PRELIMINARY EDITION VOLUME 1

# Conversational CALCULUS


David W. Cohen

James M. Henle

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# PREFACE

## Our Premise

*Conversational Calculus* is organized around the premise that Calculus is a language. The best way to learn a language is to go where it is spoken, hear it employed by native speakers, and then use it in real situations. Start with a little vocabulary and a little grammar, but begin as soon as possible to listen and talk. We've arranged this book with that model in mind.

We begin with meaningful vocabulary. In the first few pages we introduce the derivative, the integral, and what is essentially the Fundamental Theorem of Calculus. With few preliminaries the student is using the language in familiar contexts—without technical details, but in ways that should seem both natural and useful. While using the language to explore complicated situations, we introduce richer vocabulary and technical details as they're needed.

## The Structure of the Book

The book is divided into two parts. The first part contains the five basic chapters, which contain **all** the rudiments for a one-semester calculus course. We expect that most classes will complete these chapters in less than a full semester.

In the second part we present the wealth of calculus in the form of individual "conversations." Here is where students have the opportunity to turn into fluency and mastery the rudiments they learned in the first part. The conversations are written in the language of calculus; they apply it, explore it, and expand it. They are the real situations. Their subjects vary tremendously. Some are historical, some mathematical, some philosophical, some playful. Most, however, deal with our physical and social world.

All the mathematics needed in the second part is presented in the first part, so the conversations needn't be assigned in any particular order. This makes it easy for teachers and students to select topics to suit their own interests and priorities.

## Computers

We expect students to use computers or graphing calculators. They are an effective means for illustrating the geometry of the calculus and for exposing its infinite character. We have made available a public domain graphing utility, *Graph*, and a differential equations solver, *Slinky*. They are exceptionally easy to use, but others (*Maple*, *Mathematica*, or many graphing calculators) may be substituted. Utilities such as these are as essential to the modern student of mathematics as microscopes are to students of biology.

## The Level of the Book

This is not "calculus made easy." Calculus can't be made easy. The material in *Conversational Calculus* is challenging and significant. It's also relevant, exciting, and immediately meaningful.

This is not "computer calculus." The dynamic, algebraic, linguistic, and geometric characters of calculus are the subject of this book. We use computers, but they are just a tool.

This is not "calculus made difficult." This is a book that any teacher or student can pick up without special preparation. No seminars, workbooks, or tutorials are required. The book has been written to be understood.

## Our Motivation

In the first half of the twentieth century scientists, physicists, and engineers came up with beautiful and exciting theories and models to explain and predict the physical world. Many of these models were expressed in the language of differential equations. Students were taught calculus so that they might eventually understand enough about differential equations to explore the physical world.

Unfortunately, the differential equations of significance were unsolvable, even by the scientists who needed the solutions. So students never got the chance to explore real–life problems. Instead, either they became interested enough in the theory to study the differential equations themselves or they lost interest. There were few opportunities to apply what they learned in their calculus course to the real world.

All that changed with the development of inexpensive, high-speed computers. Today's computers can solve differential equations with very high numerical accuracy, can draw graphs of solutions, and can even find closed-form solutions instantly for many differential equations. The computer puts a tremendous amount of power into the hands of biologists, chemists, economists, medical researchers, conservationists, and calculus students.

Thus, in the past few years the calculus curriculum has begun a significant course correction. The study of differential equations is no longer the distant goal of introductory calculus; it's at the heart of it. Now computers and graphing calculators solve initial value problems and provide beautifully accurate graphs. Beginning calculus students now can address significant problems that were beyond the scope of advanced researchers less than twenty-five years ago.

This book shows students the way to address some of those problems, and in addition provides them with the vocabulary, mathematical tools, and structural framework to address problems that won't even be posed until years after they've completed this book.

Calculus has always been a language, but only in recent years has it become so important for so many students, headed in so many directions, to learn to speak it. Calculus is no longer the "pinnacle" math course for future scientists and social scientists. It belongs in the basic vocabulary of every educated person. We wrote *Conversational Calculus* to help put it there.


