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Xuehou Tan (Eds.)

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Preface

This volume consists of the refereed proceedings of the Japan Conference on Discrete and Computational Geometry (JCDCG 2004) held at Tokai University in Tokyo, Japan, October, 8–11, 2004, to honor János Pach on his 50th year. János Pach has generously supported the efforts to promote research in discrete and computational geometry among mathematicians in Asia for many years. The conference was attended by close to 100 participants from 20 countries.

Since it was first organized in 1997, the annual JCDCG has attracted a growing international participation. The earlier conferences were held in Tokyo, followed by conferences in Manila, Philippines, and Bandung, Indonesia. The proceedings of JCDCG 1998, 2000, 2002 and IJCCGGT 2003 were published by Springer as part of the series *Lecture Notes in Computer Science* (LNCS) volumes 1763, 2098, 2866 and 3330, respectively, while the proceedings of JCDCG 2001 were also published by Springer as a special issue of the journal *Graphs and Combinatorics*, Vol. 18, No. 4, 2002.

The organizers of JCDCG 2004 gratefully acknowledge the sponsorship of Tokai University, the support of the conference secretariat and the participation of the principal speakers: Ferran Hurtado, Hiro Ito, Alberto Márquez, Jiří Matoušek, János Pach, Jonathan Shewchuk, William Steiger, Endre Szemerédi, Géza Tóth, Godfried Toussaint and Jorge Urrutia.

July 2005

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Table of Contents

Matching Points with Circles and Squares

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Abstract. Given a class $\mathcal{C}$ of geometric objects and a point set P, a $\mathcal{C}$-*matching* of P is a set $M = \{C_1, \ldots, C_k\}$ of elements of $\mathcal{C}$ such that each C_i contains exactly two elements of P. If all of the elements of P belong to some C_i, M is called a *perfect matching*; if in addition all the elements of M are pairwise disjoint we say that this matching M is *strong*. In this paper we study the existence and properties of $\mathcal{C}$-matchings for point sets in the plane when $\mathcal{C}$ is the set of circles or the set of isothetic squares in the plane.

1 Introduction

Let $\mathcal{C}$ be a class of geometric objects and let P be a point set with n elements $p_1, \ldots, p_n$ in general position, with n even. A $\mathcal{C}$-*matching* of P is a set $M = \{C_1, \ldots, C_k\}$ of elements of $\mathcal{C}$, such that every C_i contains exactly two elements of P. If all of the elements of P belong to some C_i, M is called a *perfect matching.* If in addition all of the elements of M are pairwise disjoint we say that the matching M is *strong.*

Let $G_{\mathcal{C}}(P)$ be the graph whose vertices are the elements of P, two of which are adjacent if there is an element of $\mathcal{C}$ containing them and no other element from P. Then, a perfect matching in $G_{\mathcal{C}}(P)$ in the usual graph-theoretic sense corresponds naturally with our definition of $G_{\mathcal{C}}(P)$-matchings.

There are several natural classes $\mathcal{C}$ of geometric objects of interest, including line segments, isothetic rectangles, isothetic squares, and disks. For these cases, we refer to the corresponding matching M as a *segment-matching*, a *rectangle-matching*, a *square-matching*, or a *circle-matching*, respectively. Notice that these four classes of objects have in common the *shrinkability* property: if there is an object C' in the class that contains exactly two points p and q in P, then there is an object C'' in the class such that $C'' \subset C'$, p and q lie on the boundary of C'', and the relative interior of C'' is empty of points from P. In the case of rectangles we can even assume the points p and q to be opposite corners of C''.

It is easy to see that P always admits a strong segment-matching and a strong rectangle-matching, which in fact are respectively non-crossing matchings in the

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complete geometric graph induced by P (in the sense in which geometric graphs are defined in [4]) and in the rectangle of influence graph associated with P [3].

On the contrary, the situation is unclear for circles and squares, and gives rise to interesting problems, That is the topic of this paper, in which we study the existence of perfect and non-perfect, strong and non-strong matchings for point sets in the plane when C is the set of circles or the set of isothetic squares in the plane.

