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Representation Theory of Lie Groups

Proceedings of the SRC/LMS Research Symposium on Representations of Lie Groups, Oxford, 28 June - 15 July 1977

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1 · Introduction

M.F. ATIYAH

Lie groups and their representations occupy an important place in mathematics, with applications and repercussions over a wide front. The connections with various aspects of physics are of long-standing, as are the intimate relations with differential equations and differential geometry. More recently the global topology of Lie groups has provided a deep link with questions of number theory. Finally, when viewed as 'non-commutative harmonic analysis' the theory of representations is a branch of linear analysis.

The symposium held in Oxford in July 1977 was designed to provide an introduction to the representation theory of Lie groups on as wide a front as possible. The main lectures, which are reproduced in this volume, should give the reader some indication of the scope and results of the subject. Inevitably there are gaps in various directions, and some areas are treated in greater detail than others. This reflects the particular interests of the participants and is not to be taken as a measure of relevant importance. Broadly speaking the symposium centred on the classical case of real Lie groups and treated only briefly the p-adic and finite fields.

In Part I of these notes we have collected together the introductory material and in Part II, the more advanced lectures.

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Part I

2 · Origins and early history of the theory of unitary group representations

G.W. MACKEY
Harvard University

The theory of group representations was created by Frobenius in 1896 in a more or less deliberate attempt to generalize the theory of characters of finite abelian groups. The latter notion was only formally defined in full generality by Weber in 1881. Weber's definition was an abstraction of one given three years earlier by Dedekind and Dedekind was more or less directly inspired by Gauss' implicit use of characters of order two in his *Disquisitiones Arithmeticae* published in 1801.

To go back a bit further, Lagrange in the early 1770's wrote a two-part memoir making a systematic study of equations of the form

$$Ax^2 + Bxy + Cy^2 = n \quad .$$

Here A, B, C and n are integers and the problem is to find all integer pairs x,y satisfying the equation. Various special cases had been studied by Fermat in the seventeenth century and by Euler in the eighteenth and Lagrange's aim was to construct a systematic general theory. He observed that the transformation

$$x^1 = ax + by \ , \quad y^1 = cx + dy$$

where a, b, c and d are integers with $ad-bc = 1$ carries the equation into an equivalent one having the same values for the 'discriminant' $B^2 - 4AC$ and proved that there can be at most a finite number of inequivalent equations with a given value $D = B^2-4AC$. This number, called the class number, is of key importance in the developed theory. Gauss in the work cited above defined a notion of 'composition' for equivalence classes of forms of a given discriminant (his definition of equivalence was not quite the same as that of Lagrange) and showed in effect that under this composition law the equivalence classes form a group. We say 'in effect' because the concept of 'group' did not then exist. In developing the theory of equations whose class number is greater than one, he used what amounted to characters of order two of the group of equivalence classes and in this connection introduced the word 'character'.

As defined by Weber, a character of a finite abelian group A is a homomorphism $x \to \chi(x)$ of A into the multiplicative group of complex numbers of modulus one. It is evident that the set $\hat{A}$ of all characters of A is itself a finite abelian group under multiplication. Moreover, it is not hard to see that every complex-valued function f on A may be written uniquely as a linear combination of characters

$$f = \sum_{\chi \in \hat{A}} C_\chi \cdot \chi \quad \text{where} \quad C_\chi = \frac{1}{o(A)} \sum_{x \in A} f(x)\, \overline{\chi(x)}$$

and $o(A)$ is the order of A. The analogy with Fourier series expansions is evident and many arguments in nineteenth century number theory may be interpreted as Fourier analysis on finite abelian groups. Dirichlet, in particular, used characters on the multiplicative group of units in the ring of integers mod m and finite Fourier analysis is the key to one step in his celebrated proof that there are an infinite number of primes in any arithmetic progression which can not be extended to contain zero.

The primary impetus to the development of group theory itself was provided by another long memoir of Lagrange published shortly after the one mentioned above. In it he made a penetrating study of the solutions of polynomial equations by radicals. He managed to understand in a unified way the known methods for solving equations of the second, third and fourth degrees and tried (nearly successfully) to understand why the fifth degree equations had proved so intractible. In particular, he saw that the key to the question lay in studying what happened to rational functions of the roots when the roots were permuted amongst themselves. Inspired by this work of Lagrange, Cauchy founded the theory of permutation groups in 1815 and by 1831, Ruffini and Abel had proved the impossibility of solving the general quintic and Galois had worked out his beautiful theory relating solvability to the structure of the 'Galois group' of the equation. It is to Galois that we owe the term group and the concept of normal subgroup. On the other hand, the theorem that the order of a subgroup divides the order of the group is already implicit in Lagrange's paper.

