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COMPLEX ANALYSIS

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An Introduction to the Theory of Analytic Functions of One Complex Variable

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PREFACE

In American universities a course covering roughly the material in this book is ordinarily given in the first graduate year. The way of presenting the material differs widely: in some schools the emphasis is on teaching a certain indispensable amount of classical function theory; in others the course is used to confront the student, for the first time, with mathematical rigor. In Harvard, for instance, the course is also traditionally used to review advanced calculus with complete rigor in view.

The author's ambition has been to write a text which is at once concise and rigorous, teachable and readable. Such a goal cannot be reached, it can only be approximated, and the author is aware of many shortcomings. No attempt has been made to make the book self-contained. On the contrary, a basic knowledge of real numbers and calculus, including the definition and properties of definite integrals, is taken for granted. Questions concerning limits and continuity are reviewed in connection with their application to complex numbers, and an effort is made not to rely on results which in elementary teaching are commonly derived in a sloppy or insufficient manner. If the teacher decides that real numbers or the definition of integral should be included in his course, there are a dozen or so reliable texts that he can consult. The author has omitted these topics mainly to keep down the bulk of this volume.

Even apart from the starting point, the writer of a textbook has a difficult task in deciding what to include and what to omit. The present author has wished to give the reader a solid foundation in classical complex-function theory by emphasizing the general principles on which it rests. He believes that a person who is thoroughly inculcated with the fundamental methods will not experience any new difficulty if he wishes to go on to a specialized topic in function theory. Nevertheless, it is with great regret that the author has omitted, for instance, the theory of elliptic functions. One of the main reasons is that it is hardly possible to improve on the beautiful treatment in E. T. Copson's book ("An Introduction to the Theory of Functions of a Complex Variable," London, 1935).

In the opposite direction some topics have been included which are usually not felt to be part of elementary function theory. Such is the case with the theory of subharmonic function and Perron's method for solving the Dirichlet problem, which are certainly as elementary as they are important.

The book begins with an elementary discussion of complex numbers and ends up on a note of sophistication with the theory of multiple-valued analytic functions. In between, the progress is gradual. From his venerated teacher, Ernst Lindelöf, the author has learned to postpone the use of complex integration until the student is entirely familiar with the mapping properties of analytic functions. Geometric visualization is a source of knowledge as well as a didactic tool whose value cannot be disputed.

There are many other acknowledgments to be made. For instance, the appearance of Carathéodory's "Funktionentheorie" has, of course, not been without influence on the final form of this manuscript, which was half-finished at the time. Above all, the author has adopted without significant change E. Artin's splendid idea of basing homology theory on the notion of winding number. This approach makes it possible to present a complete and rigorous proof of Cauchy's theorem and all its immediate applications with a minimum amount of topology. Of course, to by-pass topology is no merit in itself, but in a book on function theory it is highly desirable to concentrate on that part of topology which is truly basic in the study of analytic functions. For the same reason no proof is included of the Jordan curve theorem, which, to the author's knowledge, is never needed in function theory.

The exercises in the book are to be taken as samples. The author has not had the inclination to relieve the teacher from making up more and better exercises; it is for him to decide what methods should be drilled, what alternative proofs the student should be asked to give, and what ingenuity he should be given the opportunity to show. It is to be hoped that no teacher will follow this book page by page, for nothing could be more deadening. A text is a guide for the teacher which saves him from the necessity of making up a detailed plan in advance, but the continuous contact with his class makes him the authority on desirable deviations and cuts.

One more point: the author makes abundant and unblushing use of the words clearly, obviously, evidently, etc. They are not used to blur the picture. On the contrary, they test the reader's understanding, for if he does not agree that the omitted reasoning is clear, obvious, and evident, he had better turn back a few pages and make a fresh start. There are also a few places, easily spotted, in which a voluntary gap serves the purpose of saving half a page of unconstructive and dull reasoning.

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CHAPTER I

COMPLEX NUMBERS

1. The Algebra of Complex Numbers

It is fundamental that real and complex numbers obey the same basic laws of arithmetic. We begin our study of complex function theory by stressing and implementing this analogy.