It is worth mentioning that our results on square-matchings prove, as a side effect, the fact that Delaunay triangulations for the L_1 and L_∞ metrics contain a Hamiltonian path, a question that, to best of our knowledge, has remained unsolved since it was posed in the conference version of [2].

2 Matching with Disks

In this section we study circle-matchings. We show that a perfect circle-matching is always possible, but that there are collections of points for which there is no perfect strong circle-matching. We give bounds on the size of the largest strong circle-matching that any set P of n points admits. We also consider the special case in which the point set P is in convex position.

2.1 Existence of Circle-Matchings

First, notice that the fact that two points from P can be covered by a disk that contains no other point in P is equivalent to fact that the two points are neighbors in the Delaunay triangulation of P, $DT(P)$. In other words, when C is the set of all circles on the plane, the graph $G_C(P)$ is $DT(P)$. Thus, a point set admits a circle-matching if and only if the graph $DT(P)$ contains a perfect matching, which is the case if and only if P has an even number of points, as proved by Dillencourt in 1990 [2]. Therefore we get the following result, which is a direct consequence of Dillencourt's result:

Theorem 1. *Every point set with an even number of elements admits a circle-matching.*

Nevertheless, for even n, a perfect strong circle-matching is not always possible, as we show next. Consider a circle C with unit radius and a point set P with n elements $p_1, \ldots, p_n$, where $p_1 = a$ is the center of C and $p_2, \ldots, p_n$ are points evenly spaced on the boundary of C. The point a must be matched with some point $b \in \{p_2, \ldots, p_n\}$; this forces the rest of points to be matched consecutively (see Figure 1). In particular, the following and preceding neighbors of b on the boundary must be matched using "large" circles, with centers outside of C, and these circles are forced to overlap for n large enough. In fact, elementary trigonometric computations show that this happens exactly for $n \geq 74$.

We do not describe here the details of the preceding construction. The underlying idea of the construction, however, can be extended to yield an arbitrarily large set of points such that at most a certain fraction of the points can be strongly matched. More precisely, the following result holds:

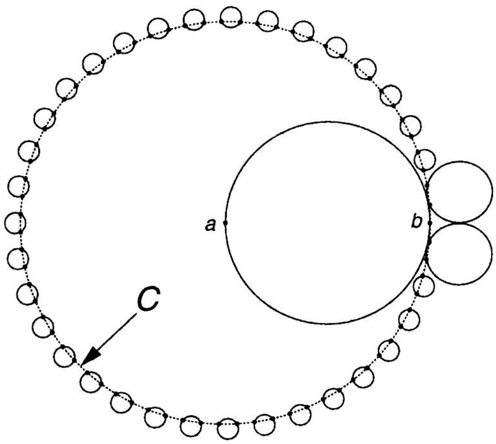

Fig. 1. The elements of a set S are $n-1$ points evenly distributed on C and the center of C. For $n \geq 74$ this point set does not admit a strong perfect circle-matching.

Theorem 2. *There is an n-element point set in the plane, where n can be arbitrarily large, such that at most a fraction* $\frac{72}{73}n$ *of its points can be strongly circle-matched.*

The proof of this result is omitted from this extended abstract, since it is very long and requires several technical lemmas.

2.2 Subsets That Can Be Strongly Matched

According to Theorem 2, not every point set P admits a strong circle-matching. In this subsection, we prove that for any point set P one can always find a linear number of disjoint disks each one covering exactly two points from P:

Theorem 3. *For any set P of n points in general position, there is a strong circle-matching using at least* $2\lceil (n-1)/8 \rceil$ *points of P.*

The proof utilizes several lemmas. Let M be a matching of $m = \lfloor n/2 \rfloor$ pairs of (distinct) points of P that minimizes the sum of the squared Euclidean distances between pairs of matched points. Let the m pairs in the matching M be $(p_1, q_1), (p_2, q_2), \ldots, (p_m, q_m)$. We let $|p_i q_i|$ denote the Euclidean distance between p_i and q_i. We let $D_i = DD(p_i q_i)$ be the diametrical disk, with segment $p_i q_i$ as diameter, and let o_i denote the center of this disk. Let $\mathcal{C} = \{D_1, D_2, \ldots, D_m\}$ be the set of diametrical disks determined by the pairs (p_i, q_i) of the matching M.

