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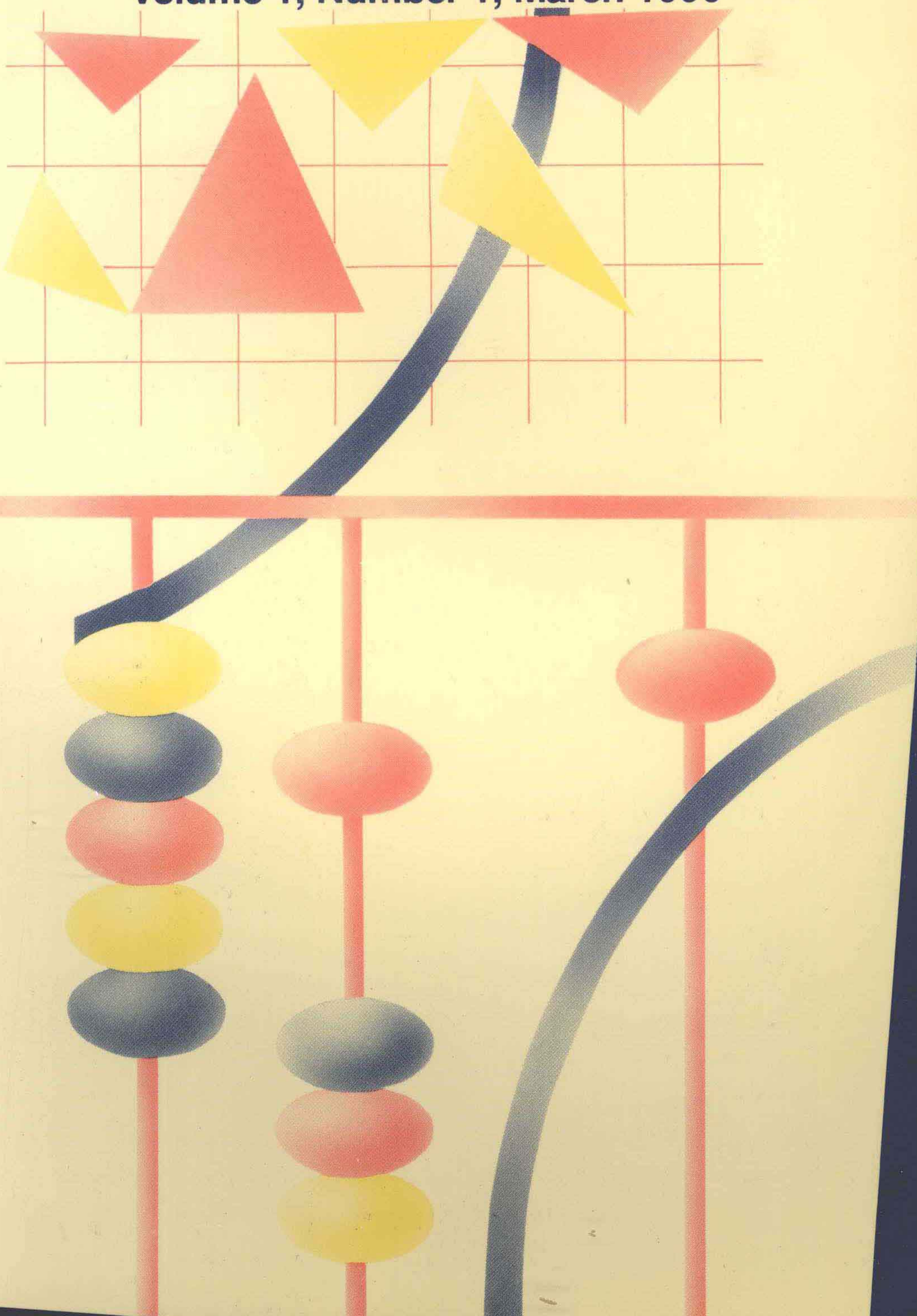

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## CONTENTS



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# HARMONIC MAPS BETWEEN SPHERES AND ELLIPSOIDS*

Dedicated to René Thom

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## 1. Introduction

In this paper we establish several existence theorems for harmonic maps between Euclidean ellipsoids of the type

$$Q^n(a,b) = \{(x,y) \in \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} : |x|^2/a^2 + |y|^2/b^2 = 1, \text{ with } p+q+1 = n, a, b > 0\}.$$

For instance, take $p = 1$, $q = n - 2$: we have

**Corollary 5.8.** *If* $n \geq 3$, *assume* $d^2/c^2 > (n-3)^2/4(n-2)$. *Then any map* $\varphi_0 : Q^n(c,d) \to Q^n(c,d)$ *can be deformed to a harmonic map.*

Note that for $n \leq 7$ we can take $d = c$, thereby recovering Smith's theorem [27, 28]. The values $n = 1, \ldots, 7, 9$ are the only dimensions where the conclusion of Corollary 5.8 is known for Euclidean spheres.