1.1. Arithmetic Operations. From elementary algebra the reader is acquainted with the *imaginary unit* i with the property $i^2 = -1$. If the imaginary unit is combined with two real numbers α, β by the processes of addition and multiplication, we obtain a *complex number* $\alpha + i\beta$. α and β are the *real* and *imaginary part* of the complex number. If $\alpha = 0$, the number is said to be *purely imaginary;* if $\beta = 0$, it is of course *real.* Zero is the only number which is at once real and purely imaginary. Two complex numbers are equal if and only if they have the same real part and the same imaginary part.

Addition and multiplication do not lead out from the system of complex numbers. Assuming that the ordinary rules of arithmetic apply to complex numbers we find indeed

$$(\alpha + i\beta) + (\gamma + i\delta) = (\alpha + \gamma) + i(\beta + \delta) \tag{1}$$

and

$$(\alpha + i\beta)(\gamma + i\delta) = (\alpha\gamma - \beta\delta) + i(\alpha\delta + \beta\gamma). \tag{2}$$

In the second identity we have made use of the relation $i^2 = -1$.

It is less obvious that division is also possible. We wish to show that $(\alpha + i\beta)/(\gamma + i\delta)$ is a complex number, provided that $\gamma + i\delta \neq 0$. If the quotient is denoted by $x + iy$, we must have

$$\alpha + i\beta = (\gamma + i\delta)(x + iy).$$

By (2) this condition can be written

$$\alpha + i\beta = (\gamma x - \delta y) + i(\delta x + \gamma y),$$

and we obtain the two equations

$$\begin{aligned} \alpha &= \gamma x - \delta y \\ \beta &= \delta x + \gamma y. \end{aligned}$$

This system of simultaneous linear equations has the unique solution

$$x = \frac{\alpha\gamma + \beta\delta}{\gamma^2 + \delta^2}$$
$$y = \frac{\beta\gamma - \alpha\delta}{\gamma^2 + \delta^2},$$

for we know that $\gamma^2 + \delta^2$ is not zero. We have thus the result

$$(3) \qquad \frac{\alpha + i\beta}{\gamma + i\delta} = \frac{\alpha\gamma + \beta\delta}{\gamma^2 + \delta^2} + i\frac{\beta\gamma - \alpha\delta}{\gamma^2 + \delta^2}.$$

Once the existence of the quotient has been proved, its value can be found in a simpler way. If numerator and denominator are multiplied with $\gamma - i\delta$, we find at once

$$\frac{\alpha + i\beta}{\gamma + i\delta} = \frac{(\alpha + i\beta)(\gamma - i\delta)}{(\gamma + i\delta)(\gamma - i\delta)} = \frac{(\alpha\gamma + \beta\delta) + i(\beta\gamma - \alpha\delta)}{\gamma^2 + \delta^2}.$$

As a special case the reciprocal of a complex number $\neq 0$ is given by

$$\frac{1}{\alpha + i\beta} = \frac{\alpha - i\beta}{\alpha^2 + \beta^2}.$$

We note that i^n has only four possible values: $1, i, -1, -i$. They correspond to values of n which divided by 4 leave the remainders 0, 1, 2, 3.

EXERCISES

1. Find the values of

$$(1 + 2i)^3, \qquad \frac{5}{-3 + 4i}, \qquad \left(\frac{2 + i}{3 - 2i}\right)^2, \qquad (1 + i)^n + (1 - i)^n.$$

2. If $z = x + iy$ (x and y real), find the real and imaginary parts of

$$z^4, \qquad \frac{1}{z}, \qquad \frac{z - 1}{z + 1}, \qquad \frac{1}{z^2}.$$

3. Show that

$$\left(\frac{-1 \pm i\sqrt{3}}{2}\right)^3 = 1 \quad \text{and} \quad \left(\frac{\pm 1 \pm i\sqrt{3}}{2}\right)^6 = 1$$

for all combinations of signs.

