Mathematics and Its Applications

A. T. Fomenko

Integrability and Nonintegrability in Geometry and Mechanics

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A. T. Fomenko

Department of Mathematics and Mechanics,
Moscow State University, Moscow, U.S.S.R.

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Mathematics and Its Applications (*Soviet Series*)

SERIES EDITOR'S PREFACE

Approach your problems from the right end and begin with the answers. Then one day, perhaps you will find the final question.

'The Hermit Clad in Crane Feathers' in R. van Gulik's *The Chinese Maze Murders.*

It isn't that they can't see the solution. It is that they can't see the problem.

G.K. Chesterton. *The Scandal of Father Brown* 'The point of a Pin'.

Growing specialization and diversification have brought a host of monographs and textbooks on increasingly specialized topics. However, the "tree" of knowledge of mathematics and related fields does not grow only by putting forth new branches. It also happens, quite often in fact, that branches which were thought to be completely disparate are suddenly seen to be related.

Further, the kind and level of sophistication of mathematics applied in various sciences has changed drastically in recent years: measure theory is used (non-trivially) in regional and theoretical economics; algebraic geometry interacts with physics; the Minkowsky lemma, coding theory and the structure of water meet one another in packing and covering theory; quantum fields, crystal defects and mathematical programming profit from homotopy theory; Lie algebras are relevant to filtering; and prediction and electrical engineering can use Stein spaces. And in addition to this there are such new emerging subdisciplines as "experimental mathematics", "CFD", "completely integrable systems", "chaos, synergetics and large-scale order", which are almost impossible to fit into the existing classification schemes. They draw upon widely different sections of mathematics. This programme, Mathematics and Its Applications, is devoted to new emerging (sub)disciplines and to such (new) interrelations as exempla gratia:

- a central concept which plays an important role in several different mathematical and/or scientific specialized areas;
- new applications of the results and ideas from one area of scientific endeavour into another;
- influences which the results, problems and concepts of one field of enquiry have and have had on the development of another.

The Mathematics and Its Applications programme tries to make available a careful selection of books which fit the philosophy outlined above. With such books, which are stimulating rather than definitive, intriguing rather than encyclopaedic, we hope to contribute something towards better communication among the practitioners in diversified fields.

Until about 20 years ago, 1967 to be precise, the year of the discovery of the inverse spectral transform, the number of interesting (model) equations which could be solved completely was four. Now there are 40 or so. They include some of the more important equations of mathematical physics such as the Korteweg-de Vries equation, the Sine-Gordon equation and the nonlinear (or cubic) Schrödinger equation and most of them are the first members of infinite hierarchies of such socalled completely integrable equations. It is in fact not a total surprise that these completely integrable equations turn up often in applications: there is a good solid argument that says that in a wide variety of circumstances when dealing with wave phenomena the next step after the linear one is likely to involve the KdV or cubic Schrödinger equation.

Since its birth, the field of integrable systems has grown enormous; it now intertwines with most of the other fields in mathematics and accounts for many hundreds, quite possibly over a thousand, articles a year in mathematics, physics, chemistry, biology, geology and other journals. There is fair selection of introductory and basic books on the subject and a large collection of proceedings volumes. The field is now certainly far too large to be treated comprehensively in one volume; even 10 would probably not suffice to survey it in all its aspects.

Thus the time has come for a second generation of books on the topic, which concentrate in depth on certain main topics within the field. This is one of the first such books and by an author who has made fundamental contributions. Some of the unique features of the present volume are: a comprehensive treatment of surgery for completely integrable systems, a thorough treatment of how to recognize, analyse, and prove nonintegrability, and a discussion of 'noncommutative integrability'. The book contains a substantial number of results from the author's seminar at Moscow state university which are not, or not readily, available elsewhere. All in all it is a book that seems to fit the concept of this series very well and which should be of great interest to a large variety of mathematicians and physicists.

The unreasonable effectiveness of mathematics in science ...

Eugene Wigner

Well, if you know of a better 'ole, go to it.

Bruce Bairnsfather

What is now proved was once only imagined.

William Blake

As long as algebra and geometry proceeded along separate paths, their advance was slow and their applications limited.

But when these sciences joined company they drew from each other fresh vitality and thenceforward marched on at a rapid pace towards perfection.

Joseph Louis Lagrange.

Bussum, September 1988

Michiel Hazewinkel

PREFACE

In recent years, a new branch of science has appeared which originates in the classical theoretical mechanics, mathematical physics, the theory of Hamiltonian systems, and symplectic geometry. This branch may be conditionally outlined as follows: new methods of integration of Hamiltonian systems on symplectic manifolds. New profound relations between integrability of many systems and their implicit algebraic properties have been revealed. Among these properties, of primary importance are "system symmetries" which are not merely understood as groups of system invariance but in a more general sense as a set of algebraic properties of a system of differential equations which enable this system to be naturally "embedded," for instance, in the Lie algebra of a certain Lie group with preservation of its dynamical structure (for instance, with preservation of its Hamiltonian properties).