With other methods,

**Corollary 6.8.** *A map* $\varphi_0 : Q^n(a,b) \to S^n$ *of given degree* $k \in \mathbb{Z}$ *can be deformed to a harmonic map, provided that the dilatation* $b/a$ *is sufficiently small* ($b/a$ *may depend upon both* $n$ *and* $k$).

**Corollary 7.8.** *If* $n \geq 3$, *assume* $d^2/c^2 > (n-3)^2/4(n-2)$. *Then any map* $\varphi_0 : S^n \to Q^n(c,d)$ *can be deformed to a harmonic map.*

More generally, we will show that the join of any two harmonic homogeneous polynomial maps of spheres can always be deformed to a harmonic map provided that suitable ellipsoidal metrics are introduced (see Theorems 5.1, 6.1, 7.1 below).

In the context of the Hopf construction, we obtain

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**Theorem 8.3.** *For any* $k,\ l \in \mathbb{Z}$ *there is an equivariant harmonic map* $\varphi_{k,l} = \varphi : Q^3(a,b) \to S^2$ *with Hopf invariant* $k \cdot l$ *iff* $Q^3(a,b)$ *has dilatation*

$$b/a = |l/k| \tag{8.4}$$

*Furthermore,* $\varphi_{k,l}$ *is a harmonic morphism.*

We work in the equivariant context of the Thesis of Smith [27], as re-interpreted by Ding [7].

In the terminology and notations of Sec. 2, we look for a suitable join of two eigenmaps between spheres. That amounts to reducing the energy integral to a 1-dimensional integral $J$. The required solutions are critical points of $J$, with specified limits.

Our Euler-Lagrange equation has the form of an exotic spherical pendulum, with damping $D$ and variable gravity $G$:

$$\ddot{\alpha}(s) + D(s)\dot{\alpha}(s) + G(s, \alpha, \dot{\alpha}) \sin(\alpha(s)) \cos(\alpha(s)) = 0 .$$

If the range is a sphere, then $G$ is a function of $s$ alone. By way of contrast, for ellipsoidal ranges

$$G(s, \alpha, \dot{\alpha}) = \frac{d^2 - c^2}{k^2(\alpha)} \dot{\alpha}^2 - \frac{h^2(s)}{k^2(\alpha)} \left( \frac{c^2 \lambda_u}{a^2 \sin^2 s} - \frac{d^2 \lambda_v}{b^2 \cos^2 s} \right); \text{ see (2.10)}.$$

Several different methods are needed in appropriate contexts to produce solutions which provide harmonic joins. Amongst them:

1) The direct method, based on weak lower semicontinuity of $J$—together with a special argument involving second variations of $J$ (see Secs. 3, 5, 7).
2) Morse Theory on closed convex subsets in Hilbert spaces; and in particular the Mountain Pass Lemma (see Secs. 4, 6, 7).
3) Qualitative analysis of trajectories, using subsolutions, comparison theorems and *a priori* estimates (see Secs. 4, 8).

**Acknowledgements**

The authors acknowledge gratefully their indebtedness to Prof. W-Y Ding, whose fine paper [7] provided the impetus for the present one.

They also greatly enjoyed the hospitality of the Institut des Hautes Etudes Scientifiques, where this paper was written.

## 2. Basic Constructions and Formulas

2.1. We shall be concerned with ellipsoids with axes of at most two different lengths, of the form:

$$Q^{p+q+1}(a,b) = \{(x,y) \in \mathbb{R}^{p+1} \times \mathbb{R}^{q+1} : |x|^2/a^2 + |y|^2/b^2 = 1\}$$

where $a$, $b > 0$ and vertical bars designate the indicated Euclidean norm. Sometimes we will write $Q^n(a,b)$ for $Q^{p+q+1}(a,b)$, slurring over the important dependence on the decomposition $p + q + 1 = n$.

We call $b/a$ the *dilatation of* $Q^{p+q+1}(a,b)$. The ellipsoid $Q^{p+q+1}(a,b)$ is parametrized by

$$z = a \sin s \cdot x + b \cos s \cdot y$$

for $x \in S^p$, $y \in S^q$ and $0 \leq s \leq \pi/2$.

The induced Riemannian metric on $Q^{p+q+1}(a,b)$ is:

$$g = (a^2 \sin^2 s) g_p + (b^2 \cos^2 s) g_q + h^2(s)\, ds^2 \tag{2.2}$$

where $g_p$, $g_q$ denote the Euclidean metrics of $S^p$, $S^q$ and

$$h(s) = [b^2 \sin^2 s + a^2 \cos^2 s]^{1/2}.$$

Its volume density is

$$v_g = a^p b^q \sin^p s \cos^q s\, h(s) \cdot v_{S^p} \cdot v_{S^q}$$

where $v_{S^p}$, $v_{S^q}$ are volume densities of the indicated Euclidean spheres.

Also we will write

$$v = a^p b^q \sin^p s \cos^q s\, h(s).$$

We shall refer to $(Q^{p+q+1}(a,b), g)$ as an *ellipsoidal join* of $S^p$, $S^q$. We observe that $\mathrm{SO}(p+1) \times \mathrm{SO}(q+1)$ is a group of isometries of $Q^{p+q+1}(a,b)$; and that $Q^{p+q+1}(a,b)$ and $Q^{q+p+1}(b,a)$ are isometric.