1.2. Square Roots. We shall now show that the square root of a complex number can be found explicitly. If the given number is $\alpha + i\beta$, we are looking for a number $x + iy$ such that

$$(x + iy)^2 = \alpha + i\beta.$$

This is equivalent with the system of equations

$$(4)\qquad \begin{aligned} x^2 - y^2 &= \alpha \\ 2xy &= \beta. \end{aligned}$$

From these equations we obtain

$$(x^2 + y^2)^2 = (x^2 - y^2)^2 + 4x^2y^2 = \alpha^2 + \beta^2.$$

Hence we must have

$$x^2 + y^2 = \sqrt{\alpha^2 + \beta^2},$$

where the square root is positive or zero. Together with the first equation (4) we find

$$(5)\qquad \begin{aligned} x^2 &= \tfrac{1}{2}(\alpha + \sqrt{\alpha^2 + \beta^2}) \\ y^2 &= \tfrac{1}{2}(-\alpha + \sqrt{\alpha^2 + \beta^2}). \end{aligned}$$

Observe that these quantities are positive or zero regardless of the sign of α.

The equations (5) yield, in general, two opposite values for x and two for y. But these values cannot be combined arbitrarily, for the second equation (4) is not a consequence of (5). We must therefore be careful to select x and y so that their product has the sign of β. This leads to the general solution

$$(6)\qquad \sqrt{\alpha + i\beta} = \pm\left(\sqrt{\frac{\alpha + \sqrt{\alpha^2 + \beta^2}}{2}} + (\operatorname{sign}\beta)i\sqrt{\frac{-\alpha + \sqrt{\alpha^2 + \beta^2}}{2}}\right),$$

where $\operatorname{sign}\beta = \pm 1$ according as $\beta > 0$ or $\beta < 0$. For $\beta = 0$ the value of $\operatorname{sign}\beta$ does not matter in our formula, but it is customary to set $\operatorname{sign} 0 = 0$. It is understood that all square roots of positive numbers are taken with the positive sign.

We have found that the square root of any complex number exists and has two opposite values. They coincide only if $\alpha + i\beta = 0$. They are real if $\beta = 0$, $\alpha \geqq 0$ and purely imaginary if $\beta = 0$, $\alpha \leqq 0$. In other words, except for zero only positive numbers have real square roots and only negative numbers have purely imaginary square roots.

Since both square roots are in general complex, it is not possible to distinguish between the positive and negative square root of a complex number. We could of course distinguish between the upper and lower sign in (6), but this distinction is artificial and should be avoided. The correct way is to treat both square roots in a symmetric manner.

EXERCISES

1. Compute

$$\sqrt{i}, \qquad \sqrt{-i}, \qquad \sqrt{1+i}, \qquad \sqrt{\frac{1-i\sqrt{3}}{2}}.$$

2. Find the four values of $\sqrt[4]{-1}$.

3. Compute $\sqrt[4]{i}$ and $\sqrt[4]{-i}$.
4. Solve the quadratic equation

$$z^2 + (\alpha + i\beta)z + \gamma + i\delta = 0.$$

1.3. Justification. So far our approach to complex numbers has been completely uncritical. We have not questioned the existence of a number system in which the equation $x^2 + 1 = 0$ has a solution while all the rules of arithmetic remain in force.

We begin by recalling the characteristic properties of the real-number system which we denote by $\mathfrak{R}$. In the first place, $\mathfrak{R}$ is a *field.* This means that addition and multiplication are defined, satisfying the *associative, commutative*, and *distributive laws.* The numbers 0 and 1 are neutral elements under addition and multiplication, respectively: $\alpha + 0 = \alpha$, $\alpha \cdot 1 = \alpha$ for all α. Moreover, the equation of subtraction $\beta + x = \alpha$ has always a solution, and the equation of division $\beta x = \alpha$ has a solution whenever $\beta \neq 0$.†

One shows by elementary reasoning that the neutral elements and the results of subtraction and division are unique. Also, every field is an *integral domain:* $\alpha\beta = 0$ if and only if $\alpha = 0$ or $\beta = 0$.

These properties are common to all fields. In addition, the field $\mathfrak{R}$ has an *order relation* $\alpha < \beta$ (or $\beta > \alpha$). It is most easily defined in terms of the set $\mathfrak{R}^+$ of *positive* real numbers: $\alpha < \beta$ if and only if $\beta - \alpha \in \mathfrak{R}^+$. The set $\mathfrak{R}^+$ is characterized by the following properties: (1) 0 is not a positive number; (2) if $\alpha \neq 0$ either α or $-\alpha$ is positive; (3) the sum and the product of two positive numbers are positive. From these conditions one derives all the usual rules for manipulation of inequalities. In particular one finds that every square α^2 is either positive or zero; therefore $1 = 1^2$ is a positive number.

