INVERSE AND ILL-POSED PROBLEMS SERIES

Inverse Problems of Mathematical Physics

M.M. Lavrentiev, A.V. Avdeev, M.M. Lavrentiev, Jr. and V.I. Priimenko

VSP

UTRECHT • BOSTON

INVERSE AND ILL-POSED PROBLEMS SERIES

Inverse Problems of Mathematical Physics

M.M. Lavrentiev, A.V. Avdeev,
M.M. Lavrentiev, Jr. and V.I. Priimenko

///VSP///
UTRECHT • BOSTON
2003

VSP
an imprint of Brill Academic Publishers
P.O. Box 346
3700 AH Zeist
The Netherlands

Tel: +31 30 692 5790
Fax: +31 30 693 2081
vsppub@compuserve.com
www.brill.nl
www.vsppub.com

First published in 2003

ISBN 90-6764-396-3

Printed in The Netherlands by Ridderprint bv, Ridderkerk.

INVERSE AND ILL-POSED PROBLEMS SERIES

Inverse Problems of Mathematical Physics

Also available in the Inverse and Ill-Posed Problems Series:

Ill-Posed Boundary-Value Problems
S.E. Temirbolat
Linear Sobolev Type Equations and Degenerate Semigroups of Operators
G.A. Sviridyuk and V.E. Fedorov
Ill-Posed and Non-Classical Problems of Mathematical Physics and Analysis
Editors: M.M. Lavrent'ev and S.I. Kabanikhin
Forward and Inverse Problems for Hyperbolic, Elliptic and Mixed Type Equations
A.G. Megrabov
Nonclassical Linear Volterra Equations of the First Kind
A.S. Apartsyn
Poorly Visible Media in X-ray Tomography
D.S. Anikonov, V.G. Nazarov, and I.V. Prokhorov
Dynamical Inverse Problems of Distributed Systems
V.I. Maksimov
Theory of Linear Ill-Posed Problems and its Applications
V.K. Ivanov, V.V. Vasin and V.P. Tanana
Ill-Posed Internal Boundary Value Problems for the Biharmonic Equation
M.A. Atakhodzhaev
Investigation Methods for Inverse Problems
V.G. Romanov
Operator Theory. Nonclassical Problems
S.G. Pyatkov
Inverse Problems for Partial Differential Equations
Yu.Ya. Belov
Method of Spectral Mappings in the Inverse Problem Theory
V. Yurko
Theory of Linear Optimization
I.I. Eremin
Integral Geometry and Inverse Problems for Kinetic Equations
A.Kh. Amirov
Computer Modelling in Tomography and Ill-Posed Problems
M.M. Lavrent'ev, S.M. Zerkal and O.E. Trofimov
An Introduction to Identification Problems via Functional Analysis
A. Lorenzi
Coefficient Inverse Problems for Parabolic Type Equations and Their Application
P.G. Danilaev
Inverse Problems for Kinetic and Other Evolution Equations
Yu.E. Anikonov
Inverse Problems of Wave Processes
A.S. Blagoveshchenskii
Uniqueness Problems for Degenerating Equations and Nonclassical Problems
S.P. Shishatskii, A. Asanov and E.R. Atamanov
Uniqueness Questions in Reconstruction of Multidimensional Tomography-Type Projection Data
V.P. Golubyatnikov
Monte Carlo Method for Solving Inverse Problems of Radiation Transfer
V.S. Antyufeev
Introduction to the Theory of Inverse Problems
A.L. Bukhgeim
Identification Problems of Wave Phenomena - Theory and Numerics
S.I. Kabanikhin and A. Lorenzi
Inverse Problems of Electromagnetic Geophysical Fields
P.S. Martyshko
Composite Type Equations and Inverse Problems
A.I. Kozhanov
Inverse Problems of Vibrational Spectroscopy
A.G. Yagola, I.V. Kochikov, G.M. Kuramshina and Yu.A. Pentin
Elements of the Theory of Inverse Problems
A.M. Denisov
Volterra Equations and Inverse Problems
A.L. Bughgeim
Small Parameter Method in Multidimensional Inverse Problems
A.S. Barashkov
Regularization, Uniqueness and Existence of Volterra Equations of the First Kind
A. Asanov
Methods for Solution of Nonlinear Operator Equations
V.P. Tanana
Inverse and Ill-Posed Sources Problems
Yu.E. Anikonov, B.A. Bubnov and G.N. Erokhin
Methods for Solving Operator Equations
V.P. Tanana
Nonclassical and Inverse Problems for Pseudoparabolic Equations
A. Asanov and E.R. Atamanov
Formulas in Inverse and Ill-Posed Problems
Yu.E. Anikonov
Inverse Logarithmic Potential Problem
V.G. Cherednichenko
Multidimensional Inverse and Ill-Posed Problems for Differential Equations
Yu.E. Anikonov
Ill-Posed Problems with A Priori Information
V.V. Vasin and A.L. Ageev
Integral Geometry of Tensor Fields
V.A. Sharafutdinov
Inverse Problems for Maxwell's Equations
V.G. Romanov and S.I. Kabanikhin

