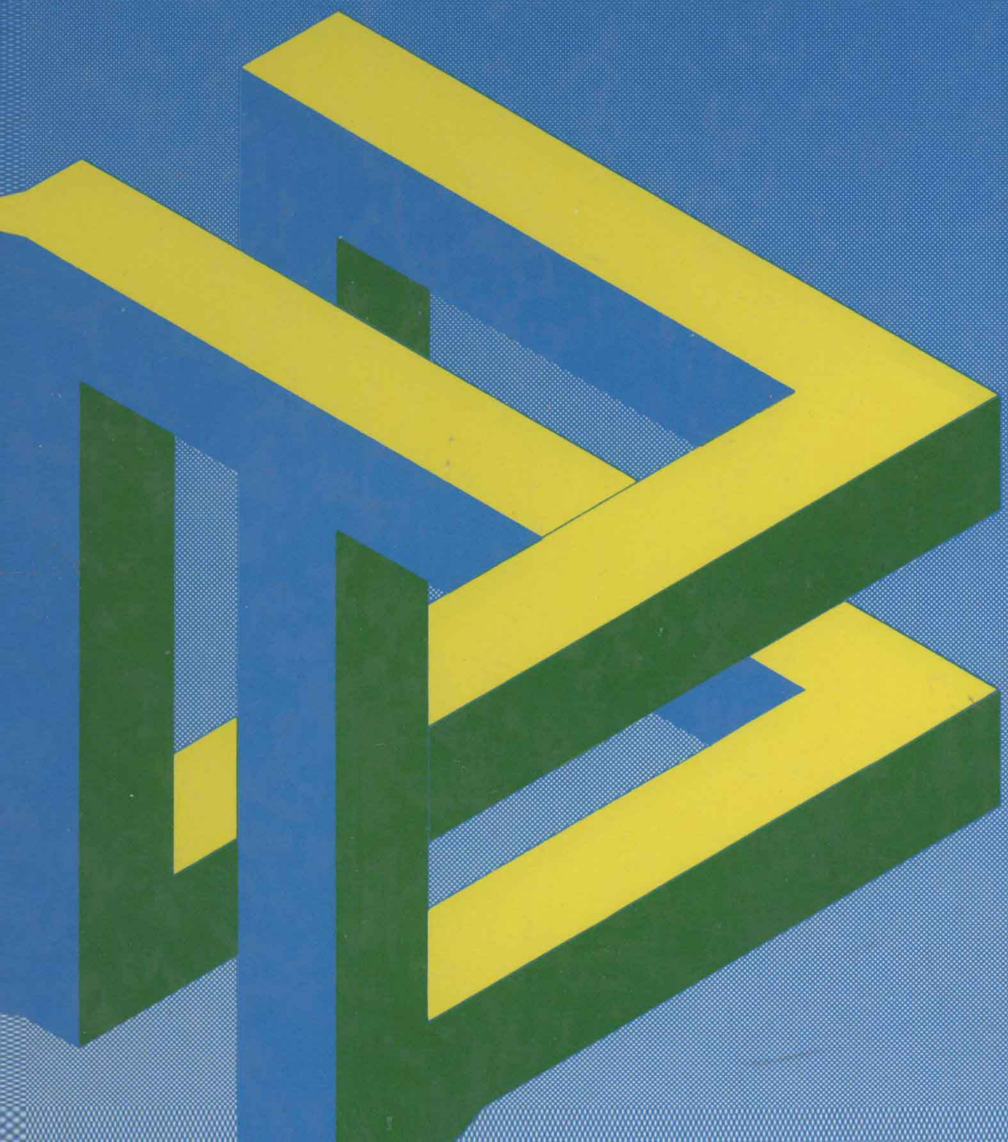
Ellis Horwood Series MATHEMATICS AND ITS APPLICATIONS

# RETHINKING MATHEMATICAL CONCEPTS

R. F. WHEELER



Ellis Horwood Series in
**MATHEMATICS AND ITS APPLICATIONS**
Series Editor: Professor G. M. Bell
Chelsea College, University of London