These mechanisms turned out to control integrals of many interesting Hamiltonian systems of equations which arise in geometry as well as in mechanics and physics. It is known that finding integrals for a concrete system is not a simple task. Furthermore, systems in "general position" do not generally have a sufficient number of integrals to integrate the equations. Therefore, sufficiently effective methods should be worked out to help search for rare integrable cases in the boundless ocean of all Hamiltonian systems (the "majority" of which are certainly nonintegrable). One of the aims of the present book is to acquaint the reader with several algorithms for finding integrals. Special attention is given to "formula-type" algorithms, that is, those algorithms which make it possible to efficiently represent integrals (whenever they are found) in an explicit form, for instance, in the form of polynomials or rational functions. This sometimes yields formulae explicitly for solutions (that is, integral trajectories) of a Hamiltonian system.

It is clear that we cannot omit the general mechanism responsible for nonintegrability of systems in general position. We believe it is instructive to present the methods of integration and the methods of proving nonintegrability of Hamiltonian systems in one book. Therefore, in the second part of the book, we give substantial attention to nonintegrability problems, especially to the qualitative aspect of the discussed effects, leaving out calculations and replacing them by references to the corresponding literature.

The book is aimed at acquainting a wide range of readers with geometric and algebraic mechanisms of integrability and nonintegrability. This aim determines the style of presentation, which is made as simple as possible. For the reader's convenience, we introduce the necessary information from related fields of science, such as the theory of Lie groups, symplectic geometry, topology, etc.

Thus, the main topics elucidated in the book are formulated as follows: 1) some mechanical systems and the corresponding Hamiltonian systems of differential equations; 2) fundamentals of symplectic geometry; 3) qualitative topological theory of integrable systems, classification of constant-energy surfaces of integrable systems, and Morse theory of integrable systems; 4) classification of Liouville torus surgery at the moment of intersection of critical energy levels; 5) general principles for integrating Hamiltonian systems: commutative and noncommutative integration, and applications; 6) integration of some concrete dynamical systems on symplectic manifolds. General methods and applications; 7) general mechanisms of nonintegrability of Hamiltonian systems.

This book is based on a two-year lecture course delivered by the author to students at the Department of Mechanics and Mathematics at Moscow State University in 1983–1985. In addition, the results of recent studies are included. Among the original material involved are the results obtained by the author and by participants of the seminar "Modern Geometric Methods," headed by the author at Moscow University in 1984–1987.

The author planned the program of the study of maximal involutive sets of functions on the orbits of Lie algebras. Investigations in this direction were started and first discussed at the above-mentioned seminar in 1980. The most interesting results obtained are also included in the book.

The book is intended for a wide range of readers, for those who are interested in applications of modern geometry to Hamiltonian mechanics and in the theory of integration of differential equations.

In the framework of a comparatively small book, it is practically impossible to give an exhaustive review of the modern state of integrability and nonintegrability of Hamiltonian systems of differential equations. The list of refereces includes some papers which will help the reader to study the problem. In particular, we recommend the papers by the following authors.

I. M. Gel'fand, S. P. Novikov, L. D. Faddeev, R. Bott, M. F. Atiyah, V. I. Arnold, J. Moser, V. P. Maslov, M. Adler, van Moerbeke, H. P. McKean, F. Calogero, O. I. Bogoyavlensky, A. M. Perelomov, A. V. Brailov, V. V. Trofimmov, A. M. Vinogradov, B. A. Kupershmidt, M. Vergne, C. S. Gardner, C. Godbillon, Yu. I. Manin, V. G. Drinfeld, V. E. Zakharov, B. Kostant, A. G.Reyman, D. Kazhdan, S. Sternberg, V. V. Koslov, P. D. Lax, J. Dixmier, D. Mumford, J. Marsden, A. Weinstein, M. D. Kruskall, F. Margi, M. A. Olshanetsky, M. A. Semenov-Tian-Shansky, T. Ratiu, A. V. Bolsinov, Ya. V. Tatarinov, A. Thimm, S. J. Takiff, S. L. Ziglin, V. Guillemin, M. Rais, M. Duflo, C. Conley, E. Zehnder, D. Ebin. G. Wilson, M. V. Karasev, B. A. Dubrovin, V. L. Golo, A. P. Veselov.