2.3. A map $\varphi : Q^m(a,b) \to Q^n(c,d)$ between ellipsoids is *harmonic* if it is an extremal of the energy functional

$$E(\varphi) = 1/2 \int_{Q^m(a,b)} |d\varphi|^2 * v_g$$

where, at each point $x$, $|d\varphi(x)|$ is the Hilbert-Schmidt norm of the linear transformation $d\varphi(x)$; and $* v_g$ is the volume form of $Q^m(a,b)$.

The Euler-Lagrange equation of $E$ can be expressed as follows: firstly, denote by $\Phi = i \circ \varphi$ the composition of $\varphi$ with the canonical embedding $i$ of $Q^n(c,d)$ into $\mathbb{R}^{n+1}$. Then *$\varphi$ is harmonic iff*

$$\Delta\Phi = (|P^{-1/2}\, d\Phi|^2 / |P^{-1}\Phi|^2) P^{-1}\Phi \tag{2.4}$$

where

$$P = \begin{pmatrix} c^2 & & & & & & 0 \\ & \ddots & & & & & \\ & & c^2 & & & & \\ & & & d^2 & & & \\ & & & & \ddots & & \\ 0 & & & & & & d^2 \end{pmatrix}$$

following the ellipsoidal join structure of $Q^n(c, d)$; and $\Delta$ denotes the Laplacian of $(Q^m(a, b), g)$. At each point, the right-hand member of (2.4) is the orthogonal projection of $\Delta\Phi$ onto the normal to $Q^n(c, d)$.

If we write $\Phi = (\Phi_1, \Phi_2)$, the components being the projections on the factors of the ambient space $\mathbb{R}^{n+1}$ following the join construction of $Q^n(c, d)$, then (2.4) becomes the system

$$\begin{cases} \Delta\Phi_1 = (\Lambda/c^2)\Phi_1 \\ \Delta\Phi_2 = (\Lambda/d^2)\Phi_2 \end{cases}$$

with

$$\Lambda = \frac{|d\Phi_1|^2/c^2 + |d\Phi_2|^2/d^2}{|\Phi_1|^2/c^4 + |\Phi_2|^2/d^4}.$$

Such harmonic maps are *real analytic* [12].

2.5. An *eigenmap* $u : S^p \to S^r$ is one whose components (as a map into $\mathbb{R}^{r+1}$) are harmonic $k$-homogeneous polynomials; its associated eigenvalue is $\lambda_u = k(k + p - 1)$.

It is easy to calculate that $|du(x)|^2 = \lambda_u$ for all $x \in S^p$; and that $u$ is a harmonic map. We refer to [8] for further details and examples.

Given two eigenmaps $u : S^p \to S^r$ and $v : S^q \to S^s$, we consider their join $u * v$, a map between ellipsoids

$$\varphi = u * v : Q^{p+q+1}(a, b) \to Q^{r+s+1}(c, d);$$

indeed, for any continuous function $\alpha : [0, \pi/2] \to [0, \pi/2]$ with $\alpha(0) = 0$, $\alpha(\pi/2) = \pi/2$, we can define

$$\varphi(z) = c \sin\alpha(s) \cdot u(x) + d \cos\alpha(s) \cdot v(y)$$

for $x \in S^p$, $y \in S^q$, and $0 \le s \le \pi/2$. We assume $p, q \ge 1$.

For such equivariant maps the energy functional $E$ reduces (up to a constant factor) to

$$J(\alpha) = 1/2 \int_0^{\pi/2} \left[ \frac{k^2(\alpha)}{h^2} \dot{\alpha}^2 + \frac{c^2 \sin^2\alpha}{a^2 \sin^2} \lambda_u + \frac{d^2 \cos^2\alpha}{b^2 \cos^2} \lambda_v \right] v \, ds \tag{2.6}$$

where $h = [b^2 \sin^2 + a^2 \cos^2]^{1/2}$, $k(\alpha) = [d^2 \sin^2\alpha + c^2 \cos^2\alpha]^{1/2}$.

Here and henceforth we have abbreviated $\sin s$ by sin, $\alpha(s)$ by $\alpha$, etc.

2.7. We define the Hilbert space

$$X = \left\{ \alpha \in L_1^2([0, \pi/2], \mathbb{R}) : \|\alpha\|^2 = \int_0^{\pi/2} [\dot\alpha^2 + \alpha^2] v \, ds < \infty \right\}.$$

For $p, q > 1$ the functional $J$ is defined and smooth on $X$. That is a consequence of the fact that $h$ is bounded above and below by positive constants and of the following Sobolev inequality (for the Riemannian manifolds $([0, \pi/2], \sin^{p-2} \cos^q)$, $([0, \pi/2], \sin^p \cos^{q-2})$):

**Lemma 2.8.** *There is a constant such that for all $\alpha \in X$*

$$\left. \begin{aligned} &\int_0^{\pi/2} \alpha^2 \sin^{p-2} \cos^q ds \\ &\int_0^{\pi/2} \alpha^2 \sin^p \cos^{q-2} ds \end{aligned} \right\} \leq \text{const.} \int_0^{\pi/2} [\dot\alpha^2 + \alpha^2] \sin^p \cos^q ds.$$

If either $p = 1$ or $q = 1$, we extend the definition of $J$, allowing it to assume the value $+\infty$.