Preface

The present book describes a part of the theory of the so-called *Inverse Problems of Mathematical Physics* and some applications of such problems. Mostly the theoretical aspects of Inverse Problems are discussed. Besides, we also consider some applications and numerical methods of solving the problems under study. Descriptions of particular numerical experiments are also included.

The theory of *Inverse Problems of Mathematical Physics* is a vast and intensively developing field of modern mathematics. Plenty of publications appear, and even a number of specialized journals are published. Because of extended area of applications, many various statements of problems are considered, and diverse methods are used for their solution.

We stress attention at providing a concept of versatility and complexity of inverse problems arising in applications. We did not pursue the aim of giving the complete review of literature, instead we pointed out the most popular textbooks and characteristic statements of the problems. Also, we often pointed out the connections of such problems with various applications of methods of mathematical simulation. The references cited are mainly of illustrative character. At the same time, we gave references to the most frequently cited monographs which contain further references and a more complete account of the history of this field. Meaning to provide an introduction intended for specialists in other fields, we tried to emphasize the basic general principles and approaches to solution of various problems, supplying them with concrete examples of results obtained.

The monograph is arranged as follows. In Introduction we explain our understanding of the concept of Mathematical Modeling, outline the general differences between direct and inverse problems, and give strict mathematical definitions of correct and ill-posed problems.

In Chapter 1, we show up a list of applied areas, where inverse problems have been successfully used for years. Of course, it is impossible to list all such applications, so we took a liberty to mention somehow the *basic fields*, some of which historically developed along with the theory of inverse problems.

In practice, almost all problems are solved approximately, in mathematical sense. So, in Chapter 2, we give some of the basic definitions related to various regularizations of inverse problems. We hope that the reader who does not need strict definitions can get all the general ideas just browsing the *text part* of the chapter.

Chapter 3 is dedicated to the problems of Integral Geometry, which are both classical (accounting Radon transform, e.g.) and non standard. Some recent results, previously available only through specialized journals, are included.

Chapter 4 is arranged in non traditional way providing the reader with a sort of overview of one-dimensional inverse problems. In spite of usual separation of model equations of hyperbolic and parabolic type, the chapter is compiled as follows. First, Lamé system is described from physical statement to model simplifications and uniqueness and stability results. Second, the so-called "quasi-stationary approximation" of Maxwell system is concerned. The point is that the complete proof through analytic relations between solutions to equations of hyperbolic and parabolic types is given. Then, a concept of relations among inverse problems for different type governing equations are discussed. Such concept is not that new, but perhaps is not well known. The next sections is dedicated to brief descriptions of such fundamental methods of investigation as the separation of singularities and the reduction of one-dimensional inverse problem with a focused source of disturbances to a linear integral equation. Finally, the determination of the piece-wise constant coefficient for wave equation is considered. The example of inverse problem (arising in applications) is given, in which case recurrent algorithm for exact determination of equation coefficients is available.

In Chapter 5 we consider some inverse problems for the coupled Maxwell and Lamé systems. First of all, the solution of the one-dimensional inverse problems for the equations of electromagnetoelasticity in the case of seismomagnetic interaction is studied. Then we give some results of the solution of inverse problems for the system of electromagnetoelasticity in the case of piezoelectric effect. In the next section, the linear process of interaction of electromagnetic and elastic waves in a weakly conducting elastic medium is considered. Finally, we give some results of solution of direct and inverse problems for the system of electromagnetoelasticity in the case of nonlinear interactions between electromagnetic and elastic fields.