For various reasons, including Galois' premature death at the age of 20, his paper was not published until 1846. At this time Hermite and Kronecker were young men at the beginning of their careers and both became quite active in developing Galois' ideas. However, group theory did not begin to be widely known or to be applied outside of a rather narrow context until around 1870. At that time, three events occurring in the space of as many years, stimulated a considerable expansion in the scope of group theory as well as an increased awareness of the existence and importance of this new branch of mathematics. In 1869 Sophus Lie began to apply the ideas of Galois to differential equations and initiated the systematic study of continuous (actually differentiable) groups. In 1870 C. Jordan published the first book ever to be written on group theory. His *Traités des substitutions et des equations algébriques* contained among other things a clear exposition of Galois theory. Finally, in 1872 Felix Klein announced his celebrated Erlanger program for unifying geometry through group theory and shortly thereafter began a sort of publicity campaign to convince mathematicians of the fruitfulness and wide applicability of the group theoretic point of view.

The parallelism between Fourier analysis on finite commutative groups as indicated above and Fourier analysis as more commonly understood arises of course because the functions $x \to e^{inx}$ are precisely the continuous characters on the compact continuous group obtained from the additive group of the real line by factoring out the discrete subgroup of all integer multiples of 2π. However, the fact that such a connection exists does not seem to have been explicitly noticed until the middle 1920's. The theory of Fourier series and integrals arose in the early nineteenth century to meet the needs of mathematical physics. In the middle of the eighteenth century D. Bernoulli, D'Alembert and Euler succeeded in extending Newton's analysis of particle motion to an analysis of the motion of fluids and deformable solids. More precisely, they found the analogues of Newton's equations of motion. These turned out to be differential equations in which partial derivatives of functions of several variables replaced the ordinary derivatives in Newton's work. Such partial differential equations presented mathematicians with a new and difficult challenge which was by no means met immediately. Progress was slow until Fourier submitted his celebrated memoir on heat conduction to the French academy in 1807. The methods which Fourier used and which are now taught to every mathematics and physics student were quickly seen to apply to many of the partial differential equations arising in physical problems and by the time Fourier's book on heat conduction appeared in 1822, Poisson and Cauchy had been active for years in applying them to a variety of problems. Actually Fourier's expansibility theorem was nearly discovered half a century earlier in connection with studies of the one dimensional wave equation

$$\frac{1}{\nu^2}\frac{\partial^2\Psi}{\partial t^2} = \frac{\partial^2\Psi}{\partial x^2} .$$

However, prejudices of the time made the result implausible to many and in the end the key clues were ignored. Lagrange who developed and systematized the work of Euler, Bernoulli and D'Alembert and incorporated it into his great synthesis of 1787 *Mêcanique Analytique* came close to finding Fourier's theorem but he also refused to accept it. In fact he was one of the referees who at first rejected Fourier's memoir of 1807.

A group representation as defined by Frobenius is a homomorphism $x \to L_x$ of a finite group G into the multiplicative group of all $n \times n$ non-singular complex matrices for some $n = 1,2,\ldots$. Its character χ^L is the complex valued function on G defined by $\chi^L(x) = \text{Trace}\ (L_x)$. This definition evidently reduces to that of Dedekind and Weber when $n = 1$. More generally one shows that $\chi^L(x) \equiv \chi^M(x)$ if and only if L and M have the same dimension (or degree) n and there exists a non-singular $n \times n$ matrix W such that $W^{-1}L_xW = M_x$ for all x. One then says that L and M are equivalent. One also shows that for each finite group G there exists a unique finite set $\chi_1, \chi_2, \ldots, \chi_r$ of linearly independent characters on G such that the finite linear combinations $n_1\chi_1 + n_2\chi_2 + \ldots + n_r\chi_r$ (where the n_j are non-negative integers) are precisely the characters of G . Here r is the number of distinct conjugacy classes of G and finding the χ_j (the so-called irreducible characters) can be a highly non-trivial problem.

The immediate stimulus for Frobenius' introduction of group representations and their characters was a problem of Dedekind concerning a little-known concept - the group determinant - which he began to work on in the 1880's. He could solve it in some cases using characters of finite groups and solicited the help of Frobenius in dealing with more general ones. Apparently the problem in group determinants was suggested by the study of the discriminant of an algebraic number field. Frobenius succeeded using his new generalized characters - which he invented expressly for the purpose. The exact story of the relationship between Dedekind's problem and the introduction of higher dimensional characters is complicated and has only recently been elucidated. For further details the reader is referred to three recent articles by Thomas Hawkins in *Archiv for the history of the exact sciences*.