2.9. The directional derivative of $J$ at $\alpha$ in the direction $\xi \in X$ is

$$dJ(\alpha)\xi = \int_0^{\pi/2} \left\{ \frac{k^2(\alpha)}{h^2} \dot\alpha \dot\xi + \left[ \frac{k(\alpha)k'(\alpha)}{h^2} \dot\alpha^2 + \left( \frac{c^2 \lambda_u}{a^2 \sin^2} - \frac{d^2 \lambda_v}{b^2 \cos^2} \right) \sin\alpha \cos\alpha \right] \xi \right\} v \, ds.$$

2.10. The Euler-Lagrange equation associated with the reduced energy $J$ is

$$\ddot\alpha + \left( p \frac{\cos}{\sin} - q \frac{\sin}{\cos} - \frac{\dot h}{h} \right) \dot\alpha + \frac{k'(\alpha)}{k(\alpha)} \dot\alpha^2 = \frac{h^2}{k^2(\alpha)} \left( \frac{c^2 \lambda_u}{a^2 \sin^2} - \frac{d^2 \lambda_v}{b^2 \cos^2} \right) \sin\alpha \cos\alpha.$$

Here $\dot h = dh/ds$, and $k' = dk/d\alpha$.

This has the form

$$\ddot\alpha + D(s)\dot\alpha + G(s, \alpha, \dot\alpha) \sin\alpha \cos\alpha = 0,$$

which is a sort of spherical pendulum with damping $D$ and variable gravity $G$.

2.10′. An equivalent form of (2.10) is

$$\frac{d}{ds} \left( \frac{k(\alpha)}{h^2} \dot\alpha v \right) = \left( \frac{c^2 \lambda_u}{a^2 \sin^2} - \frac{d^2 \lambda_v}{b^2 \cos^2} \right) \left( \frac{\sin\alpha \cos\alpha}{k(\alpha)} \right) v$$

2.11. For any critical point $\alpha \in X$ and variation $\xi$, the corresponding second variation is

$$\nabla^2 J(\alpha)(\xi,\xi) = \int_0^{\pi/2} \left\{ \left[ \frac{(k'(\alpha))^2 + k(\alpha)k''(\alpha)}{h^2}\dot{\alpha}^2 + \left( \frac{c^2\lambda_u}{a^2 \sin^2} - \frac{d^2\lambda_v}{b^2\cos^2} \right)\cos 2\alpha \right]\xi^2 \right.$$
$$\left. + \frac{k(\alpha)}{h^2}(4k'(\alpha)\dot{\alpha}\xi\dot{\xi} + k(\alpha)\dot{\xi}^2) \right\} v\, ds.$$

## 3. Existence Methods

3.1. The following are standard properties of integrals $I : L_1^2(M,N) \to \mathbb{R}$ of the form

$$I(\varphi) = \int_M [A(x,\varphi(x))|d\varphi(x)|^2 + B(x,\varphi(x))] * v_M$$

where $* v_M$ is the volume form of $M$; $M, N$ are compact, $A, B : M \times N \to \mathbb{R}$ are smooth functions, and $A > 0$.

3.2. *For $p$, $q \geq 1$ the functional $J : X \to \mathbb{R}$ is weakly lower semicontinuous.* I.e., for any sequence $\alpha_0$, $(\alpha_i)_{i\geq 1}$ in $X$ such that the inner products $\langle \alpha_i, \beta \rangle \underset{i}{\to} \langle \alpha_0, \beta \rangle$ for all $\beta \in X$, then

$$J(\alpha_0) \leq \liminf_i J(\alpha_i).$$

Consequently, $J$ assumes its minimum on weakly compact subsets of $X$; these are just those subsets which are weakly closed and bounded in norm. In particular, *$J$ assumes its minimum on the closed convex set*

$$X_0 = \{\alpha \in X : 0 \leq \alpha(s) \leq \pi/2 \text{ for all } s \in [0, \pi/2]\}.$$

Let $\underline{\alpha} \in X_0$ realize that minimum

$$J(\underline{\alpha}) = \inf\{J(\alpha) : \alpha \in X_0\}. \tag{3.3}$$

**Proposition 3.4.** *Assume that $p$, $q > 1$. Then $J : X_0 \to \mathbb{R}$ satisfies the compactness condition of Palais-Smale: If $(\alpha_i)_{i\geq 1} \subset X_0$ is a sequence on which $J$ is bounded and for which $dJ(\alpha_i) \to 0$ as $i \to +\infty$, then a subsequence of $(\alpha_i)$ converges in $X_0$.*

**Proof.** First we assume $c = 1 = d$ and follow [7]: we have noted in (2.7) that $J$ is smooth for $p$, $q > 1$. Now we observe that $(\|\alpha_i\|)_{i\geq 1}$ is bounded, because $(J(\alpha_i))_{i\geq 1}$ is and $\alpha_i \in X_0$. Thus a subsequence, still denoted by $(\alpha_i)$, converges weakly to some $\alpha_0 \in X_0$.