Chapter 6 contains some examples of numerical solution of the different type inverse problems arising in applications. The first section is a small survey in numerical methods for inverse problems. The next section represents the numerical solution of a 3D inverse kinematic problem of seismics. Then we describes how the proposed algorithm of numerical solution could be used for the determination of the structure of the Earth's upper mantle. The numerical solutions of two inverse problems of electromagnetoelasticity are discussed in the next section. We would like to draw attention on the last section, which represents the results of numerical modeling of coastal profile evolution. This application of inverse problems is new and not well developed at the moment.

Acknowledgement

The authors are grateful to Academician of RAS, Professor A. S. Alekseev, Corresponding Members of RAS, Professors V. G. Romanov, V. V. Vasin, and B. G. Mikhailenko, and to Professors Yu. E. Anikonov, S. I. Kabanikhin, V. G. Yakhno, and V. A. Cheverda for useful discussions and kind permission to use some their results.

Also, the authors are grateful to A. Nazarov and D. Nechaev for their kind and helpful association in preparation of the camera-ready.

Finally, we would like to mention that this book was written under financial support, in part, of the Russian Foundation for Basic Research (projects Nos. 01-05-64704, 02-05-64939) and the U.S. Civil Research Development Foundantion (project No. RG1-2415-NO-2).

Mikhail M. Lavrentiev,
Alexander V. Avdeev,
Mikhail M. Lavrentiev, Jr.,
Viatcheslav I. Priimenko

Contents

Introduction

0.1. THE CONCEPT OF MATHEMATICAL SIMULATION

The statement of direct and inverse problems of mathematical physics implies preliminary schematic representation of the real process in a certain mathematical form. At present, modern technology is inconceivable without mathematical modeling, which is understood as replacement of the original object under investigation by its *model*. The aim of such a replacement is to study the properties of the object with the help of its model.

There are *experimental* (physical) modeling and *theoretical* (mathematical) simulation. In the former case the process is studied on the basis of real experiments on mock-ups, physical models, laboratory installations, etc. Mathematical simulation consists in construction and investigation of quantitative values for physical parameters through a simplified model of the process under study, which is formulated in mathematical terms. As a rule, a mathematical model should be *adequate* to the physical process.

Mathematical models contain unknown characteristics. The choice and evaluation of these characteristics is a difficult problem which is solved on the basis of accumulated experience, available experimental data, physical laws, etc. To develop a mathematical model it is necessary to pass through two main stages: *identification*, i. e., the choice of the type (structure) of the model, and *determination* of the numerical values of the model parameters and characteristics.

The structure of a model is conceived as a qualitative character of the mathematical description of the processes under investigation. Thus, physical laws are most frequently represented in the form of differential equations. One can distinguish models with *concentrated parameters*, in which case physical parameters are independent of space and only their evolution in time is the subject of study. Such models are described by systems of

Ordinary Differential Equations. On the contrary, models with *distributed parameters* involve spatial distribution of physical fields in addition to temporal evolution. Such models are based on *Partial Differential Equations* (equations in partial derivatives).

Such classification is very rough. Indeed, one can distinguish stationary (steady-state) and non-stationary (dynamic) models, linear and nonlinear models, one-dimensional and multidimensional (in space variables) models, etc.

Henceforth we shall deal with models represented in the form of differential equations in partial derivatives (PDE's). According to the general theory of such equations, a number of additional conditions should be attached to the equation itself to single out a unique element from the whole of the set of solutions to this equation.

We shall mostly speak about *linear governing equations of second order.* It means that the changes of physical variables (displacement, temperature, concentration, field intensity, etc.) are described by linear differential equations, whose coefficients represent the physical properties of the medium where the process being modeled takes place.

Example 0.1.1. The equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial z^2}, \tag{0.1.1}$$

describes small oscillations of a string. In this case the function $u(z,t)$ is the displacement of the string, at a point z and at a moment t, from the equilibrium state, which is supposed to coincide with the z-axis. The coefficient c characterizes the speed of disturbance propagation along the string. If the string is bounded, $z \in [0,l]$, and fixed at the endpoints $z=0$ and $z=l$, then the boundary conditions for (0.1.1) have the form

$$u(0,t) = u(l,t) = 0, \qquad t \in \overline{\mathbb{R}}_+, \tag{0.1.2}$$

where $\overline{\mathbb{R}}_+ \doteq [0,\infty)$. To set the boundary conditions is not sufficient to single out a unique solution to (0.1.1). We arrive at a unique solution only if some additional initial conditions are used. In this case they are the initial string displacement and the initial velocities of its points:

$$u(z,0) = \varphi(z), \qquad \frac{\partial u}{\partial t}(z,0) = \psi(z), \qquad z \in [0,l]. \tag{0.1.3}$$

One can prove that conditions (0.1.2) and (0.1.3) define a unique solution to (0.1.1).