# RETHINKING MATHEMATICAL CONCEPTS

R. F. WHEELER, Department of Mathematics, University of Leicester

Too many university lecturers show a regrettable tendency to criticize school mathematics teaching and then do nothing constructive to help the beleaguered teacher, who is probably working hard to come to grips with an unfamiliar syllabus. This book is the first (of which we are aware) to attack the problem in a sympathetic way. It discusses basic concepts in mathematics likely to be introduced to secondary pupils and considers the strengths and weaknesses of various approaches and notations. It looks extensively at the confusion in many presentations of elementary pure mathematics, and will do much to clarify modern trends.

The author (himself a former schoolmaster) is anxious to promote the consistent use of good mathematical symbols and terminology, and urges the need for forethought and rigour in presentation. He provides the background to many of the decisions a teacher will have to make, particularly at the upper end of the secondary school, and shows the consequences such decisions have upon the development of the subject. He considers, for instance, the definition of a function: describes what alternative definitions exist, why there is a somewhat bewildering variety, and the reasons why some are preferable to others.

Discussion is lively and thought-provoking and gives existing experience sharper focus. The book encourages a critical attitude on the part of the reader, presenting the evidence on which informed action can be based, rather than offering dogmatic solutions to the problems. Sufficient reminders of the relevant mathematics are included to make each chapter self-contained, especially where arguments range beyond the usual school syllabus, but prior knowledge of the essentials of each topic is assumed.

**Readership:** Appropriate for a variety and depth in mathematical education — undergraduates and teachers in university departments of education, teachers and undergraduates in teachers' training colleges — and also all teachers in other subject disciplines who need to teach mathematics, e.g. in physics, mathematical physics, engineering and so on.

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# Introduction

This book is addressed to the student mathematics teacher. Its aim is to encourage you to think about the subject and its presentation.

The book may also be of interest to other students in higher education and to all those called on to teach mathematics, especially if this subject has not been your main area of specialization. Indeed, any educator who is keen for students, at whatever level, to acquire a sound grasp of mathematical concepts may find something of novelty in the discussions.

The process of learning mathematics is one of evolution. No topic is ever finally exhausted: as one's experience grows, one continually discovers thrilling new levels at which an idea can be investigated. Ultimately, the challenge of having to teach a topic to someone else reveals a further hierarchy of unsuspected subtleties in the concept every time it is presented. It is important to avoid teaching anything that needs contradicting later: what you say may need further amplification and expansion; hidden assumptions may need to be exposed and examined; the more mature student may demand a more elaborate or precise enunciation. But a wise teacher aims never to have to tell his pupils to forget what they were told last year because he is now going to use a conflicting approach. Textbooks also must be chosen carefully if the teacher is not going to have to say: 'Ignore what the book says: that's not a good way of doing it: I want you to look at it this way'.

This book examines some of the many problems that arise when our subject is presented to pupils. It does not set out to teach you any new mathematics—each chapter assumes you are already acquainted with the essentials of the topic being discussed—and what it certainly does *not* try to do is to develop any theories of mathematics learning or concept formation.

In any sequence of lessons, the teacher has to make many decisions on matters of exposition, particularly of definition and notation, some of which have far-reaching consequences for later work, not always appreciated at the time. As any experienced teacher will tell you, one of the most difficult things in teaching is to get pupils to unlearn something they have been taught; to discard a cherished habit once acquired. If you doubt the truth of this maxim, consider the case of the student who evaluates $\int_0^1 dx/(1 + x^2)$ as 45° (and who probably feels aggrieved when you mark it wrong). Having first met the tangent in trigonometry as a function of an angle and grown accustomed to this, he has obviously found it too traumatic to abandon this established mode of thought and to accept that the tangent has long since become a function of a *number*: that $\tan^{-1} 1$ is the number $\frac{1}{4}\pi$, and not an angle. Perhaps this particular hurdle is unavoidable, but many others are not and the fewer obstacles we create for our pupils, the better.