For the next quarter of a century or so the theory of group representations was a branch of pure algebra concerned more or less exclusively with the development of Frobenius' ideas by Frobenius himself, by Burnside and by I. Schur and others. There were striking applications to the structure theory of finite groups (for example, the theorem of Burnside that a group whose order is divisible by only two primes is solvable) but none outside of group theory. However, in the 1920's the situation changed radically. The scope of the theory was enlarged so as to apply to compact Lie groups by work of Hurwitz, Schur, Cartan and Weyl and at the same time important applications were found to number theory and to the new quantum physics.

In 1924 Schur observed that one could apply earlier ideas of Hurwitz on integration over manifolds to define integration of continuous functions defined on compact Lie groups. Using this as a substitute for summing over the group he was able to extend the main ideas of group representation theory from finite groups to compact Lie groups. He also was able to determine all of the irreducible representations of the orthogonal groups. In the next three years Weyl determined the irreducible representations (and their characters) of all the classical compact semisimple Lie groups and in collaboration with F. Peter proved the celebrated Peter-Weyl theorem. This asserts in essence that the matrix coefficients of the irreducible representations of a compact Lie group are plentiful enough so that every continuous function on the group can be uniformly approximated by their linear combinations. It follows that one can obtain an orthonormal basis for the square integrable functions on the groups whose members are such matrix elements. When the group is commutative the basis elements are necessarily complex multiples of characters and the Riesz-Fischer theorem in the theory of

Fourier series is a special case. It was in this work of Weyl that the group theoretical character of classical harmonic analysis was first clearly pointed out. Weyl also pointed out that the classical theory of expansions in spherical harmonics has a group theoretical interpretation and moreover one demanding consideration of higher dimensional representations of a non-commutative group. This observation of Weyl was generalized and further developed by E. Cartan. On the other hand Weyl made heavy use of earlier work of Cartan on the Lie algebras of the classical compact Lie groups in his determination of their representations. Cartan in earlier work had given what amounted to an infinitesimal version of Weyl's results.

The first application of the theory of higher dimensional group representations outside of group theory itself seems to have been made by E. Artin in 1923. Let K be an algebraic number field; that is, a finite extension of the field Q of all rational numbers and let R_K be the ring of all 'algebraic integers' in K. Let H be any subgroup of the group G of all automorphisms of K and let k_H be the subfield of all elements of K which are carried into themselves by all members of H. The so-called zeta function ζ_K of K is defined for all $\mathrm{Re}(s) > 1$ by the convergent Dirichlet series

$$\sum_{n=1}^{\infty} \frac{\phi(n)}{n^s}$$

where $\phi(n)$ is the number of ideals I in R_K whose 'norm' $N(I)$ is n. By definition $N(I)$ is the number of elements in the quotient ring R_K/I. As shown by Hecke a few years earlier ζ_K is always continuable to a meromorphic function defined in the whole complex plane and satisfying a simple functional equation relating the values of $\zeta_K(s)$ to those of $\zeta_K(1-s)$. Of course, one can define ζ_{k_H} in the same way and the question arises as to the relationship between these two zeta functions. In the special case when H is commutative it was known from previous work of Takagi that ζ_K factors as a product of ζ_{k_H} and o(H)-1 'L functions'. These L functions are also analytically continuable Dirichlet series of the form

$$\sum_{n=1}^{\infty} \frac{\Psi(n)}{n^s}$$

which satisfy simple functional equations and have no poles. Artin extended this result in two ways. First he obtained a completely different factorization of ζ_K/ζ_{k_H} valid for all H, commutative or not, and with the factors parameterized by the irreducible characters of H other than the trivial one. The corresponding factorization of ζ_K reflects the decomposition of the regular representation of H in that each factor occurs as many times as the corresponding irreducible representation occurs in the regular representation of H. Secondly when H is commutative he showed that his (conceptually) completely different factorization was the same as that of Takagi. In other words he showed that Takagi's results implied a reinterpretation of the classical L functions in terms of one dimensional characters of H and that the theory of group representations could be used to remove the restriction that H be commutative. Artin's generalized L functions are known to share many properties of the classical ones. However it is still an open question as to whether they are entire.

A rather different application of the theory of group representations to number theory was made by E. Hecke in 1928. Actually this application was made indirectly via the theory of modular forms - a theory having extremely close connections with number theory. Let ℓ and k be positive integers and let Γ_ℓ be the group of all 2×2 matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ for which ad-bc = 1 and a-1, b, c and d-1 are all integer multiples of ℓ. Then a modular form of weight k and level ℓ is an