By virtue of the order relation the sums $1, 1 + 1, 1 + 1 + 1, \ldots$ are all different. Hence $\mathfrak{R}$ contains the natural numbers, and since it is a field it must contain the subfield formed by all rational numbers.

Finally, $\mathfrak{R}$ satisfies the following *completeness condition:* every increasing and bounded sequence of real numbers has a limit. Let $\alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_n < \cdots$, and assume the existence of a real number B such that $\alpha_n < B$ for all n. Then the completeness condition requires the existence of a number $A = \lim_{n\to\infty} \alpha_n$ with the following property: given any $\varepsilon > 0$ there exists a natural number n_0 such that $A - \varepsilon < \alpha_n < A$ for all $n > n_0$.

Our discussion of the real-number system is incomplete inasmuch as we have not proved the existence and uniqueness (up to isomorphisms) of

† We assume that the reader has a working knowledge of elementary algebra. Although the above characterization of a field is complete, it obviously does not convey much to a student who is not already at least vaguely familiar with the concept.

a system $\mathfrak{R}$ with the postulated properties.† The student who is not thoroughly familiar with one of the constructive processes by which real numbers can be introduced should not fail to fill this gap by consulting any textbook in which a full axiomatic treatment of real numbers is given.

The equation $x^2 + 1 = 0$ has no solution in $\mathfrak{R}$, for $\alpha^2 + 1$ is always positive. Suppose now that a field $\mathfrak{F}$ can be found which contains $\mathfrak{R}$ as a subfield, and in which the equation $x^2 + 1 = 0$ can be solved. Denote a solution by i. Then $x^2 + 1 = (x + i)(x - i)$, and the equation $x^2 + 1 = 0$ has exactly two roots in $\mathfrak{F}$, i and $-i$. Let $\mathfrak{C}$ be the subset of $\mathfrak{F}$ consisting of all elements which can be expressed in the form $\alpha + i\beta$ with real α and β. This representation is unique, for $\alpha + i\beta = \alpha' + i\beta'$ implies $\alpha - \alpha' = -i(\beta - \beta')$; hence $(\alpha - \alpha')^2 = -(\beta - \beta')^2$, and this is possible only if $\alpha = \alpha'$, $\beta = \beta'$.

The subset $\mathfrak{C}$ is a subfield of $\mathfrak{F}$. In fact, except for trivial verifications which the reader is asked to carry out, this is exactly what was shown in Sec. 1.1. What is more, the structure of $\mathfrak{C}$ is independent of $\mathfrak{F}$. For if $\mathfrak{F}'$ is another field containing $\mathfrak{R}$ and a root i' of the equation $x^2 + 1 = 0$, the corresponding subset $\mathfrak{C}'$ is formed by all elements $\alpha + i'\beta$. There is a one-to-one correspondence between $\mathfrak{C}$ and $\mathfrak{C}'$ which associates $\alpha + i\beta$ and $\alpha + i'\beta$, and this correspondence is evidently a field isomorphism. It is thus demonstrated that $\mathfrak{C}$ and $\mathfrak{C}'$ are isomorphic.

We now define the field of *complex numbers* to be the subfield $\mathfrak{C}$ of an arbitrarily given $\mathfrak{F}$. We have just seen that the choice of $\mathfrak{F}$ makes no difference, but we have not yet shown that there exists a field $\mathfrak{F}$ with the required properties. In order to give our definition a meaning it remains to exhibit a field $\mathfrak{F}$ which contains $\mathfrak{R}$ (or a subfield isomorphic with $\mathfrak{R}$) and in which the equation $x^2 + 1 = 0$ has a root.

There are many ways in which such a field can be constructed. The simplest and most direct method is the following: Consider all expressions of the form $\alpha + i\beta$ where α, β are real numbers while the signs + and i are pure symbols (+ does *not* indicate addition, and i is *not* an element of a field). These expressions are elements of a field $\mathfrak{F}$ in which addition and multiplication are defined by (1) and (2) (observe the two different meanings of the sign +). The elements of the particular form $\alpha + i0$ are seen to constitute a subfield isomorphic to $\mathfrak{R}$, and the element $0 + i1$ satisfies the equation $x^2 + 1 = 0$; we obtain in fact $(0 + i1)^2 = -(1 + i0)$. The field $\mathfrak{F}$ has thus the required properties; moreover, it is identical with the corresponding subfield $\mathfrak{C}$, for we can write

$$\alpha + i\beta = (\alpha + i0) + \beta(0 + i1).$$

† An *isomorphism* between two fields is a one-to-one correspondence which preserves sums and products. The word is used quite generally to indicate a correspondence which is one to one and preserves all relations that are considered important in a given connection.