Mathematicians like to feel they are consistent people, but the inherent consistency of their subject does not always carry over to their methods of exposition. One trivial example will suffice for the moment: plenty of more significant examples are discussed later in the book. Ask yourself how you read aloud (as if to a class) the expression $(2x + 3)(x + 5)$. Decide precisely what you actually say before reading on. Did you say '$2x + 3$ into $x + 5$' or '$2x + 3$ times $x + 5$'? You have only to listen to mathematics lecturers and teachers to find out that most people have been brought up to use the reading 'into'. Does this not strike you as an extraordinary reading? Would you use it yourself when teaching a pupil just starting algebra? One hopes not, yet that is just what many teachers are doing. Having spent time in arithmetic lessons clearly associating in the child's mind the word 'into' with the operation of division ('4 into 12 goes 3' and that sort of thing), all of a sudden the teacher expects him in algebra to associate 'into' with the inverse operation of multiplication. The stupidity of this is obvious, yet the conventional presentation of mathematics is shot through with examples like this. One sometimes feels that the surprising thing is not that some children abandon mathematics but that any persevere at all.

During your period of professional training, you will be mainly concerned with trying to evolve exciting ways of presenting your mathematics to your future pupils. But you will also have the opportunity and leisure to reflect on the subject itself. This book will try to give you the background to just a few of the decisions you will have to make once you get to the chalk-face (particularly if you teach pure mathematics at the upper end of a secondary school) and to show you the consequences these have for the development of the subject. The one unforgivable sin in mathematics teaching is to go on following bad models without having any definite reason for doing so. The author dare not expect that you will agree with everything he has written, although he naturally hopes that most of his arguments will be sufficiently cogent to compel assent. If, however, you are not convinced by a particular piece of reasoning and decide to reject the evidence offered, the writer will not be unduly disappointed. At least you will then be acting because of a positive conviction you have arrived at that what you have *chosen* to do provides the best possible approach, and not just continuing to do something in a particular way because 'that's the way I was taught myself' or 'that's the way this textbook does it'. Indeed, one of the things one would most like you to develop is a certain critical faculty in appraising mathematics books (including this one). As a newcomer to teaching, you should certainly *start* by recognizing that the textbook writer will have been more experienced than you and may have had good reasons for the presentation he has chosen but, if I share with you the secret that many books in circulation are seriously misleading in their mathematics, whatever their pedagogic felicities, you will perhaps not feel the need slavishly to follow their authors if you genuinely think you have sound reasons for objecting to their treatment.

This book will try not just to tell you what, in the opinion of the author, the definition of, say, a function should be, but to consider what the alternative definitions are, why the books offer such a bewildering variety, why some are demonstrably unsuitable and, among those that are at least acceptable, what the relative merits are and the reasons why mathematicians are likely to prefer some definitions to others. In other words, it attempts to fill in some of the explanations of *why* certain things are done, whereas books aimed at exposition only tell you *what* is done and *how* it is done. It also considers, for example, questions such as why mathematicians choose *not* to write $\infty = 1/0$: it does not tell you what to say to your pupil when he or she does this; it tries to give you the background you need to have, so that you yourself can devise an explanation that will be appropriate to present to the child. Nor does this book give you any guidance on how to react when you find that your school's chosen examination board has elected to promote the 'arcsin' notation—as, at the time of writing, one board sadly does—if you have become convinced by what you have read that this is manifestly inferior to '$\sin^{-1}$'!

There is one particular trap into which a new teacher can easily fall. He is lured into it because his recollection of the fads and foibles of his mathematics lecturers in tertiary education are necessarily much fresher in his mind than are his memories of the practices of his teachers at school. He is, therefore, strongly tempted to show off his own newly acquired mathematical maturity and imagined sophistication by repeating with his own classes the mannerisms he has picked up since leaving school. But this *may* not be at all desirable. All such fashions need to be inspected closely and their suitability for school use assessed before they are copied with young pupils: this, indeed, is a frequent theme throughout this book.

