UNITARY REPRESENTATIONS OF REDUCTIVE LIE GROUPS

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DAVID A. VOGAN, JR.

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To my parents, for keeping the first draft

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INTRODUCTION

Perhaps the most fundamental goal of abstract harmonic analysis is to understand the actions of groups on spaces of functions. Sometimes this goal appears in a slightly disguised form, as when one studies systems of differential equations invariant under a group; or it may be made quite explicit, as in the representation-theoretic theory of automorphic forms. Interesting particular examples of problems of this kind abound. Generously interpreted, they may in fact be made to include a significant fraction of all of mathematics. A rather smaller number are related to the subject matter of this book. Here are some of them.

Let X be a pseudo-Riemannian manifold, and G a group of isometries of X. Then X carries a natural measure, and G acts on $L^2(X)$ by unitary operators. Often (for example, if the metric is positive definite and complete) the Laplace-Beltrami operator Δ on X is self-adjoint. In that case, G will preserve its spectral decomposition. Conversely, if the action of G is transitive, then any G-invariant subspace of $L^2(X)$ will be preserved by Δ. The problem of finding G-invariant subspaces therefore refines the spectral problem for Δ.

The prototypical example of this nature is the sphere S^{n-1}, with G the orthogonal group O(n). If n is at least 2, the minimal invariant subspaces for O(n) acting on $L^2(S^{n-1})$ are precisely the eigenspaces of the spherical Laplacian. (This is the abstract part of the theory of spherical harmonics.) If n is 2, we are talking about Fourier series. The fundamental importance of these is clear; but they may of course be analyzed without explicit discussion of groups. For $n = 3$, the theory of spherical harmonics leads to the solution of the Schrödinger equation for the hydrogen atom. Here the clarifying role of the group is less easy to overlook, and it was in this connection that the "Gruppenpest" entered quantum physics in an explicit way.

A second example, still in the framework of pseudo-Riemannian manifolds, is the wave operator. Viewed on a four-dimensional space-time manifold, this is just the Laplace-Beltrami operator for a metric of signature (3,1). If the manifold has a large isometry group (for instance, if it is Minkowski space), then the space of solutions can often be described terms of this group action.

An example with a rather different flavor is the space X of lattices (that is, discrete subgroups isomorphic to $\mathbb{Z}^n$) in $\mathbb{R}^n$. An automorphic form for $G = GL(n,\mathbb{R})$ is a smooth function on X, subject to some technical growth and finiteness conditions. (Actually it is convenient to consider at the same time various covering spaces of X, such as (for fixed p) the space of lattices L endowed with a basis of L/pL.) It is easy to imagine that functions on X have something to do with number theory, and this is the case. One goal of the representation-theoretic theory of automorphic forms is to understand the action of G on the space of automorphic forms. Because the G-invariant measure on X has finite total mass (although X is not compact), this problem is closely connected to the corresponding L^2 problem. An introduction to this problem may be found in [Arthur, 1979].

Finally, suppose X is a compact locally symmetric space. (Local symmetry means that $-\mathrm{Id}$ on each tangent space exponentiates to a local isometry of X. An example is a compact Riemann surface.) We seek to understand the deRham cohomology groups of X. Here there is no group action in evidence, and no space of functions. However, Hodge theory relates the cohomology to harmonic forms on X, so the latter defect is not serious. For the former, we consider the bundle Y over X whose fiber at p is the (compact) group K_p of local isometries of X fixing p. Harmonic forms on X pull back to Y as certain vector-valued functions. On the other hand, Y has a large transitive group G acting on it. (G may be taken to be the isometry group of the universal cover of X; Y is then the quotient of G by the fundamental group of X.) The cohomology of X can now be studied in terms of the action of G on functions on Y. Perhaps surprisingly, this has turned out to be a useful approach (see [Borel-Wallach, 1980]).

With these examples in mind, we recall very briefly the program for studying such problems which had emerged by 1950 or so. The first idea was to formalize the notion of group actions on function spaces. In accordance with the general philosophy of functional analysis, the point is to forget where the function space came from.

Definition 0.1. Suppose G is a topological group. A *representation* of G is a pair (π, V) consisting of a complex topological vector space V, and a homomorphism π from G to the group of automorphisms of V. We assume that the map from $G \times V$ to V, given by

$$(g,v) \to \pi(g)v$$

is continuous. An *invariant subspace* of the representation is a subspace W of V which is preserved by all the operators $\pi(g)$ (for g in G). The representation is called *reducible* if there is a closed invariant subspace W other than V itself and $\{0\}$. We say that π is *irreducible* if V is not zero, and π is not reducible.

The problem of understanding group actions on spaces of functions can now be formalized in two parts: we want first to understand how general representations are built from irreducible representations, and then to understand irreducible representations. This book is concerned almost exclusively with the second part. Nevertheless, we may hope to gain a little insight into the first part along the way, much as one may study architecture by studying bricks.

If we take G to be $\mathbb{Z}$, then a representation is determined by a single bounded invertible operator, $\pi(1)$. The only interesting irreducible representations of G are the one-dimensional ones (sending 1 to a non-zero complex number). The decomposition problem in this case amounts to trying to diagonalize the operator $\pi(1)$. There are some