## Acknowledgements

This book bears an enormous debt to the vision of Jim Callahan and Ken Hoffman who first imagined a better calculus course. We are especially grateful for Jim's kindness and encouragement over many years.

Over several summers we have been assisted by very talented undergraduates: Kristy Anderson, Allison Crawford, Mollee Dissell, Aimee Foreman, Alison Ra, Rebecca Rouselle, Liz Stuart, and Marisa Wallace. The illustrations that illuminate the text, indeed they almost animate it, are their special contribution.


David Cohen
James Henle

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# Annotated CONTENTS for the Instructor


## 1 WORDS ... 1

Introduces two symbols of the language, $'$ and $\int$, and the words derivative and integral.

## 2 PICTURES ... 10

Discusses the geometric significance of derivative (rising and falling graphs) and integral (change in graph height).

### Conversation: Looking for the Best ... 29

Introduces extremal problems, solving them approximately by graphing with computers or graphing calculators.

### Conversation: Expecting the Worst ... 32

Uses pictures to compare linear and exponential growth ($y' = k$ vs. $y' = ky$)

## 3 NUMBERS ... 38

Computes derivatives using the difference quotient (which we call the "Newton quotient") and computes definite integrals using Euler's method (which we call "Euler sums").

### Conversation: Growth and Decay ... 53

Introduces the number $e$, and explores exponential growth and decay.

### Conversation: Paying Back Your Student Loan ... 59

Uses "integrators" (computers or graphing calculator utilities that solve differential equations numerically) to explore loan payment plans.

## 4 FUNCTIONS ... 62

Introduces slope fields to picture solutions to differential equations and applies graphers to the Newton quotient to picture derivative functions.

### Conversation: Exact Answers ... 82

Discusses the philosophical and mathematical problem of approximation vs. exactness

### Conversation: Baking Bread ... 90

Models the logistic growth of yeast.

## 5 FORMULAS ... 95

Covers the basic algebraic formulas for differentiation and integration.

### Conversation: Glottochronology ... 115

Models the evolution of language using exponential decay.

### Conversation: If Curves Were Straight ... 120

Describes Newton's method for finding roots of functions.

## 6 CONVERSATIONS


**Conversation: "Boston Mystery—Found Dead" ... 126**

Models the spread of disease with a system of differential equations. (The "SIR" model)

**Conversation: Chaos ... 131**

Describes mathematical chaos and generates chaotic behavior by applying Newton's method to a function with no roots.

**Conversation: Discrete vs. Continuous ... 137**

Derives a formula for the Fibonacci sequence using a differential equation; introduces Riemann sums by computing $\int_0^1 x^2 dx$ using the formula for $\Sigma_{i=1}^n i^2$.

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**Conversation: The Error in Euler ... 152**

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**Conversation: Fractals ... 159**

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**Conversation: The Gentle Rain from Heaven ... 168**

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**Conversation: How Old Is the Shroud of Turin? ... 178**

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**Conversation: An Intelligent Beam of Light ... 181**

Explains the reflection and refraction of light as an extremal problem.

**Conversation: Just Noticeable Difference ... 186**

Derives Weber's law of sensation, an application of calculus to psychology.

**Conversation: Making the Most Money ... 188**

Solves extremal problems by setting the derivative to zero; applies the method to maximizing profit.

**Conversation: A Murder of Quantity ... 194**

Newton's law for cooling bodies.

**Conversation: Ozone ... 196**

Models chemical reaction in the atmosphere with a system of differential equations.

**Conversation: Tangents and Areas ...** 199

Relates tangents and derivatives, and areas and integrals, and presents the Fundamental Theorem of Calculus.

**Conversation: Why? ...** 208

Justifies the differentiation rules in FORMULAS: trigonometric functions, the power rule, product rule, chain rule, and the derivative of ln.