The point being made is that a lecturer presenting mathematics to a 19 or 20 year old student often expresses himself in a way that he would not (or, at least, should not) if teaching a 15 or 16 year old. As an example, the ways in which he uses the word 'infinity' or the sign '$\infty$' may be particularly lax, because he believes—one hopes the belief is justified!—that he can rely on his listeners' wisdom to translate his sloppy shorthand into taut mathematics. But it will probably be quite inappropriate to copy such imprecise abbreviated statements with schoolchildren, who will not have had a similar depth of experience.

It must be stressed that not all lecturers would subscribe to the above philosophy: this writer, for one, *certainly* would not. If it is wrong to use language or notation in a cavalier way with a pupil, it is no less wrong to do so with an undergraduate. But it is no good shutting one's ears and eyes to the liberties that are taken in practice—one hears and reads them at every turn. It is unrealistic, therefore, to ignore the influence these habits are likely to have been exerting on a candidate new to teaching.

It is hoped that sufficient reminders of the relevant mathematics have been included to make each discussion self-contained, especially where it has been necessary to range beyond the school syllabus in order to support the argument. But since an ordered, logical presentation was neither appropriate nor intended, it may be that some readers will need to refer to explanatory texts to remind themselves of some facts the author has taken for granted. To make the book as useful as possible as a vade-mecum, generous cross-references, both forward and backward, have been provided.

Even if you are a recently qualified teacher of secondary mathematics, you may find the discussions useful and thought-provoking. Indeed, the very experience you have been acquiring may give them a sharper focus, as you may now appreciate, even more than you would have done as a student, the relevance of some of the problems raised. Dare one hope that a few textbook writers and examiners might also find that reading and evaluating the arguments was not a complete waste of time?

Any experienced teacher who dips into this book will recognize the author's debt to the various Mathematical Association Reports, which were a seminal influence in his early years as a schoolmaster and can be warmly recommended to every new teacher. The further debt to my own gifted schoolmasters is one that will not be obvious to the reader, but is just as real.

Chapter **1**

# Relations and operations

## 1.1

There are probably three main arguments that would be put forward to justify the introduction of the various changes in the school pure mathematics syllabus during recent decades: (1) that new topics have widened the scope of the curriculum by, for example, applying algebra to interesting structures other than the usual number systems—structures governed by novel rules of operation, such as sets, vectors, matrices, finite arithmetics, permutations and geometric transformations—so that the pupil gets a broader view of the nature of mathematics and its potential areas of application; (2) that there has been an attempt to increase the understanding of mathematical concepts, even at the expense of some loss of proficiency in certain techniques of limited value; and (3) that the idea of mathematics as a language has been promoted by trying to encourage clarity of thought and precision of expression.

In the furtherance of these last two objectives there has been a quite dramatic sharpening of the presentation of fundamental ideas on, for example, operations and relations and functions. You have only to look at the woolly accounts of these notions in some old school textbooks to see what advances in lucidity have been achieved. There are, however, many points worth bringing to your attention before you try your hand at discussing such topics with your classes and the first two chapters are, therefore, devoted to these basic concepts. These chapters may require rather more concentration on your part than some of the later topics in the book. But remember that the more fundamental an idea is for the development of mathematics, the more important it is to make sure that one's own grasp of the concept is secure, so that one's teaching, even with the youngest pupils, will be laying sound foundations.

Not so very long ago, the occurrence in students' work of phrases like

| | | |
|---|---|---|
| '$x$ is = to $y$' | '$x$ and $y$ are =' | 'the = numbers $x$ and $y$' |
| 'AB is $\parallel$ to CD' | 'triangles ABC, DEF are $\equiv$' | 'the $\perp$ lines $l$, $m$' |

was quite common (although it is only fair to point out that perceptive teachers have *always* discouraged any such misuse of mathematical language). But, even if these grammatical mistakes are becoming rarer, they are by no means extinct and, until they are, one cannot be entirely happy about the teaching of relations. One's disquiet is reinforced by the number of student teachers who have been known to say, for example, that the binary relation $\rho$ is symmetric 'if it satisfies $a\,\rho\,b = b\,\rho\,a$'. Please pause for a moment to reflect on the enormity of this blunder!