Lemma 1. *If $DD(ab), DD(cd) \in \mathcal{C}$ then $\{c, d\} \nsubseteq DD(ab)$.*

Proof. Suppose that $c, d \in DD(ab)$. Note that $\angle dcb + \angle bdc < \pi$, so we may assume that $\angle dcb < \pi/2$. Thus $|bd|^2 < |cd|^2 + |bc|^2$, and since $c \in DD(ab)$, we know that $\angle bca \geq \pi/2$ and $bc^2 + ac^2 \leq ab^2$. Combining these inequalities we get $|bd|^2 + |ac|^2 < |ab|^2 + |cd|^2$, contradicting the optimality of M. □

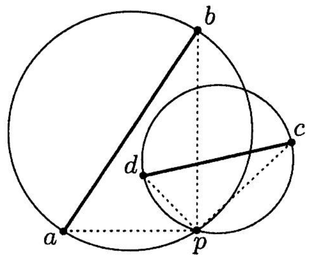

Fig. 2. Proof of Lemma 2

Lemma 2. *If $DD(ab), DD(cd) \in C$ and p is in the intersection of the bounding circles of $DD(ab)$ and $DD(cd)$, then triangles $\triangle apb$ and $\triangle dpc$ do not overlap.*

Proof. Suppose that $\triangle apb$ and $\triangle dpc$ overlap. Assume $\overrightarrow{pd}$ is between $\overrightarrow{pb}$ and $\overrightarrow{pa}$ as in Figure 2. Since $\overline{ab}$ and $\overline{cd}$ are diameters of their respective circles, we know that $\angle dpc = \angle apb = \pi/2$, implying that $\angle apd < \pi/2$ and $\angle bpc < \pi/2$. Then $|ad|^2 < |pa|^2 + |pd|^2$, $|bc|^2 < |pb|^2 + |pc|^2$, and

$$|ad|^2 + |bc|^2 < |pa|^2 + |pb|^2 + |pc|^2 + |pd|^2 = |ab|^2 + |cd|^2,$$

which contradicts the optimality of M. □

Lemma 3. *No three disks in C have a common intersection.*

Proof. Suppose $I = DD(p_1q_1) \cap DD(p_2q_2) \cap DD(p_3q_3) \neq \varnothing$. By Lemma 1, the boundary of I must contain sections of at least two of the bounding circles of $DD(p_1q_1)$, $DD(p_2q_2)$ and $DD(p_3q_3)$. Thus, we may assume there is a point $p \in I$ such that p is in the intersection of the bounding circles of $DD(p_1q_1)$ and $DD(p_2q_2)$. By Lemma 2 the triangles $\triangle pp_1q_1$ and $\triangle pp_2q_2$ do not overlap. Now we consider three cases depending on the number of triangles that overlap with $\triangle pp_3q_3$.

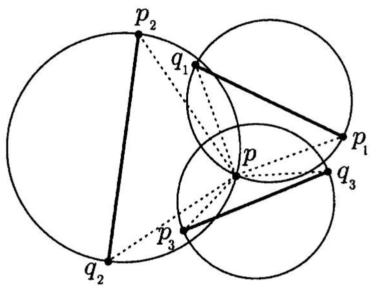

Fig. 3. Case 1: $\triangle pp_3q_3$ does not overlap with $\triangle pp_1q_1$ or with $\triangle pp_2q_2$

Case 1. $\triangle pp_3q_3$ does not overlap with $\triangle pp_1q_1$ or with $\triangle pp_2q_2$.

