PRINCETON LECTURES IN ANALYSIS III

REAL ANALYSIS

MEASURE THEORY, INTEGRATION, & HILBERT SPACES

ELIAS M. STEIN & RAMI SHAKARCHI

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III

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REAL ANALYSIS

Princeton Lectures in Analysis

I Fourier Analysis: An Introduction

II Complex Analysis

III Real Analysis: Measure Theory, Integration, and Hilbert Spaces

IV Functional Analysis: Introduction to Further Topics in Analysis

TO MY GRANDCHILDREN
CAROLYN, ALISON, JASON

E.M.S.

TO MY PARENTS
MOHAMED & MIREILLE
AND MY BROTHER
KARIM

R.S.

Foreword

Beginning in the spring of 2000, a series of four one-semester courses were taught at Princeton University whose purpose was to present, in an integrated manner, the core areas of analysis. The objective was to make plain the organic unity that exists between the various parts of the subject, and to illustrate the wide applicability of ideas of analysis to other fields of mathematics and science. The present series of books is an elaboration of the lectures that were given.

While there are a number of excellent texts dealing with individual parts of what we cover, our exposition aims at a different goal: presenting the various sub-areas of analysis not as separate disciplines, but rather as highly interconnected. It is our view that seeing these relations and their resulting synergies will motivate the reader to attain a better understanding of the subject as a whole. With this outcome in mind, we have concentrated on the main ideas and theorems that have shaped the field (sometimes sacrificing a more systematic approach), and we have been sensitive to the historical order in which the logic of the subject developed.

We have organized our exposition into four volumes, each reflecting the material covered in a semester. Their contents may be broadly summarized as follows:

I. Fourier series and integrals.

II. Complex analysis.

III. Measure theory, Lebesgue integration, and Hilbert spaces.

IV. A selection of further topics, including functional analysis, distributions, and elements of probability theory.

However, this listing does not by itself give a complete picture of the many interconnections that are presented, nor of the applications to other branches that are highlighted. To give a few examples: the elements of (finite) Fourier series studied in Book I, which lead to Dirichlet characters, and from there to the infinitude of primes in an arithmetic progression; the X-ray and Radon transforms, which arise in a number of problems in Book I, and reappear in Book III to play an important role in understanding Besicovitch-like sets in two and three dimensions; Fatou's theorem, which guarantees the existence of boundary values of bounded holomorphic functions in the disc, and whose proof relies on ideas developed in each of the first three books; and the theta function, which first occurs in Book I in the solution of the heat equation, and is then used in Book II to find the number of ways an integer can be represented as the sum of two or four squares, and in the analytic continuation of the zeta function.

A few further words about the books and the courses on which they were based. These courses where given at a rather intensive pace, with 48 lecture-hours a semester. The weekly problem sets played an indispensable part, and as a result exercises and problems have a similarly important role in our books. Each chapter has a series of "Exercises" that are tied directly to the text, and while some are easy, others may require more effort. However, the substantial number of hints that are given should enable the reader to attack most exercises. There are also more involved and challenging "Problems"; the ones that are most difficult, or go beyond the scope of the text, are marked with an asterisk.

Despite the substantial connections that exist between the different volumes, enough overlapping material has been provided so that each of the first three books requires only minimal prerequisites: acquaintance with elementary topics in analysis such as limits, series, differentiable functions, and Riemann integration, together with some exposure to linear algebra. This makes these books accessible to students interested in such diverse disciplines as mathematics, physics, engineering, and finance, at both the undergraduate and graduate level.

It is with great pleasure that we express our appreciation to all who have aided in this enterprise. We are particularly grateful to the students who participated in the four courses. Their continuing interest, enthusiasm, and dedication provided the encouragement that made this project possible. We also wish to thank Adrian Banner and José Luis Rodrigo for their special help in running the courses, and their efforts to see that the students got the most from each class. In addition, Adrian Banner also made valuable suggestions that are incorporated in the text.

We wish also to record a note of special thanks for the following individuals: Charles Fefferman, who taught the first week (successfully launching the whole project!); Paul Hagelstein, who in addition to reading part of the manuscript taught several weeks of one of the courses, and has since taken over the teaching of the second round of the series; and Daniel Levine, who gave valuable help in proof-reading. Last but not least, our thanks go to Gerree Pecht, for her consummate skill in typesetting and for the time and energy she spent in the preparation of all aspects of the lectures, such as transparencies, notes, and the manuscript.

We are also happy to acknowledge our indebtedness for the support we received from the 250th Anniversary Fund of Princeton University, and the National Science Foundation's VIGRE program.