The existence of the complex-number field is now proved, and we can go back to the simpler notation $\alpha + i\beta$ where the $+$ indicates addition in $\mathfrak{C}$ and i is a root of the equation $x^2 + 1 = 0$.

EXERCISES (For students with a background in algebra)

1. Show that the system of all matrices of the special form

$$\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix},$$

combined by matrix addition and matrix multiplication, is isomorphic to the field of complex numbers.

2. Show that the complex-number system can be thought of as the field of all polynomials with real coefficients modulo the irreducible polynomial $x^2 + 1$.

1.4. Conjugation, Absolute Value. A complex number can be denoted either by a single letter a, representing an element of the field $\mathfrak{C}$, or in the form $\alpha + i\beta$ with real α and β. Other standard notations are $z = x + iy$, $\zeta = \xi + i\eta$, $w = u + iv$, and when used in this connection it is tacitly understood that x, y, ξ, η, u, v are real numbers. The real and imaginary part of a complex number a will also be denoted by Re a, Im a.

In deriving the rules for complex addition and multiplication we used only the fact that $i^2 = -1$. Since $-i$ has the same property, all rules must remain valid if i is everywhere replaced by $-i$. Direct verification shows that this is indeed so. The transformation which replaces $\alpha + i\beta$ by $\alpha - i\beta$ is called *complex conjugation*, and $\alpha - i\beta$ is the *conjugate* of $\alpha + i\beta$. The conjugate of a is denoted by $\bar{a}$. A number is real if and only if it is equal to its conjugate. The conjugation is an *involutory* transformation: this means that $\bar{\bar{a}} = a$.

The formulas

$$\text{Re } a = \frac{a + \bar{a}}{2}, \qquad \text{Im } a = \frac{a - \bar{a}}{2i}$$

express the real and imaginary part in terms of the complex number and its conjugate. By systematic use of the notations a and $\bar{a}$ it is hence possible to dispense with the use of separate letters for the real and imaginary part. It is more convenient, though, to make free use of both notations.

The fundamental property of conjugation is the one already referred to, namely, that

$$\overline{a + b} = \bar{a} + \bar{b}$$
$$\overline{ab} = \bar{a} \cdot \bar{b}.$$

The corresponding property for quotients is a consequence: if $ax = b$, then $\bar{a}\bar{x} = \bar{b}$, and hence $\overline{(b/a)} = \bar{b}/\bar{a}$. More generally, let $R(a,b,c, \ldots)$ stand for any rational operation applied to the complex numbers a, b, c, $\ldots$. Then

$$\overline{R(a,b,c, \ldots)} = R(\bar{a},\bar{b},\bar{c}, \ldots).$$

As an application, consider the equation

$$c_0z^n + c_1z^{n-1} + \cdots + c_{n-1}z + c_n = 0.$$

If ζ is a root of this equation, then $\bar{\zeta}$ is a root of the equation

$$\bar{c}_0z^n + \bar{c}_1z^{n-1} + \cdots + \bar{c}_{n-1}z + \bar{c}_n = 0.$$

In particular, if the coefficients are *real*, ζ and $\bar{\zeta}$ are roots of the same equation, and we have the familiar theorem that the nonreal roots of an equation with real coefficients occur in pairs of conjugate roots.

The product $a\bar{a} = \alpha^2 + \beta^2$ is always positive or zero. Its nonnegative square root is called the *modulus* or *absolute value* of the complex number a; it is denoted by $|a|$. The terminology and notation are justified by the fact that the modulus of a real number coincides with its numerical value taken with the positive sign.