We may assume that the relative order of the triangles $\triangle pp_iq_i$ is as in Figure 3. Then, since $\angle q_1pp_1$, $\angle q_2pp_2$, $\angle q_3pp_3 \geq \pi/2$, we have that

$$\angle p_2pq_1 + \angle p_3pq_2 + \angle p_1pq_3 \leq \pi/2.$$

Thus, each of these angles is at most $\pi/2$, and at least two of them strictly acute (or zero). Thus,

$$|q_1p_2|^2 + |q_2p_3|^2 + |q_3p_1|^2 < |pq_1|^2 + |pp_2|^2 + |pq_2|^2 + |pp_3|^2 + |pq_3|^2 + |pp_1|^2.$$

Also, since none of the angles $\angle q_1pp_1, \angle q_2pp_2, \angle q_3pp_3$ is acute,

$$|pp_1|^2 + |pq_1|^2 + |pp_2|^2 + |pq_2|^2 + |pp_3|^2 + |pq_3|^2 \leq |p_1q_1|^2 + |p_2q_2|^2 + |p_3q_3|^2.$$

Thus, $|q_1p_2|^2 + |q_2p_3|^2 + |q_3p_1|^2 < |p_1q_1|^2 + |p_2q_2|^2 + |p_3q_3|^2$, which contradicts the optimality of M.

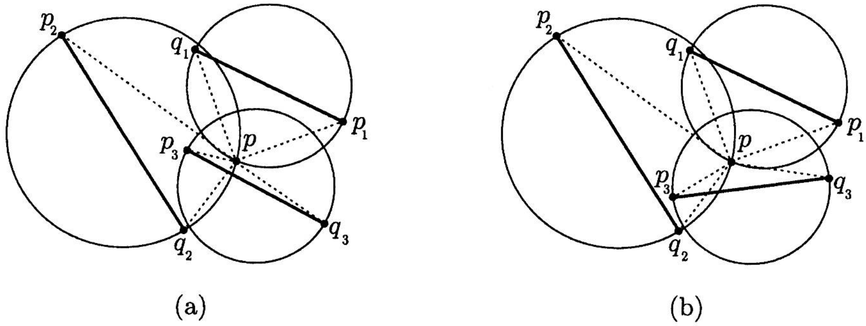

Fig. 4. Case 2: $\triangle pp_3q_3$ overlaps with $\triangle pp_2q_2$, but not with $\triangle pp_1q_1$

Case 2. $\triangle pp_3q_3$ overlaps with $\triangle pp_2q_2$, but not with $\triangle pp_1q_1$.

Assume $\overrightarrow{pp_3}$ is between $\overrightarrow{pp_2}$ and $\overrightarrow{pq_2}$. We may also assume that $\angle q_3pp_3 > \pi/2$; otherwise, p is in the bounding circle of $DD(p_3q_3)$ and then, by Lemma 2, $\triangle pp_3q_3$ and $\triangle pp_2q_2$ do not overlap. Since $\angle q_3pp_3 > \pi/2$, $|p_3q_3|^2 > |pp_3|^2 + |pq_3|^2$.

If $\angle q_3pq_2 \leq \pi/2$ (Figure 4a) then, as in the proof of Lemma 2, $|q_2q_3|^2 \leq |pq_2|^2 + |pq_3|^2$, $|p_2p_3|^2 \leq |pp_2|^2 + |pp_3|^2$, and thus

$$|q_2q_3|^2 + |p_2p_3|^2 \leq |pp_2|^2 + |pq_2|^2 + |pp_3|^2 + |pq_3|^2 < |p_2q_2|^2 + |p_3q_3|^2,$$

which contradicts the optimality of M.