I express my gratitude to S. P. Novikov, V. I. Arnold, D. V. Anosov, Yu. I. Manin, H. Zieschang, V. V. Golo, O. I. Bogoyavlensky, Ya. V. Tatarinov, S. V. Matveev, V. V. Trofimov, A. V. Brailov, and A. V. Bolsinov for stimulating scientific discussions which resulted in improvement and extension of some sections of the book.

I would like to thank the Reidel Publishing Company and Professor M. Hazewinkel and Dr. D. J. Larner personally for their support and interest in my work.

I am also grateful to M. V. Tsaplina for much effort put in the translation of the book.

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CHAPTER 1

Some Equations of Classical Mechanics and Their Hamiltonian Properties

§1 Classical Equations of Motion of a Three-Dimensional Rigid Body

1.1 *The Euler–Poisson Equations Describing the Motion of a Heavy Rigid Body around a Fixed Point*

Let us consider a rigid body located in a gravity field and moving around a fixed point 0. To described this motion, it is convenient to use two Cartesian coordinate systems. The *first* one is *fixed* in an enveloping three-dimensional Euclidean space $\mathbb{R}^3$, and the coordinates of the point with respect to the system are denoted as x_1, y_1, z_1. The *second, moving* coordinate system is rigidly connected with the body ("frozen-in" in it), and the coordinates of the point with respect to this system are denoted by x, y, z. It is convenient to assume that the axes of the moving ("frozen-in" in the body) coordinate system are directed along the principal axes of inertia of the body for the point 0. Consider a unit vector e directed along the z_1-axis fixed in the space of the coordinate system. Let the coordinates of this vector be α, β, γ [with respect to the moving coordinate system (rotating with the body)]. It is clear that these coordinates are the functions of time t (Fig. 1).
Let us denote the centre of mass of the rigid body by P and let (x_0, y_0, z_0) be the coordinates of the body with respect to the moving coordinate system. Since this system is rigidly connected with the body, the numbers x_0, y_0, z_0 do not depend on time (the centre of mass does not change its position in the body with respect to the frozen-in coordinate axes). Let m be *the mass of the rigid body* and A, B, C the *principal moments of inertia of the body with respect to the point* 0. By virtue of the choice of the moving coordinate system, the inertia tensor of the rigid body may be written as the following quadratic diagonal matrix $\begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix}$. The tensor is taken in the diagonal form because the axes of the frozen-in coordinate system are directed along the principal axes of inertia of the body. Let ω be the vector of the instantaneous angular velocity of the body, whose coordinates with respect to

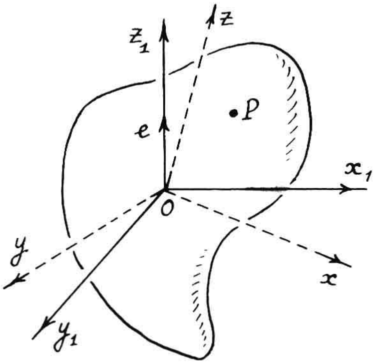

Figure 1

the moving coordinate system will be denoted by (p, q, r). It is clear that p, q, r are functions of time. Thus, the motion of a rigid body is described by six functions $p(t), q(t), r(t), \alpha(t), \beta(t), \gamma(t)$. The free fall acceleration will be denoted by g. It is obvious that if d is a gravitational vector applied in the centre of mass of the body, then d is directed vertically downward (along the z_1- axis) and has the form $d = -\,\text{mge}$.

From classical mechanics it is known that if K is the angular momentum of the body with respect to the point 0 and M is the moment of external forces (in our case, the moment of the gravitational force d) then, according to the theorem on the change of angular momentum, the following relation $\frac{dK}{dt} = M$, written with respect to the fixed coordinate system (x_1, y_1, z_1), holds. The quantities K and M can be explicitly calculated; however, the calculations are omitted. Projecting this vector equation onto the axes of the moving coordinate system (x, y, z), we can derive the following three scalar equations which are usually called *Euler dynamic equations*:

$$A\frac{dp}{dt} + (C - B)qr = mg(z_0\beta - y_0\gamma),$$
$$B\frac{dq}{dt} + (A - C)rp = mg(x_0\gamma - z_0\alpha),$$
$$C\frac{dr}{dt} + (B - A)pq = mg(y_0\alpha - x_0\beta).$$

These equations contain six unknown functions of time: $p, q, r, \alpha, \beta, \gamma$ and six constant quantities: A, B, C, x_0, y_0, z_0 characterizing body mass distribution with respect to the principal axes of inertia for the point 0. To make the system of equations closed, one should add three more equations which will imply that the velocity of the endpoint of the unit vector e in the fixed coordinate system (x_1, y_1, z_1) is equal to zero (the vector is fixed). Projecting the obtained vector equation onto the axes of the moving coordinate system, we can derive the three missing scalar