Elias M. Stein
Rami Shakarchi

Princeton, New Jersey
August 2002

In this third volume we establish the basic facts concerning measure theory and integration. This allows us to reexamine and develop further several important topics that arose in the previous volumes, as well as to introduce a number of other subjects of substantial interest in analysis. To aid the interested reader, we have starred sections that contain more advanced material. These can be omitted on first reading. We also want to take this opportunity to thank Daniel Levine for his continuing help in proof-reading and the many suggestions he made that are incorporated in the text.

November 2004

Introduction

I turn away in fright and horror from this lamentable plague of functions that do not have derivatives.
C. Hermite, 1893

Starting in about 1870 a revolutionary change in the conceptual framework of analysis began to take shape, one that ultimately led to a vast transformation and generalization of the understanding of such basic objects as functions, and such notions as continuity, differentiability, and integrability.

The earlier view that the relevant functions in analysis were given by formulas or other "analytic" expressions, that these functions were by their nature continuous (or nearly so), that by necessity such functions had derivatives for most points, and moreover these were integrable by the accepted methods of integration – all of these ideas began to give way under the weight of various examples and problems that arose in the subject, which could not be ignored and required new concepts to be understood. Parallel with these developments came new insights that were at once both more geometric and more abstract: a clearer understanding of the nature of curves, their rectifiability and their extent; also the beginnings of the theory of sets, starting with subsets of the line, the plane, etc., and the "measure" that could be assigned to each.

That is not to say that there was not considerable resistance to the change of point-of-view that these advances required. Paradoxically, some of the leading mathematicians of the time, those who should have been best able to appreciate the new departures, were among the ones who were most skeptical. That the new ideas ultimately won out can be understood in terms of the many questions that could now be addressed. We shall describe here, somewhat imprecisely, several of the most significant such problems.

1 Fourier series: completion

Whenever f is a (Riemann) integrable function on $[-\pi, \pi]$ we defined in Book I its Fourier series $f \sim \sum a_n e^{inx}$ by

$$a_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx}\, dx, \tag{1}$$

and saw then that one had Parseval's identity,

$$\sum_{n=-\infty}^{\infty} |a_n|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^2\, dx.$$

However, the above relationship between functions and their Fourier coefficients is not completely reciprocal when limited to Riemann integrable functions. Thus if we consider the space $\mathcal{R}$ of such functions with its square norm, and the space $\ell^2(\mathbb{Z})$ with its norm,[1] each element f in $\mathcal{R}$ assigns a corresponding element $\{a_n\}$ in $\ell^2(\mathbb{Z})$, and the two norms are identical. However, it is easy to construct elements in $\ell^2(\mathbb{Z})$ that do not correspond to functions in $\mathcal{R}$. Note also that the space $\ell^2(\mathbb{Z})$ is *complete* in its norm, while $\mathcal{R}$ is not.[2] Thus we are led to two questions:

(i) What are the putative "functions" f that arise when we complete $\mathcal{R}$? In other words: given an arbitrary sequence $\{a_n\} \in \ell^2(\mathbb{Z})$ what is the nature of the (presumed) function f corresponding to these coefficients?

(ii) How do we integrate such functions f (and in particular verify (1))?

2 Limits of continuous functions

Suppose $\{f_n\}$ is a sequence of continuous functions on $[0, 1]$. We assume that $\lim_{n\to\infty} f_n(x) = f(x)$ exists for every x, and inquire as to the nature of the limiting function f.

If we suppose that the convergence is uniform, matters are straightforward and f is then everywhere continuous. However, once we drop the assumption of uniform convergence, things may change radically and the issues that arise can be quite subtle. An example of this is given by the fact that one can construct a sequence of continuous functions $\{f_n\}$ converging everywhere to f so that

[1] We use the notation of Chapter 3 in Book I.

[2] See the discussion surrounding Theorem 1.1 in Section 1, Chapter 3 of Book I.

(a) $0 \leq f_n(x) \leq 1$ for all x.

(b) The sequence $f_n(x)$ is montonically decreasing as $n \to \infty$.

(c) The limiting function f is not Riemann integrable.[3]

However, in view of (a) and (b), the sequence $\int_0^1 f_n(x)\,dx$ converges to a limit. So it is natural to ask: what method of integration can be used to integrate f and obtain that for it

$$\int_0^1 f(x)\,dx = \lim_{n\to\infty} \int_0^1 f_n(x)\,dx\,?$$

It is with Lebesgue integration that we can solve both this problem and the previous one.