## 1.2

It will be a good idea to start by looking at some of the (binary) relations that are met in fairly elementary mathematics, together with the symbols conventionally used to represent them, in order to remind ourselves that in all cases the symbol has a *verbal*, not an adjectival, function in a mathematical sentence. (In teaching the subject, of course, valuable use can be made in the early stages of more colloquial relations, such as 'is a brother of', 'lives in the same street as', 'is taught by', and so on.)

$=$ 'is equal to'
$<$ 'is less than'
$>$ 'is greater than'
$\leqslant$ 'is less than or equal to'
$\geqslant$ 'is greater than or equal to'
$|$ 'divides' [or, less shortly, 'is a divisor of']
$\equiv$ 'is congruent to' [in number theory or in geometry]
R 'is a quadratic residue of'
N 'is a quadratic non-residue of'
$\parallel$ 'is parallel to'
$\perp$ 'is perpendicular to'
$\in$ 'belongs to' [or, less shortly, 'is an element of']
$\subseteq$ 'is a subset of', 'is contained in', or 'is included in'†
$\subset$ 'is a proper subset of', or 'is strictly (properly) contained in'
$\supseteq$ 'contains', or 'includes'

†This choice of symbol ($\subseteq$) is *greatly* to be preferred to the extremely unfortunate selection of '$\subset$' to mean 'is a subset of'. Not only does that leave no symbol available for the relation 'is a proper subset of' but the analogy is lost between the partial order relations $\leqslant$ and $\subseteq$ (which are reflexive, transitive and antisymmetric) and between the dominance relations $<$ and $\subset$ (which are transitive and irreflexive).

$\supset$ 'strictly (properly) contains'
$\Rightarrow$ 'implies'
$\Leftarrow$ 'follows from' [this reading is shorter than 'is implied by']
$\Leftrightarrow$ 'implies and follows from'

These last three relations ($\Rightarrow$, $\Leftarrow$, $\Leftrightarrow$) raise such interesting questions that they have been allotted a chapter to themselves [Chapter 17].

Associated with each of these relations is a *complementary* relation, which holds whenever the original relation does not; thus, $\neq$ ('is not equal to'), $\nmid$ ('does not divide'), $\not\equiv$, $\notin$, and so on. The phrases defining a pair of complementary relations are mutual negations. Obviously, R and N are complementary and $\nless$ is equivalent to $\geqslant$; but, of course, $\not\subset$ is not equivalent to $\supseteq$! In practice, some of the relation *symbols* are seldom negated.

All these mathematical relations have very precise meanings and are sharply defined within their contexts, so that no liberties are possible with any of the signs (although the tense or mood of the verb may change in subordinate clauses, as explained in Section 1.7). It is their very precision that makes them a vital part of the language of mathematics: and the less *that* language is abused the better. Sometimes lazy pupils try to get the best of both worlds by obligingly using $\perp$ for 'is perpendicular to', but inventing a symbol like $\perp^{r}$ or $\perp^{ar}$ for the adjective or noun 'perpendicular'. This is still thoroughly bad practice, because it debases the currency of the relation symbol '$\perp$'.

## 1.3

The long-suffering sign for the relation of equality always has been (and probably always will be) subject to the worst mangling, with pupils using it as an all-purpose link to mean 'I think there is some connexion, possibly rather tenuous, between what I have written just before and just after this sign, which I hope you may be shrewd enough to discover'. Many are the crimes that have been committed in primary schools, when $(2 + 7) \times 3 - 8$ has been 'worked out' as '$2 + 7 = 9 \times 3 = 27 - 8 = 19$'. In the days when geometry involved formal proofs, there were pupils who started each line with the sign '=' (instead of the then-fashionable '$\therefore$'), hence using the verb '=' as a conjunction.† The same thing was (and still is) perpetrated in algebra

