absence and quality–quantity of prior instruction. In Carpenter and Moser's early number research (Chapter 2), in Karplus, Pulos, and Stage's proportional reasoning research (Chapter 3), and in Vergnaud's multiplicative structures research (Chapter 5), the explicit assumption was made that task behavior was as much a function of internal models as of external stimuli, and that these internal models (i.e., mathematical ideas) were, in large part, the products of explicit instruction.

Care must be taken in using instruction-related research methodologies because changes are induced in the subjects that the research is intended to describe, and the research environment is modified differentially to fit individual subjects, thus raising questions about standardization. On the other hand, the entire notion of a *standardized question* may be an inappropriate construct if the theory on which the research is based assumes that two students frequently interpret a single problem situation or stimulus in quite different ways (as a function of the internal models that are selected and superimposed onto the problem). In mathematics education research, it is common to assume not only that different students may interpret a single seemingly unambiguous problem in completely different ways, but also that two responses that appear identical may be produced using completely different solution paths.

Clinical interviewing techniques have begun to reflect a diminished concern with standardization in order to obtain a more complete picture of children's developing understandings of mathematical ideas and the processes that are used to produce answers. Investigators often find it useful to tailor an interview to the subject either in a relatively spontaneous manner or by beginning with a standard series of questions and then making modifications based on the interviewee's responses. Such stochastic questioning procedures might use *sets* or *sequences* of problems to identify the *processes* used to arrive at answers, and to identify the relational–operational systems that are used to make judgments based on the underlying ideas.

The projects discussed in this book frequently used properties of formal mathematical systems to generate tasks for investigating the structural properties of the conceptual models, mathematical understandings, or internal "programs" students used to interpret and manipulate problem situations. For example, in Lesh, Landau, and Hamilton (Chapter 9), sets of tasks were generated that were presumably characterized by isomorphic structures; sources of variability across tasks were investigated. Vergnaud (Chapter 5) emphasizes the importance of attending to formal mathematical structures when studying the acquisition of interconnected concepts. His analysis and classification of multiplicative structures shaped his investigation of products and proportions in seventh-grade classrooms in France. Carpenter and Moser's chapter (Chapter 2) represents another example of a body of research that has investigated relationships between process use and content understanding. In early number research, relevant processes have included counting or imaging capabilities as well as various computational processes or problem-solving procedures.

The goals of much of the research reported in this book were (*a*) to identify students' primitive conceptualizations of various mathematical ideas and processes (e.g., rational numbers, proportions, early number concepts, and spatial–geometric concepts); (*b*) to investigate similarities and differences between students' conceptual structures associated with these ideas and the formal mathematical structures that characterize them; (*c*) to describe how these conceptualizations are gradually modified into mature understandings; and (*d*) to identify factors that influence this development. These *idea analyses* are quite distinct from the *task analyses* and *analyses of children's cognitive characteristics* that tend to be used in general (i.e., not subject-matter-oriented) psychological research. These three types of analyses are clearly interrelated. However, the research procedures that are appropriate and the generalizations that result often depend on which of these analyses is chosen as central.

Child development research tends to focus on (*a*) cognitive capabilities that are (or are assumed to be) invariant across large bodies of subject matter, (*b*) changes in general cognitive capabilities before and after major cognitive reorganizations (i.e., at approximately 2 years, 6–7 years, and early adolescence), and (*c*) ideas that most students acquire "naturally," without specific instruction.

By contrast, mathematics education research designed to trace the development of a given idea is likely to focus on one of the following: (*a*) ideas that are acquired at intermediate levels between concrete and formal stages of thought, and on idea-related variations in operational ability; (*b*) factors and processes that produce or facilitate transitions from one level of conceptualization to another for individual concepts; (*c*) processes and capabilities that are linked to content understanding; and (*d*) ideas that do not develop naturally, that is, without artificial (e.g., instructional) experiences to facilitate their acquisition. Thus, research focusing on analyses of students' cognitive characteristics tends to generate labels (e.g., preoperational, impulsive, field dependent) that are associated with a given child, or on characteristics that are assumed to be difficult or impossible to change through instruction. Variability across concepts (or across tasks based on the same concept) is frequently ignored as an uninteresting *décalage*.

Idea analyses tend to result in generalizations about children's *ideas* (i.e., about behaviors that can be expected, given a particular conceptualization of an idea) rather than about children per se. Also, investigating variability across tasks is important for idea analysis. For researchers whose ultimate purpose is to design better instructional experiences for students, it is necessary (in situations in which empirical evidence is not always available) to generate theories for predicting the appropriate sequencing of ideas, tasks, and instructional models.