We repeat the definition

$$a\bar{a} = |a|^2,$$

where $|a| \geqq 0$, and observe that $|\bar{a}| = |a|$. For the absolute value of a product we obtain

$$|ab|^2 = ab \cdot \overline{ab} = ab\bar{a}\bar{b} = a\bar{a}b\bar{b} = |a|^2|b|^2,$$

and hence

$$|ab| = |a| \cdot |b|$$

since both are $\geqq 0$. In words:

The absolute value of a product is equal to the product of the absolute values of the factors.

It is clear that this property extends to arbitrary finite products:

$$|a_1a_2 \cdots a_n| = |a_1| \cdot |a_2| \cdots |a_n|.$$

The quotient a/b, $b \neq 0$, satisfies $b(a/b) = a$, and hence we have also $|b| \cdot |a/b| = |a|$, or

$$\left|\frac{a}{b}\right| = \frac{|a|}{|b|}.$$

The formula for the absolute value of a sum is not as simple. We find

$$|a + b|^2 = (a + b)(\bar{a} + \bar{b}) = a\bar{a} + (a\bar{b} + b\bar{a}) + b\bar{b}$$

or

(7) $$|a + b|^2 = |a|^2 + |b|^2 + 2\ \mathrm{Re}\ a\bar{b}.$$

The corresponding formula for the difference is

(7′) $$|a - b|^2 = |a|^2 + |b|^2 - 2\ \mathrm{Re}\ a\bar{b},$$

and by addition we obtain the identity

(8) $$|a + b|^2 + |a - b|^2 = 2(|a|^2 + |b|^2).$$

EXERCISES

1. Verify by calculation that the values of

$$\frac{z}{z^2+1}$$

for $z = x + iy$ and $z = x - iy$ are conjugate.

2. Find the absolute values of

$$-2i(3+i)(2+4i)(1+i) \quad \text{and} \quad \frac{(3+4i)(-1+2i)}{(-1-i)(3-i)}.$$

3. Prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right| = 1$$

if either $|a| = 1$ or $|b| = 1$. What exception must be made if $|a| = |b| = 1$?

4. Prove Lagrange's identity in the complex form

$$\left|\sum_{i=1}^{n} a_i b_i\right|^2 = \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2 - \sum_{1 \leqq i < j \leqq n} |a_i \bar{b}_j - a_j \bar{b}_i|^2.$$

1.5. Inequalities. We shall now prove some important inequalities which will be of constant use. It is perhaps well to point out that there is no order relation in the complex-number system, and hence all inequalities must be between real numbers.

From the definition of the absolute value we deduce the inequalities

$$(9) \qquad \begin{aligned} -|a| &\leqq \operatorname{Re} a \leqq |a| \\ -|a| &\leqq \operatorname{Im} a \leqq |a|. \end{aligned}$$

The equality $\operatorname{Re} a = |a|$ holds if and only if a is real and $\geqq 0$.

If (9) is applied to (7), we obtain

$$|a+b|^2 \leqq (|a| + |b|)^2$$

and hence

$$(10) \qquad |a+b| \leqq |a| + |b|.$$

This is called the *triangle inequality* for reasons which will emerge later. By induction it can be extended to arbitrary sums:

$$(11) \qquad |a_1 + a_2 + \cdots + a_n| \leqq |a_1| + |a_2| + \cdots + |a_n|.$$

The absolute value of a sum is at most equal to the sum of the absolute values of the terms.

The reader is well aware of the importance of the estimate (11) in the real case, and we shall find it no less important in the theory of complex numbers.

Let us determine all cases of equality in (11). In (10) the sign of equality holds if and only if $a\bar{b} \geqq 0$ (it is convenient to let $c > 0$ indicate that c is *real* and *positive*). If $b \neq 0$ this condition can be written in the form

$|b|^2(a/b) \geqq 0$, and it is hence equivalent with $a/b \geqq 0$. In the general case we proceed as follows: Suppose that equality holds in (11); then

$$|a_1| + |a_2| + \cdots + |a_n| = |(a_1 + a_2) + a_3 + \cdots + a_n| \\ \leqq |a_1 + a_2| + |a_3| + \cdots + |a_n| \leqq |a_1| + |a_2| + \cdots + |a_n|.$$