The weak convergence insures that

$$\int_0^{\pi/2} (\alpha_i - \alpha_j)^2 v\, ds \to 0 \qquad \text{as } i, j \to \infty.$$

From (2.9) (with $k \equiv 1$) we see that

$$dJ(\alpha_i)(\alpha_i - \alpha_j) = \int_0^{\pi/2} \left[\frac{1}{h^2} \dot{\alpha}_i(\dot{\alpha}_i - \dot{\alpha}_j) + L \sin \alpha_i \cos \alpha_i (\alpha_i - \alpha_j)\right] v \, ds,$$

where

$$L = \frac{\lambda_u}{a^2 \sin^2} - \frac{\lambda_v}{b^2 \cos^2}.$$

Expressing $dJ(\alpha_j)(\alpha_i - \alpha_j)$ similarly, taking their difference, and using the hypothesis that these directional derivatives are $0(1)$ (i.e., they go to zero as $i, j \to +\infty$), we have

$$\begin{aligned} 0(1) &= \big(dJ(\alpha_i) - dJ(\alpha_j)\big)(\alpha_i - \alpha_j) \\ &= \int_0^{\pi/2} \frac{(\dot{\alpha}_i - \dot{\alpha}_j)^2}{h^2} v \, ds + \int_0^{\pi/2} [L(\sin \alpha_i \cos \alpha_i - \sin \alpha_j \cos \alpha_j)(\alpha_i - \alpha_j)] \, v \, ds. \end{aligned}$$

The second integral is $0(1)$; that is seen by writing it as a sum over $[0, \varepsilon]$, $[\pi/2 - \varepsilon, \pi/2]$ and $[\varepsilon, \pi/2 - \varepsilon]$, and estimating each separately.

Because $h^2$ is bounded above and below by positive constants, we conclude that $\int_0^{\pi/2} (\dot{\alpha}_i - \dot{\alpha}_j)^2 v \, ds$ is $0(1)$; i.e., $(\alpha_i)_{i \geq i}$ is a Cauchy sequence in $X_0$, and hence convergent. In order to handle the case $d/c \neq 1$, it is convenient to express the energy functional (2.6) in terms of different coordinates on $Q^{r+s+1}(c, d)$: let

$$t = P(s) = \int_0^s k(r) \, dr \qquad 0 \leq s \leq \pi/2.$$

In terms of coordinates $(x, y, t)$, $x \in S^r$, $y \in S^s$ and $0 \leq t \leq P(\pi/2)$, the metric on $Q^{r+s+1}(c, d)$ is expressed by

$$g = c^2 f_1^2(t) g_r + d^2 f_2^2(t) g_s + dt^2$$

where

$$f_1(t) = \sin\big(P^{-1}(t)\big), \qquad f_2(t) = \cos\big(P^{-1}(t)\big).$$

Thus the reduced energy functional (2.6) takes the form

$$\hat{J}(\beta) = \int_0^{\pi/2} \left[\frac{\dot{\beta}^2}{h^2} + \frac{c^2 f_1^2(\beta)}{a^2 \sin^2} \lambda_u + \frac{d^2 f_2^2(\beta)}{b^2 \cos^2} \lambda_v\right] v \, ds.$$

By construction, $J(\alpha) = \hat{J}(P(\alpha))$.

Because $f_1$ and $f_2$ behave qualitatively like sin and cos, the Palais-Smale condition can be proved easily, using the same arguments as in the case $c = d = 1$.

3.5. The qualitative theory of critical points of differentiable functions has been adjusted to include domains which are closed convex subsets of Banach spaces ([6], [29]). Proposition 3.4 enables us to apply that theory: in particular, we have a

**Mountain Pass Lemma 3.6.** *Assume $p, q > 1$. Let $0 \in X_0$ be an isolated local minimum of $J : X_0 \to \mathbb{R}$, and assume there is an $\alpha \in X_0 - \{0\}$ such that $J(\alpha) = J(0)$. Then there is a critical point $\beta \in X_0$ with $J(\beta) > J(0)$. In particular, if $J$ has two isolated local minima, then it has another critical point in $X_0$ (which is not an absolute minimum).*

3.7. Proposition 3.4 also provides a version of the Morse inequalities, provided the critical points of $J : X_0 \to \mathbb{R}$ are isolated. We refer to [6] for further details.

**Remark 3.8.** In this section we have shown the existence of certain critical points of $J : X_0 \to \mathbb{R}$; they all satisfy the Euler-Lagrange equation (2.10). That can be seen by proving that they are also critical points of a simply modified functional $J^* : X \to \mathbb{R}$ [7] which also has (2.10) as its Euler-Lagrange equation.