# Chapter 1

# WORDS

> One cannot escape the feeling that these mathematical formulae have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers.
>
> *Heinrich Hertz*

Calculus is a language. Calculus is also an elegant and profound mathematical theory, and a system of calculation. But the soul of calculus is language. Its greatest importance lies in its ability to describe the universe. This ability is manifold and far-reaching. Calculus can give tongue to biology, economics, physics, sociology, chemistry, and finance. In these and other areas it can detail the most subtle and intricate ideas.

## 1. Rates

Calculus adds new words, symbols, and grammar to mathematics. But at the core of this new language is one significant stroke:

$$'$$

The idea is simple. If a variable $x$ represents a certain quantity, then $x'$ represents the rate at which that quantity is changing. (We read $x'$ as "x prime.") Calculus is about change, and the prime ($'$) is what we use to discuss change.

**Example:** *We were about 20 miles from Detroit when we realized we'd left the lasagna in the oven.*

Not much to translate here. We can let $d$ be the distance from Detroit and write:

$$d = 20 \text{ (miles)}.$$

*That was at 11:30.*

To be precise, $d$ is a function of time, and what we really know is that at a specific time (11:30), $d$ was 20. When we want to emphasize that $d$ is a function of time, we write $d(t)$ for the function instead of just $d$, and we can write

$$d\left(11\frac{1}{2}\right) = 20$$

to specify the value of $d$ at a particular time.

*At 11:30 we were doing the speed limit, 55, and Minnie was drawing on the window with crayons.*

The 55 represents how fast the distance $d$ is changing.

$$d'\left(11\frac{1}{2}\right) = 55 \text{ (miles per hour).}$$

*So we turned around and raced back. By 11:40 we were heading back to Detroit, and exceeding the speed limit by 7 mph.*

Our speed and direction have changed. Now we have

$$d'\left(11\frac{2}{3}\right) = -62 \text{ (miles per hour).}$$

The rate of change of $d$ is negative because $d$ is the distance from Detroit, and now it is decreasing.

*We sped along at 62 for the next 20 minutes*

$$d'(t) = -62 \text{ for } 11\frac{2}{3} \leq t \leq 12$$

We call $d'$ the **derivative** of $d$. The derivative of $d$ is the rate of change of $d$.

## Exercises: Translation Practice

The object here is to translate the mathematical content of English sentences into equations. We've shown you how to answer some. We leave others to you. *Warning:* Some sentences do not use the derivative ($'$). Some sentences use *several* derivatives. Also, some sentences have absolutely no mathematical content!

1. *My net worth* (call it $W$) *was about $55,000 last year.* [ Answer: $W = 55,000$ ]
2. *My net worth is growing at about $6,000 each year.* [ Answer: $W' = 6000$ ]
3. *The population of Malawi was increasing by 110,000 per year in 1973.*
4. *The population of Malawi was 5.04 million in 1975.*
5. *It was a dark and stormy night.*
6. *The world's rain forests are decreasing constantly at a rate of 300 square miles each year.*
7. *Simon weighs about 50 pounds more than Mona.*
8. *But Mona runs twice as fast as Simon.*
9. *She also shucks corn faster. For every two ears Simon shucks, Mona shucks three.*
10. *I'm holding a book 5 feet off the ground.* [Answer: $h = 5$ ft.]
11. *At the moment I let the book go, its velocity is 0.* (*Hint:* Velocity is the rate of change of distance.)
12. *During its fall, the velocity is increasing in absolute value at the constant rate of 32 feet per second per second.* (*Hint:* The velocity is $h'$, and the rate of change of $h'$ is $(h')'$, which we write as $h''$ and call *acceleration*. Note that the book is falling *down*, not up. This affects the signs of $h'$ and $h''$.)
13. *The national debt is 2 trillion dollars!*
14. *This year, the annual budget deficit is 350 billion dollars!* (*Hint:* The annual deficit is the rate at which the debt is growing.)
15. *The good news is that the deficit is decreasing by about 50 billion dollars a year.*

Most of these exercises involved either a quantity or its derivative. In many applications it's necessary to use *both* the quantity and its derivative in one equation.