†When the use of '$\Leftrightarrow$' and its associated symbols first became fashionable in elementary work, the writer predicted that it would not be long before it too would have degenerated into yet another universal link, written down without any serious thinking taking place: he has seen many instances of this that have amply justified his apprehension. Perhaps it is the case that children *need* a sign to represent some vague connexion that they are not sufficiently articulate to express but, if that really is so, let us invent a special sign for that purpose and not keep grasping symbols that have already been invested with precise meanings and then allowing them to be used with anything less than perfect precision.

with solutions like

$$x^2 = 2x - 1 = x^2 - 2x + 1 = 0 = (x - 1)^2 = 0 = x - 1 = 0 = x = 1,$$

(which may be written on one line or on several). The reader should be able to sort this one out more easily than some examples of the art.

**Any use of '=' to mean anything other than 'is equal to' must be unreservedly deplored.**

It is the pupils who have been allowed to be sloppy in using the balancing sign 'is equal to' who have difficulty in solving equations; who, because they do not clearly perceive that '$3x - 5 = 7$' is making the absolutely precise statement that '$3x - 5$ *is equal to* 7', cannot readily proceed to '$3x - 5 + 5 = 7 + 5$' and so on.

Any teacher who has ever encouraged (or even condoned) the writing of foolishness like

'A bicycle = £35' or 'Tom = 13 years'

instead of

'A bicycle costs £35' and 'Tom is 13 years old'

must share some of the blame for all the difficulties children experience. Note that even writing

'Tom's age = 13 years',

although not as culpable as the previous monstrosity, is still NOT to be recommended: just write

'Tom's age is 13 years'.

A disservice is done to the cause of mathematical language every time the sign for 'is equal to' is devalued by such cavalier and unnecessary use.

Similarly, a statement such as

'the gradient when $x = \alpha$ is 3'

clearly becomes quite unacceptable if the pupil replaces 'is' by '=', and although (provided the writer can be trusted to have written gramatically)

'the gradient at $\alpha = 3$'

can have only one meaning, it is discourteous to subject the reader to that unnecessary jolt before he realizes that the sentence has ended. That is the sort of discomfort which, in other contexts, punctuation is designed to prevent. This analogy is not frivolous: the enclosure of a clause between commas or dashes or parentheses closes it up, so to speak, in relation to the other elements in the syntax; the presence of the sign '=' between $\alpha$ and 3 closes up the phrase '$\alpha = 3$' in a similar way. In the above statement, the unwanted bunching would have been avoided if 'is' had been used.

How then do you react to the student (or teacher) who writes

'when $x = \alpha$, the gradient = 3' or 'at $\alpha$, the gradient = 3'?

These sentences must be acknowledged to be completely unambiguous but, even so, do you not feel that 'the gradient is 3' would have been incomparably better?

Teachers set poor examples to their pupils by unnecessarily replacing the simple word 'is' by a sign meaning 'is equal to', thereby encouraging their young charges to believe that the merit of a piece of mathematics is proportional to the number of occurrences per square metre of the sign '='. Mathematical symbols have evolved to promote concision of expression: in some hands, these servants seem to be in danger of becoming our masters.

It is not yet clear whether the trend in the early years of arithmetic towards writing things like

$$4 + 5 \rightarrow 9 \quad \text{or} \quad 4, 5 \overset{+}{\curvearrowright} 9$$

to mean that 'under the operation of addition, 4 and 5 combine to become 9' will have any advantages. Presumably, it is meant to prevent misuse of the sign of equality and to present addition as an active process rather than a passive state. The arrow is intended to suggest in a very elementary way a mapping (but without, of course, using that word). Since, however, what is being gently suggested is a function† $f$ of *two* variables, $f(a, b) = a + b$, with $a + b = c$ replaced by the idea that, under $f$, the number pair $(a, b)$ maps to $c$, that is, $(a, b) \overset{f}{\mapsto} c$, it may be felt to be a rather cumbersome approach. Time, however, will be needed to tell whether this interesting method does have any advantages in improving understanding and preparing the way for later work; and, in particular, whether, if the introduction of the sign '=' is delayed, its

†The concept of function and the various notations for expressing it are discussed in Chapter 2.