## 4. Properties of Solutions

4.1. We apply the transformation $\tan s = e^t$, $t \in \mathbb{R}$ to (2.10). With the notation $A(t) = \alpha(\arctan e^t)$, and $H(t) = h(\arctan e^t)$, that equation becomes

$$A'' + \left[\frac{(p-1)e^{-t} - (q-1)e^t}{(e^t + e^{-t})} - \frac{H'}{H}\right] A'$$

$$= \left[(c^2 - d^2)(A')^2 + \frac{H^2}{(e^t + e^{-t})}\left(\frac{c^2\lambda_u e^{-t}}{a^2} - \frac{d^2\lambda_v e^t}{b^2}\right)\right]\frac{\sin A \cos A}{k^2(A)}.$$

The following is an extension of a basic lemma of [28]; the proof uses ideas from [23].

**Lemma 4.2.** *If $\alpha \in X_0$ is a non-constant solution of (2.10), then $A'(t) > 0$ for all $t \in \mathbb{R}$; and*

$$\lim_{t \to -\infty} A(t) = 0, \qquad \lim_{t \to +\infty} A(t) = \pi/2;$$
$$\text{i.e.,} \quad \lim_{s \to 0} \alpha(s) = 0, \qquad \lim_{s \to \pi/2} \alpha(s) = \pi/2. \tag{4.3}$$

**Proof.** We begin by observing that

$$0 < A(t) < \pi/2 \qquad \text{for all } t \in \mathbb{R}. \tag{4.4}$$

For if $A(\bar{t}) = 0$ for some $\bar{t} \in \mathbb{R}$, then $A'(\bar{t}) \neq 0$; for otherwise $A \equiv 0$. Thus $A$ would assume negative values, and consequently $\alpha$ could not belong to $X_0$. Similarly, $A$ does not assume the value $\pi/2$.

We proceed to show that $A' > 0$ on $\mathbb{R}$: let $t_0$ be the solution of $c^2\lambda_u e^{-t}/a^2 = d^2\lambda_v e^t/b^2$. Suppose $A'(\bar{t}) = 0$ for some $\bar{t} \le t_0$. Because $A$ is real analytic and non-constant, the zeros of $A'$ are isolated, so there is an $\varepsilon > 0$ such that $A'(t) \neq 0$ for $\bar{t} - \varepsilon < t < \bar{t}$.

Consider the linear equation:

$$Y'(t) + P_\alpha(t)Y(t) = Q_\alpha(t) \tag{4.5}$$

where

$$P_\alpha(t) = 2\left[\frac{A''}{A'} + \frac{(p-1)e^{-t} - (q-1)e^t}{(e^t + e^{-t})} - \frac{H'}{H}\right] + \frac{(d^2 - c^2)\sin A\cos AA'}{k^2(A)}$$

$$Q_\alpha(t) = 2H^2\left[\frac{c^2\lambda_u e^{-t}}{a^2} - \frac{d^2\lambda_v e^t}{b^2}\right]\frac{\sin A\cos A}{A'(e^t + e^{-t})k^2(A)}.$$

Then $P_\alpha(t) \equiv Q_\alpha(t)$ on $(\bar{t} - \varepsilon, \bar{t})$, because $\alpha$ is a solution of (2.10). Therefore the function $\bar{Y}(t) \equiv 1$ is a solution of (4.5) on $(\bar{t} - \varepsilon, \bar{t})$, expressible as

$$\bar{Y}(t) \equiv 1 \equiv \frac{\int_{\tilde{t}}^{t} Q_\alpha(r)\exp\left(\int_{\tilde{t}}^{r} P_\alpha(u)\,du\right)dr + c}{\exp\left(\int_{\tilde{t}}^{t} P_\alpha(u)\,du\right)} \tag{4.6}$$

for some $\tilde{t} \in (\bar{t} - \varepsilon, \bar{t})$ and $c \in \mathbb{R}$.

If $T$ is the first point where $-\infty \le T < \bar{t}$ and $A'(T) = 0$, then (4.6) holds for $t \in (T, \bar{t})$. The explicit formula for (4.6) is (see [23])

$$1 = N(t)/D(t) \qquad \text{for } t \in (T, \bar{t}), \tag{4.7}$$

where

$$N(t) = \int_{\tilde{t}}^{t}\left[\frac{c^2\lambda_u e^{-r}}{a^2} - \frac{d^2\lambda_v e^r}{b^2}\right]\frac{(1 + e^{-2r})^{1-p}\cdot(1 + e^{2r})^{1-q}}{(e^r + e^{-r})}\sin(2A(r))A'(r)\,dr + c$$

and

$$D(t) = (A')^2(1 + e^{-2t})^{1-p}(1 + e^{2t})^{1-q}k^2(A)(H(t))^{-2}.$$

Then, for all $t \in (T, \bar{t})$, we have

$$N(t) > 0; \tag{4.8}$$

$$N'(t) \neq 0; \quad \text{because } A'(t) \neq 0,\ 0 < A(t) < \pi/2 \text{ and } \bar{t} \le t_0. \tag{4.9}$$

Moreover

4.10. $T = -\infty$; for otherwise $D(T) = 0$, and so $N(T) = 0$ by (4.7). This, together with $N(\bar{t}) = 0$ and (4.8), tell us that $N$ must have an interior maximum on $[T, \bar{t}]$, contradicting (4.9).