If $\angle q_3pq_2 > \pi/2$ (Figure 4b), then $\angle p_1pq_3 + \angle p_2pq_1 < \pi/2$. Thus, $\angle p_1pq_3$, $\angle p_2pq_1 < \pi/2$, and thus $p_1q_3^2 < pp_1^2 + pq_3^2$ and $p_2q_1^2 < pp_2^2 + pq_1^2$. Also, since $\overrightarrow{pp_3}$ is between $\overrightarrow{pp_2}$ and $\overrightarrow{pq_2}$, $\angle q_2pq_3 < \pi/2$ and $q_2p_3^2 < pq_2^2 + pp_3^2$. Putting all of these inequalities together, we get

$$|p_1q_3|^2 + |p_2q_1|^2 + |q_2p_3|^2 < |pp_1|^2 + |pq_1|^2 + |pp_2|^2 + |pq_2|^2 + |pp_3|^2 + |pq_3|^2.$$

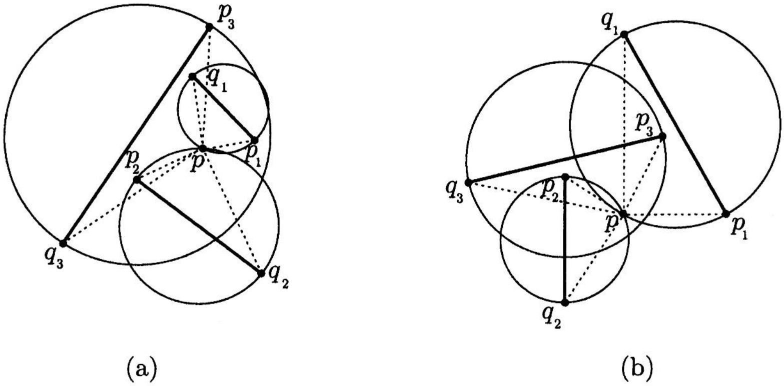

Fig. 5. Case 3: $\triangle pp_3q_3$ overlaps $\triangle pp_2q_2$ and $\triangle pp_1q_1$

Moreover, $|pp_1|^2 + |pq_1|^2 = |p_1q_1|^2$, $|pp_2|^2 + |pq_2|^2 = |p_2q_2|^2$, and $|pp_3|^2 + |pq_3|^2 < |p_3q_3|^2$. Thus,

$$|p_1q_3|^2 + |p_2q_1|^2 + |q_2p_3|^2 < |p_1q_1|^2 + |p_2q_2|^2 + |p_3q_3|^2,$$

which contradicts the optimality of M.

Case 3. $\triangle pp_3q_3$ overlaps $\triangle pp_2q_2$ and $\triangle pp_1q_1$.

We may assume that $\overrightarrow{pp_3}$ (resp., $\overrightarrow{pq_3}$) is between $\overrightarrow{pp_1}$ and $\overrightarrow{pq_1}$ (resp., $\overrightarrow{pp_2}$ and $\overrightarrow{pq_2}$). Refer to Figure 5. Again, by Lemma 2, we may assume that $\angle q_3pp_3 > \pi/2$ and $|p_3q_3|^2 > |pp_3|^2 + |pq_3|^2$.

If $\angle p_1pq_2 \leq \pi/2$ (Figure 5a), then $|p_1q_2|^2 \leq |pp_1|^2 + |pq_2|^2$. From the locations of p_3 and q_3, we have that $\angle q_1pp_3$, $\angle q_3pp_2 < \pi/2$, so $|p_3q_1|^2 < |pq_1|^2 + |pp_3|^2$ and $|p_2q_3|^2 < |pp_2|^2 + |pq_3|^2$. Thus,

$$|p_1q_2|^2 + |p_3q_1|^2 + |p_2q_3|^2 < |pp_1|^2 + |pq_1|^2 + |pp_2|^2 + |pq_2|^2 + |pp_3|^2 + |pq_3|^2.$$