**Example:** *At the start of the year I put \$357 into a money market account. They're only giving me 2% interest per year. I'll never have enough for a Lexus.*

The first sentence introduces nothing new. We can express it as

$$m(0) = 357 \text{ (dollars)},$$

where $m$ is a function of time. The second sentence introduces a new idea, growth as a percentage. Now 2% of \$357 is \$7.14. When I opened the account, my balance started growing at the rate of \$7.14 per year. We can write $m'(0)$ to stand for the growth rate of my account when I opened it (when $t = 0$):

$$m'(0) = .02 \cdot 357 = 7.14 \text{ (dollars per year)}.$$

After a while, however, $m$ will increase, hence 2% of $m$ will increase, which means that the growth rate of $m$ will also increase. We can write all of that in one general equation because $m'$ will *always* be 2% of $m$:

$$m'(t) = .02m(t) \text{ for all times } t.$$

There you have it–a quantity and its derivative (rate) in the same equation. Equations such as this, which express a relationship true for all values of $t$, are often written without the $t$:

$$m' = .02m.$$

This is an example of an equation whose derivative (the rate) is proportional to the quantity. The word *proportional* is extremely useful. When we say that one quantity $A$ is **proportional** to a quantity $B$ (or we sometimes say **directly proportional** to $B$), it means that $A$ is a multiple of $B$, or

$$A = kB$$

for some constant $k$. We call $k$ the **constant of proportionality**. So we translate the expression

$$m' = .02m$$

this way: *$m'$ is proportional to $m$, and the constant of proportionality is .02.*

## Exercises: More Translation Practice

16. *The population of Malawi was 4.9 million in 1974 and was increasing at 2.8% per year.*
17. *The population of Malawi was 4.79 million in 1973, but it was increasing at 2.3% per year.*
18. *I don't know what the population of Malawi was in 1983, but it was increasing at the rate of 2.7% per year.*
19. *I don't know what the population of Malawi is now, and I don't really know how fast it's increasing, except I know the rate of change is proportional to the population.*
20. *The number of people with O-negative blood in a certain country is proportional to the total population.*
21. *Radioactive elements tend to decay. Barium 140 is a good example. If you leave a lump of this stuff alone, it will gradually change into something else. The rate at which it does this is always proportional to the amount of barium 140.*
22. *Although the water tank holds 100,000 gallons, it's only half full right now because there's a leak in the bottom.*
23. *The sides of the tank are straight, so the volume is proportional to the height of the water in the tank.*
24. *It is an interesting fact that the rate at which water leaks out of tanks like this is directly proportional to the square root of the height of the water above the leak.*
25. *This sentence has absolutely no mathematical content.*
26. *When an object is dropped, its velocity is proportional to the length of time it has been falling.*
27. *When an object is dropped, its velocity is proportional to the square root of the distance it has fallen.*

Now let's try translating from mathematical equations to English.

The inspector absent-mindedly tapped her pencil on the desk as she studied the printout from the flight recorder, which was recovered from the wreckage. The printout showed $h(t)$, the height of the airplane (in feet) from the ground, at every second from 1:02 P.M. to 1:07 P.M., the time of the crash. It also showed $h'(t)$ at every second in that same time interval.

28. *This says that* $h(1:02:00) = 4000$ *and* $h'(1:02:00) = 0$*, she said thoughtfully. Well, that tells us something.* What does it tell us?

29. *I see that* $h(1:03:30) = 3800$, yet $h'(1:03:30) = 0$ again, although $h'(t)$ was negative for $t$ just after 1:02:00 and just before 1:03:30. Where was the aircraft and what was it doing at 1:03:30? What happened between 1:02:00 and 1:03:30?
30. *Look at this!* $h'(1:05:00) = -300$. *That looks like trouble!* Why? What was happening?
31. *But* $h(1:06) = 1000$ *and* $h'(1:06) = 200$. *That's strange!* Why? *Now what was happening?*
32. *This is interesting:* $h(1:06:40) = 200$ *and* $h'(1:06:40) = -10$. *That's why so many lives were saved.* What does the inspector mean?