We conclude from (4.10) that $A' \neq 0$ on $(-\infty, \bar{t})$ and that (4.7) holds there. But there must be points $\tilde{t} \in (-\infty, \bar{t})$ at which $A'(\tilde{t})$ is arbitrarily close to 0; for if $A'$ is bounded away from zero, the values of the solution $A$ would not remain in $[0, \pi/2]$.

Thus $D$, and consequently $N$, must have values arbitrarily close to zero. That, together with $N(\bar{t}) = 0$ and (4.8), insure that $N$ has a local maximum in $(-\infty, \bar{t})$, contradicting (4.9). That means that there is no $\bar{t} \leq t_0$ such that $A'(\bar{t}) = 0$. Similarly we find that there is no $\bar{t} > t_0$ for which $A'(\bar{t}) = 0$.

Moreover, $A' < 0$ on $\mathbb{R}$ is not possible; for otherwise $t_0$ would be a minimum of $N$, again leading to a contradiction. Therefore $A' > 0$ on $\mathbb{R}$, which guarantees the existence of the limits $\lim_{t \to +\infty} A(t)$ and $\lim_{t \to -\infty} A(t)$.

The condition $0 < A(t) < \pi/2$ insures that for any small $\varepsilon > 0$ and any large $C > 0$ there is $\tilde{t} > C$ (or $\tilde{t} < -C$) with $A'(\tilde{t}) < \varepsilon$, $|A''(\tilde{t})| < \varepsilon$: otherwise $A$ would go out of bounds. Therefore inspection of (4.1) shows that the only limits possible are those of (4.3).

Henceforth we shall say that a solution $\alpha$ of (2.10) with limits (4.3) *is* (*or provides*) *a harmonic join.*

Here is a basic *a priori estimate*:

**Lemma 4.11.** *Let $\alpha$ provide a harmonic join. Then $J(\alpha) < J(0)$.*

**Proof.**

$$J(\alpha) - J(0) = 1/2 \int_0^{\pi/2} \left[ \frac{k^2(\alpha)}{h^2} \dot{\alpha}^2 + \left( \frac{c^2 \lambda_u}{a^2 \sin^2} - \frac{d^2 \lambda_v}{b^2 \cos^2} \right) \sin^2 \alpha \right] v \, ds.$$

From (2.10′) we have

$$\left( \frac{c^2 \lambda_u}{a^2 \sin^2} - \frac{d^2 \lambda_v}{b^2 \cos} \right) (\sin^2 \alpha) v = k(\alpha) \tan \alpha \frac{d}{ds} \left( \frac{k(\alpha)}{h^2} \dot{\alpha} v \right)$$

$$= \frac{d}{ds} \left( \frac{k^2(\alpha)}{h^2} (\tan \alpha) \dot{\alpha} v \right) - \frac{k^2(\alpha)}{h^2} \left( \frac{k'(\alpha)}{k(\alpha)} \tan \alpha + \frac{1}{\cos^2 \alpha} \right) \dot{\alpha}^2 v.$$

Therefore

$$J(\alpha) - J(0) = 1/2 \int_0^{\pi/2} \left[ 1 - \frac{k'(\alpha)}{k(\alpha)} \tan \alpha - \frac{1}{\cos^2 \alpha} \right] \frac{k^2(\alpha) \dot{\alpha}^2}{h^2} v \, ds + 1/2 \frac{k^2(\alpha)}{h^2} (\tan \alpha) \dot{\alpha} v \Big|_0^{\pi/2}$$

The last term is zero: this is because the asymptotic behavior of $\alpha$ is qualitatively as in the case $k^2(\alpha) \equiv 1 \equiv h^2$; thus the well-known asymptotic estimates of [28] can be used to prove our assertion.

By using the explicit expression $k^2(\alpha) = [d^2 \sin^2 \alpha + c^2 \cos^2 \alpha]$, an elementary computation shows that

$$k^2(\alpha)\left[1 - \frac{k'(\alpha)}{k(\alpha)} \tan \alpha - \frac{1}{\cos^2 \alpha}\right] = -d^2 \tan^2 \alpha .$$

In conclusion we have

$$J(\alpha) - J(0) = -d^2/2 \int_0^{\pi/2} \tan^2 \alpha \frac{\dot{\alpha}^2 v}{h^2} ds < 0,$$

so the Lemma is established.

**Proposition 4.12.** *Let $p$, $q \geq 1$ and assume $J(\pi/2) \geq J(0)$. Then there is a harmonic join $\alpha$ iff $0 \in X_0$ is an unstable critical point of $J : X_0 \to \mathbb{R}$.*

**Proof.** If 0 is unstable, then the minimum $\underline{\alpha}$ (as in (3.3)) provides a harmonic join by Lemma 4.2. Conversely, assume first $p$, $q > 1$ and suppose that 0 were stable.