As $|pp_i|^2 + |pq_i|^2 = |p_iq_i|^2$ for $i = 1, 2$ and $|pp_3|^2 + |pq_3|^2 < |p_3q_3|^2$, we get

$$|p_1q_2|^2 + |p_3q_1|^2 + |p_2q_3|^2 < |p_1q_1|^2 + |p_2q_2|^2 + |p_3q_3|^2.$$

If $\angle p_1pq_2 > \pi/2$ (Figure 5b), then, similarly, we get

$$|p_2q_1|^2 + |p_1p_3|^2 + |q_2q_3|^2 < |p_1q_1|^2 + |p_2q_2|^2 + |p_3q_3|^2.$$

In both cases we get a contradiction to the optimality of M. □

Lemma 4. *If $D_1, D_2, D_3, D_4 \in \mathcal{C}$ with $D_1 \cap D_2 \neq \varnothing$ and $D_3 \cap D_4 \neq \varnothing$ then the segments $\overline{o_1o_2}$ and $\overline{o_3o_4}$ do not intersect.*

Proof. Suppose $\overline{o_1o_2}$ and $\overline{o_3o_4}$ intersect at a point c, and let $p \in D_1 \cap D_2 \cap \overline{o_1o_2}$, and $q \in D_3 \cap D_4 \cap \overline{o_3o_4}$. Assume that $p \in \overline{co_2}$ and $q \in \overline{co_4}$. By the triangle inequality, $|o_1q| \leq |o_1c| + |cq|$ and $|o_3p| \leq |o_3c| + |cp|$; thus,

$$|o_1q| + |o_3p| \leq |o_1c| + |cp| + |o_3c| + |cq| = |o_1p| + |o_3q|.$$

Thus, either $|o_1q| \leq |o_1p|$ or $|o_3p| \leq |o_3q|$, which implies that either $q \in D_1$ or $p \in D_3$. This is a contradiction to Lemma 3, since either $q \in D_1 \cap D_3 \cap D_4$ or $p \in D_1 \cap D_2 \cap D_3$. □

Proof of Theorem 3. Let G be the graph with vertex set equal to the set of centers of the disks in $\mathcal{C}$ and with two vertices connected by an edge if and only it the corresponding disks intersect. By Lemma 4, G is a planar graph. Then, by the Four Color Theorem, the maximum independent set of G has at least $\lceil m/4 \rceil = \lceil \lfloor n/2 \rfloor /4 \rceil = \lceil (n-1)/8 \rceil$ vertices. Thus, the corresponding diametrical disks are pairwise disjoint. Therefore, P has a circle-matching using at least $2\lceil (n-1)/8 \rceil$ points. It may happen that these diametrical disks have points of P in their interior; however, it is always possible to find a circle inside one of these diametrical disks containing only two points of P.

2.3 Convex Position

For n points on a line, with n even, it is obvious that a strong perfect matching with disks is always possible, as we can simply take the diametrical circles defined by consecutive pairs. As a consequence a strong perfect matching is also always possible when we are given any set P of n points lying on a circle C: using an inversion with center at any point in $C \setminus P$ the images of all points from P become collinear and admit a matching, which, applying again the same inversion, gives the desired matching (because inversions are involutive and map circles that do not pass through the center of inversion to circles).

This may suggest that a similar result should hold for any set of points in convex position, but this is not the case, as we show next using the same kind of arguments.

Let Q be a point set consisting of the center a of a circle C, and 73 additional points evenly distributed on C; as remarked earlier (see Figure 1), Q does not admit a strong perfect circle-matching.

Let P be the point set obtained from Q by applying any inversion with center at some point $p \in C \setminus Q$; the point set P does not admit a strong perfect circle-matching. Note that all of the points in P with the exception of the image of a lie on a line. Applying an infinitesimal perturbation to the elements of P in such a way that they remain in convex position but no three are collinear produces a point set P' in convex position for which no strong perfect circle-matching exists, since the inverse set Q' is an infinitesimal perturbation of Q and therefore does not admit a strong perfect circle-matching. Therefore we have proved the following result:

Proposition 1. *There are point sets in convex position in the plane for which there is no strong perfect circle-matching.*

3 Isothetic Squares

In this section we consider the following variation of our geometric matching problem. Let P be a set of $2n$ points in general position in the plane. As in the