0.2. DIRECT AND INVERSE PROBLEMS

In some statements, the coefficients of the governing equations are considered as given, thus, the problem is to study the properties of solutions to the model equations. In the case where the initial and boundary conditions are set "properly" (there exists a unique solution to the problem under study and this solution depends continuously on the parameters of the problem), we shall refer to such statements as *Direct Problems.*

The statement of each direct problem implies prescribing some set of functions, i. e., the coefficients of the equations, sources (right-hand sides of the equations), external actions (nonhomogeneities and the coefficients of the boundary conditions), etc. In the result of solving a direct problem, a new set of functions — the solutions to the direct problem — is placed into correspondence to that original set. Thus, the operator of a given direct problem is defined; i. e., the operator which maps the data of the problem into its solution.

Assume now that some functions among the data of a "properly stated" direct problem are unknown; and, instead, some additional information on the solution of the problem is given. Such problems will be referred to as *Inverse Problems.*

In particular, inverse problems of mathematical physics are often understood as problems of determining the internal characteristics of a medium (as a rule, they cannot be measured directly) from a certain information on the values of various physical fields (parameters) at the boundary of a certain domain.

Example 0.2.1. The wave propagation in a vertically-inhomogeneous medium can be described by the following problem:

$$\frac{\partial^2 u}{\partial t^2} = c^2(x_3)\Delta u, \qquad x \in \mathbb{R}^3, \quad t \in \mathbb{R}, \tag{0.2.1}$$

$$u\big|_{t<0} \equiv 0, \qquad x \in \mathbb{R}^3, \tag{0.2.2}$$

$$\frac{\partial u}{\partial x_3}\bigg|_{x_3=0} = f(t)g(x_1,x_2), \qquad (x_1,x_2,t) \in \mathbb{R}^2 \times \overline{\mathbb{R}}_+, \tag{0.2.3}$$

where $c(x_3)$ is a function characterizing the velocity of wave propagation in the medium, and $f(t)$ and $g(x_1,x_2)$ describe the duration of action and the space distribution of sources on the free surface, respectively.

Inverse Problem 0.2.1 (IP 0.2.1). *Let the wave propagation be described by the system* (0.2.1)–(0.2.3). *It is necessary to reconstruct the* *velocity distribution (to find the function $c(x_3)$) in the medium and/or the characteristics of the sources, $f(t)$ and $g(x_1, x_2)$, using the additional information about the vibration regime of the observation surface $x_3 = 0$*

$$u|_{x_3=0} = u_0(x_1, x_2, t),$$
$$t \in [0, T], \qquad (x_1, x_2) \in S \subset \{x \in \mathbb{R}^3 \mid x_3 = 0\}. \tag{0.2.4}$$

To clarify better the relations and distinctions between Direct and Inverse Problems of Mathematical Physics, we give their "cause-and-effect" interpretation. Consider the given physical parameters of a medium (e. g., density, conductivity, etc.) along with the boundary and initial conditions, the geometry of the domain, etc. as *causal* characteristics. As *effects* we obtain the states of physical fields (temperature or concentration distributions, velocity fields, etc.), which are determined by solving the corresponding direct problem. So, to solve a *Direct Problem* means to describe the *effect* of given *causal* factors. On the contrary, solution of an *Inverse Problem* is interpreted as reconstruction of causal characteristics from their effect.

Therefore, in contrast to Direct Problems, the statements of some Inverse Problems do not correspond to any physically realizable events. Indeed, one cannot invert the direction of time-flow (in order to reconstruct the initial distribution of a physical field from its state at a given moment); it is also impossible to reverse the process of reagent diffusion or heat propagation. In this sense, one can say that a number of inverse problems are "*physically incorrect*". In mathematical statements, naturally, this difficulty displays itself as *mathematical incorrectness*, which results in such complications as instability of a solution, multiple solutions, even absence of solutions, etc. These natural causes give rise to difficulties in development of reliable methods and algorithms to solve inverse problems.