If $\alpha_0$ provides a harmonic join, then Lemma 4.11 assures us that $J(\alpha_0) < J(0)$; moreover, because of Proposition 3.4, we can apply the Mountain Pass Lemma 3.6 to $J$ on the closed convex set

$$Y_0 = \{\alpha \in X : 0 \leq \alpha(s) \leq \alpha_0(s) \quad \text{for all } s \in [0, \pi/2]\}$$

to conclude that there is a solution $\beta$ which provides a harmonic join; and $J(\beta) > J(0)$. That contradicts Lemma 4.11.

If $p = 1$, $q > 1$ or $p = 1 = q$, then a modification [7] of the previous argument can be used to complete the proof of our Proposition.

## 5. Harmonic Maps between Ellipsoids

**Theorem 5.1.** *Let $u : S^p \to S^r$ and $v : S^q \to S^s$ be eigenmaps, $p$, $q \geq 1$. Assume that there are $a$, $b$, $c$, $d > 0$, with $a \geq b$, such that*

$$(q-1)\lambda_u b^2/a^2 \geq (p-1)\lambda_v d^2/c^2 ; \tag{5.2}$$

*and*

$$(q-1)^2 < 4\lambda_v d^2/c^2 . \tag{5.3}$$

*Then there is an equivariant harmonic map $\varphi = u *_\alpha v : Q^{p+q+1}(a, b) \to Q^{r+s+1}(c, d)$ homotopic to $u * v$.*

*Furthermore, if $p = 1$, then the assumption $a \geq b$ is unnecessary.*

**Proof.**

*Step* 1: We take the minimum $\underline{\alpha} \in X_0$, as in (3.3). If $\underline{\alpha} \not\equiv 0$ or $\pi/2$, then $\underline{\alpha}$ provides a harmonic join by Lemma 4.2.

*Step* 2: We prove that

$$J(\pi/2) \geq J(0)$$

If $p = 1$, this is obvious because $J(\pi/2) = +\infty$. If $p > 1$, then (5.2) forces $q > 1$ and integration by parts gives

$$J(\pi/2)/a^p b^q = \frac{(q-1)\lambda_u c^2}{2(p-1)a^2}\int_0^{\pi/2} \sin^p \cos^{q-2} h\, ds + \frac{\lambda_u c^2(a^2-b^2)}{2(p-1)a^2}\int_0^{\pi/2} \sin^p \cos^q h^{-1}\, ds.$$

The second term is non-negative because $a \geq b$, so

$$J(\pi/2) - J(0) \geq \frac{a^p b^q}{2}\left(\frac{(q-1)\lambda_u c^2}{(p-1)a^2} - \frac{\lambda_v d^2}{b^2}\right)\int_0^{\pi/2} \sin^p \cos^{q-2} h\, ds.$$

But (5.2) ensures that the term in parentheses is non-negative.

*Step* 3: Assume first $q > 1$. We show that $0 \in X_0$ is unstable: we calculate the second variation at 0 for Ding's variations $\xi = \sin^n \cos^{-r}$, with suitable $n$, $r$ (to be chosen in the course of the proof). Following (2.11), we obtain

$$\nabla^2 J(0)(\xi,\xi)/a^p b^q = \int_0^{\pi/2}\left(\frac{c^2 r^2 \sin^2}{h^2} - \frac{d^2\lambda_v}{b^2}\right)\sin^{p+2n}\cos^{q-2r-2} h\, ds$$

$$+ \int_0^{\pi/2}\left[\frac{c^2}{h^2}\left(n^2\frac{\cos^2}{\sin^2} + 2nr\right) + \frac{c^2\lambda_u}{a^2\sin^2}\right]\sin^{p+2n}\cos^{q-2r} h\, ds. \quad (5.4)$$

Restrict $n > 0, 0 < r < (q-1)/2$. As a function of $r$, the second integral in (5.4) remains bounded as $r \to (q-1)/2$. Now we show that the first integral in (5.4) tends to $-\infty$ as $r \to (q-1)/2$: it follows that $0 \in X_0$ is unstable. When $r$ increases to $(q-1)/2$, the first integral in (5.4) is clearly smaller than

$$\int_0^{\pi/2}\left[\frac{c^2(q-1)^2}{4h^2} - \frac{d^2\lambda_v}{b^2}\right]\sin^{p+2n}\cos^{q-2r-2} h\, ds. \quad (5.5)$$

Now we observe that $\lim_{s\to\pi/2} h^2(s) = b^2$: thus (5.3) enables us to conclude that the term in parentheses is strictly negative on $[\pi/2 - \varepsilon, \pi/2]$ for a suitable small $\varepsilon > 0$ (independent of $r$). We write the integral (5.5) as the sum of two pieces

$$\int_0^{\pi/2} = \int_0^{\pi/2-\varepsilon} + \int_{\pi/2-\varepsilon}^{\pi/2}.$$

Now we let $r$ tend to $(q-1)/2$: the first integral in the sum is clearly bounded; the second integral tends to $-\infty$, because the exponent of cos in (5.5) tends to $-1$ as