3 Length of curves

The study of curves in the plane and the calculation of their lengths are among the first issues dealt with when one learns calculus. Suppose we consider a continuous curve Γ in the plane, given parametrically by $\Gamma = \{(x(t), y(t))\}$, $a \leq t \leq b$, with x and y continuous functions of t. We define the *length* of Γ in the usual way: as the supremum of the lengths of all polygonal lines joining successively finitely many points of Γ, taken in order of increasing t. We say that Γ is *rectifiable* if its length L is finite. When $x(t)$ and $y(t)$ are continuously differentiable we have the well-known formula,

$$L = \int_a^b \left((x'(t))^2 + (y'(t))^2\right)^{1/2} dt. \tag{2}$$

The problems we are led to arise when we consider general curves. More specifically, we can ask:

(i) What are the conditions on the functions $x(t)$ and $y(t)$ that guarantee the rectifiability of Γ?

(ii) When these are satisfied, does the formula (2) hold?

The first question has a complete answer in terms of the notion of functions of "bounded variation." As to the second, it turns out that if x and y are of bounded variation, the integral (2) is always meaningful; however, the equality fails in general, but can be restored under appropriate reparametrization of the curve Γ.

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[3]The limit f can be highly discontinuous. See, for instance, Exercise 10 in Chapter 1.

There are further issues that arise. Rectifiable curves, because they are endowed with length, are genuinely one-dimensional in nature. Are there (non-rectifiable) curves that are two-dimensional? We shall see that, indeed, there are continuous curves in the plane that fill a square, or more generally have any dimension between 1 and 2, if the notion of fractional dimension is appropriately defined.

4 Differentiation and integration

The so-called "fundamental theorem of the calculus" expresses the fact that differentiation and integration are inverse operations, and this can be stated in two different ways, which we abbreviate as follows:

$$F(b) - F(a) = \int_a^b F'(x)\,dx, \tag{3}$$

$$\frac{d}{dx}\int_0^x f(y)\,dy = f(x). \tag{4}$$

For the first assertion, the existence of continuous functions F that are nowhere differentiable, or for which $F'(x)$ exists for every x, but F' is not integrable, leads to the problem of finding a general class of the F for which (3) is valid. As for (4), the question is to formulate properly and establish this assertion for the general class of integrable functions f that arise in the solution of the first two problems considered above. These questions can be answered with the help of certain "covering" arguments, and the notion of absolute continuity.

5 The problem of measure

To put matters clearly, the fundamental issue that must be understood in order to try to answer all the questions raised above is the problem of measure. Stated (imprecisely) in its version in two dimensions, it is the problem of assigning to each subset E of $\mathbb{R}^2$ its two-dimensional measure $m_2(E)$, that is, its "area," extending the standard notion defined for elementary sets. Let us instead state more precisely the analogous problem in one dimension, that of constructing one-dimensional measure $m_1 = m$, which generalizes the notion of length in $\mathbb{R}$.

We are looking for a non-negative function m defined on the family of subsets E of $\mathbb{R}$ that we allow to be extended-valued, that is, to take on the value $+\infty$. We require:

(a) $m(E) = b - a$ if E is the interval $[a, b]$, $a \leq b$, of length $b - a$.

(b) $m(E) = \sum_{n=1}^{\infty} m(E_n)$ whenever $E = \bigcup_{n=1}^{\infty} E_n$ and the sets E_n are disjoint.

Condition (b) is the "countable additivity" of the measure m. It implies the special case:

(b′) $m(E_1 \cup E_2) = m(E_1) + m(E_2)$ if E_1 and E_2 are disjoint.

However, to apply the many limiting arguments that arise in the theory the general case (b) is indispensable, and (b′) by itself would definitely be inadequate.

To the axioms (a) and (b) one adds the translation-invariance of m, namely

(c) $m(E + h) = m(E)$, for every $h \in \mathbb{R}$.

A basic result of the theory is the existence (and uniqueness) of such a measure, Lebesgue measure, when one limits oneself to a class of reasonable sets, those which are "measurable." This class of sets is closed under countable unions, intersections, and complements, and contains the open sets, the closed sets, and so forth.[4]

It is with the construction of this measure that we begin our study. From it will flow the general theory of integration, and in particular the solutions of the problems discussed above.

A chronology

We conclude this introduction by listing some of the signal events that marked the early development of the subject.

1872 – Weierstrass's construction of a nowhere differentiable function.

1881 – Introduction of functions of bounded variation by Jordan and later (1887) connection with rectifiability.

1883 – Cantor's ternary set.

1890 – Construction of a space-filling curve by Peano.

1898 – Borel's measurable sets.

1902 – Lebesgue's theory of measure and integration.

1905 – Construction of non-measurable sets by Vitali.

1906 – Fatou's application of Lebesgue theory to complex analysis.

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[4]There is no such measure on the class of all subsets, since there exist non-measurable sets. See the construction of such a set at the end of Section 3, Chapter 1.

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