Hence $|a_1 + a_2| = |a_1| + |a_2|$, and if $a_2 \neq 0$ we conclude that $a_1/a_2 \geqq 0$. But the numbering of the terms is arbitrary; thus the ratio of any two nonzero terms must be positive. Suppose conversely that this condition is fulfilled. Assuming that $a_1 \neq 0$ we obtain

$$|a_1 + a_2 + \cdots + a_n| = |a_1| \cdot \left|1 + \frac{a_2}{a_1} + \cdots + \frac{a_n}{a_1}\right| \\ = |a_1| \left(1 + \frac{a_2}{a_1} + \cdots + \frac{a_n}{a_1}\right) = |a_1| \left(1 + \frac{|a_2|}{|a_1|} + \cdots + \frac{|a_n|}{|a_1|}\right) \\ = |a_1| + |a_2| + \cdots + |a_n|.$$

To sum up: *the sign of equality holds in* (11) *if and only if the ratio of any two nonzero terms is positive.*

By (10) we have also

$$|a| = |(a - b) + b| \leqq |a - b| + |b|$$

or

$$|a| - |b| \leqq |a - b|.$$

For the same reason $|b| - |a| \leqq |a - b|$, and these inequalities can be combined to

(12) $$|a - b| \geqq ||a| - |b||.$$

Of course the same estimate can be applied to $|a + b|$.

A special case of (10) is the inequality

(13) $$|\alpha + i\beta| \leqq |\alpha| + |\beta|$$

which expresses that the absolute value of a complex number is at most equal to the sum of the absolute values of the real and imaginary part.

Many other inequalities whose proof is less immediate are also of frequent use. Foremost is *Cauchy's inequality* which states that

$$|a_1b_1 + \cdots + a_nb_n|^2 \leqq (|a_1|^2 + \cdots + |a_n|^2)(|b_1|^2 + \cdots + |b_n|^2)$$

or, in shorter notation,

(14) $$\left|\sum_{i=1}^{n} a_ib_i\right|^2 \leqq \sum_{i=1}^{n} |a_i|^2 \sum_{i=1}^{n} |b_i|^2.\dagger$$

To prove it, let λ denote an arbitrary complex number. We obtain by (7)

† i is a convenient summation index and, used as a subscript, cannot be confused with the imaginary unit. It seems pointless to ban its use.

$$(15) \qquad \sum_{i=1}^{n} |a_i - \lambda \bar{b}_i|^2 = \sum_{i=1}^{n} |a_i|^2 + |\lambda|^2 \sum_{i=1}^{n} |b_i|^2 - 2 \operatorname{Re} \bar{\lambda} \sum_{i=1}^{n} a_i b_i.$$

This expression is $\geqq 0$ for all λ. We can choose

$$\lambda = \frac{\sum_1^n a_i b_i}{\sum_1^n |b_i|^2},$$

for if the denominator should vanish there is nothing to prove. This choice is not arbitrary, but it is dictated by the desire to make the expression (15) as small as possible. Substituting in (15) we find, after simplifications,

$$\sum_1^n |a_i|^2 - \frac{\left|\sum_1^n a_i b_i\right|^2}{\sum_1^n |b_i|^2} \geqq 0$$

which is equivalent with (14).

From (15) we conclude further that the sign of equality holds in (14) if and only if the a_i are proportional to the $\bar{b}_i$.

Cauchy's inequality can also be proved by means of Lagrange's identity (Sec. 1.4, Ex. 4).

EXERCISES

1. Prove that

$$\left|\frac{a-b}{1-\bar{a}b}\right| < 1$$

if $|a| < 1$ and $|b| < 1$.

2. Prove Cauchy's inequality by induction.

3. If $|a_i| < 1$, $\lambda_i \geqq 0$ for $i = 1, \ldots, n$ and $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$, show that

$$|\lambda_1 a_1 + \lambda_2 a_2 + \cdots + \lambda_n a_n| < 1.$$

4. Determine the smallest value of $|(z-a)(z-b)|$ when a, b are given.

2. The Geometric Representation of Complex Numbers

With respect to a given rectangular coordinate system in a plane, the complex number $a = \alpha + i\beta$ can be represented by the point with coordinates (α,β). This representation is constantly used, and we shall often speak of the *point a* as a synonym of the *number a*. The first coordinate axis (x-axis) takes the name of *real axis*, and the second coordinate axis (y-axis) is called the *imaginary axis*. The plane itself is referred to as the *complex plane*.