That is why, in spite of existence of many general methods for solution of inverse problems, each concrete statement requires special theoretical treatment. Note that without such preliminary "analytic" investigation, it is practically impossible to create cost-effective and efficient numerical algorithms.

A natural approach to solving complex problems consists in constructing a series of models with increasing complexity that describe the initial statement more and more comprehensively. Consecutive study of these models allows us to determine, at initial stages, the most general qualitative properties of solutions. Later these general properties are determined more exactly in the course of study of more complex models.

0.3. ON CORRECTNESS OF DIRECT AND INVERSE PROBLEMS OF MATHEMATICAL PHYSICS

The notion of correctness is usually considered in the theory of Direct Problems of Mathematical Physics. When dealing with Inverse Problems of Mathematical Physics it is convenient to alter this notion a little. Below we give a short overview.

0.3.1. General notes about correct problems

The theory of differential equations states that a differential equation defines a whole set of its solutions which depend on a certain number of arbitrary constants or arbitrary functions. For the problem to have definite physical sense, we need to single out a unique solution. Usually it is achieved by setting initial and boundary conditions. This is illustrated by examples below.

Example 0.3.1.

Consider the so-called *heat equation*, which describes, e.g., the temperature evolution in a cooling body,

$$\frac{\partial u}{\partial t} = \Delta u \tag{0.3.1}$$

in a cylindrical domain $\mathcal{G} = \Omega \times \overline{\mathbb{R}}_+$, where $\Omega \subset \mathbb{R}^3$ is a domain bounded by a closed surface S. To single out a unique solution in this case it is sufficient to set the heat regime on the surface S, for instance,

$$u(x,t) = 0, \qquad x \in S, \qquad t \in \overline{\mathbb{R}}_+, \tag{0.3.2}$$

and the initial distribution of temperature inside Ω,

$$u(x,0) = g(x), \qquad x \in \Omega. \tag{0.3.3}$$

Setting of the initial and boundary conditions is aimed at singling out a unique solution from the whole class of solutions to a differential equation. But the number of these conditions should be minimal, for otherwise they may contradict one another, in which case a solution to such problem does not exist.

As it is known (see, e.g., Ladyženskaya, Solonnikov, and Ural'tseva, 1968), there exists a unique solution to the problem (0.3.1)–(0.3.3). Moreover, small enough perturbations of the initial profile $g(x)$ cause *arbitrarily* *small* deviations of this solution over any finite time interval $t \in [0, T]$. One should remember that the main goal of solving mathematical problems is to describe certain physical processes in mathematical terms. In this case the initial data are obtained experimentally; and since measurements cannot be absolutely precise, the data contain measurement errors. For a mathematical model to describe a real physical process, the problem should be supplemented with some additional requirements reflecting, in a physical sense, the fact that the solution should have only small variations under slight changes in initial data or, to put it conventionally, the *stability* of the solution under small perturbations in the initial data.

Generally speaking, in such case the problem is said to be *correct*, while in alternative cases, *ill-posed* or *incorrect*.

0.3.2. Mathematical definitions

Now we put the above general idea on the strict theoretical (mathematical) basis. Given a differential equation with concrete initial and boundary conditions, we can pose the problem of finding its solutions belonging to various functional spaces. The choice of a concrete function class depends on the physical interpretation of the problem. For example, we can consider the problem of finding a solution to (0.1.1)–(0.1.3) in the class of functions $C^2(D)$, where $D = \{(z,t) \mid z \in [0,l],\ t \in \overline{\mathbb{R}}_+\}$, or in other classes. In other words, one can choose a functional space of solutions to a differential equation in quite an arbitrary way.

The functions involved in the boundary and initial conditions of the problems for differential equations cannot be chosen arbitrarily; they should ensure that the solution belongs to the chosen functional space. For this, they should belong to the certain special functional space corresponding to the space of solutions. This becomes clearer if one considers problems for differential equations from the viewpoint of functional analysis. Choose a space U for the solutions of a differential equation. The differential equation together with some additional conditions defines the operator A that relates any solution $u \in U$ to the set of functions involved in the additional (initial and/or boundary) conditions. For (0.1.1) these are the functions φ and ψ, for (0.3.1) it is the function g. Considering this set of functions as an element f of a functional space F, one comes to a conclusion that solving a problem for a differential equation is equivalent to solving the formal operator equation

$$Au = f \tag{0.3.4}$$

under the condition that $u \in U$.