SCHAUM'S OUTLINE SERIES

THEORY AND PROBLEMS OF

ADVANCED CALCULUS

MURRAY R. SPIEGEL

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SCHAUM'S OUTLINE SERIES IN MATHEMATICS

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THEORY AND PROBLEMS

of

ADVANCED CALCULUS

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BY

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Preface

The subject commonly called "Advanced Calculus" means different things to different people. To some it essentially represents elementary calculus from an advanced viewpoint, i.e. with rigorous statements and proofs of theorems. To others it represents a variety of special advanced topics which are considered important but which cannot be covered in an elementary course.

In this book an effort has been made to adopt a reasonable compromise between these extreme approaches which, it is believed, will serve a variety of individuals. The early chapters of the book serve in general to review and extend fundamental concepts already presented in elementary calculus. This should be valuable to those who have forgotten some of the calculus studied previously and who need "a bit of refreshing". It may also serve to provide a common background for students who have been given different types of courses in elementary calculus. Later chapters serve to present special advanced topics which are fundamental to the scientist, engineer and mathematician if he is to become proficient in his intended field.

This book has been designed for use either as a supplement to all current standard textbooks or as a textbook for a formal course in advanced calculus. It should also prove useful to students taking courses in physics, engineering or any of the numerous other fields in which advanced mathematical methods are employed.

Each chapter begins with a clear statement of pertinent definitions, principles and theorems together with illustrative and other descriptive material. This is followed by graded sets of solved and supplementary problems. The solved problems serve to illustrate and amplify the theory, bring into sharp focus those fine points without which the student continually feels himself on unsafe ground, and provide the repetition of basic principles so vital to effective learning. Numerous proofs of theorems and derivations of basic results are included among the solved problems. The large number of supplementary problems with answers serve as a complete review of the material of each chapter.

Topics covered include the differential and integral calculus of functions of one or more variables and their applications. Vector methods, which lend themselves so readily to concise notation and to geometric and physical interpretations, are introduced early and used whenever they can contribute to motivation and understanding. Special topics include line and surface integrals and integral theorems, infinite series, improper integrals, gamma and beta functions, and Fourier series. Added features are the chapters on Fourier integrals, elliptic integrals and functions of a complex variable which should prove extremely useful in the study of advanced engineering, physics and mathematics.

Considerably more material has been included here than can be covered in most courses. This has been done to make the book more flexible, to provide a more useful book of reference and to stimulate further interest in the topics.

I wish to take this opportunity to thank the staff of the Schaum Publishing Company for their splendid cooperation in meeting the seemingly endless attempts at perfection by the author.

M. R. SPIEGEL

Rensselaer Polytechnic Institute
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Chapter 1

Numbers

SETS

Fundamental in mathematics is the concept of a *set, class* or *collection* of objects having specified characteristics. For example we speak of the set of all university professors, the set of all letters $A, B, C, D, \ldots, Z$ of the English alphabet, etc. The individual objects of the set are called *members* or *elements.* Any part of a set is called a *subset* of the given set, e.g. A, B, C is a subset of $A, B, C, D, \ldots, Z$. The set consisting of no elements is called the *empty set* or *null set.*

REAL NUMBERS

The following types of numbers are already familiar to the student.

1. **Natural numbers** $1, 2, 3, 4, \ldots$, also called *positive integers,* are used in counting members of a set. The symbols varied with the times, e.g. the Romans used I, II, III, IV, ... The *sum* $a+b$ and *product* $a \cdot b$ or ab of any two natural numbers a and b is also a natural number. This is often expressed by saying that the set of natural numbers is *closed* under the operations of *addition* and *multiplication,* or satisfies the *closure property* with respect to these operations.

2. **Negative integers and zero** denoted by $-1, -2, -3, \ldots$ and 0 respectively, arose to permit solutions of equations such as $x + b = a$ where a and b are any natural numbers. This leads to the operation of *subtraction,* or *inverse of addition,* and we write $x = a - b$.

   The set of positive and negative integers and zero is called the set of *integers.*

3. **Rational numbers** or *fractions* such as $\frac{2}{3}, -\frac{5}{4}, \ldots$ arose to permit solutions of equations such as $bx = a$ for all integers a and b where $b \neq 0$. This leads to the operation of *division,* or *inverse of multiplication,* and we write $x = a/b$ or $a \div b$ where a is the *numerator* and b the *denominator.*

   The set of integers is a subset of the rational numbers, since integers correspond to rational numbers where $b = 1$.

4. **Irrational numbers** such as $\sqrt{2}$ and π are numbers which are not rational, i.e. cannot be expressed as $\frac{a}{b}$ (called the *quotient* of a and b) where a and b are integers and $b \neq 0$.

   The set of rational and irrational numbers is called the set of *real numbers.*

DECIMAL REPRESENTATION of REAL NUMBERS

Any real number can be expressed in *decimal form,* e.g. $17/10 = 1.7$, $9/100 = 0.09$, $1/6 = 0.16666\ldots$. In the case of a rational number the decimal expansion either terminates or, if it does not terminate, one or a group of digits in the expansion will ultimately repeat as, for example, in $\frac{1}{7} = 0.142857\,142857\,142\ldots$. In the case of an irrational number such as $\sqrt{2} = 1.41423\ldots$ or $\pi = 3.14159\ldots$ no such repetition can occur. We can always consider a decimal expansion as unending, e.g. 1.375 is the same as 1.37500000... or 1.3749999.... To indicate recurring decimals we sometimes place dots over the repeating cycle of digits, e.g. $\frac{1}{7} = 0.\dot{1}\dot{4}\dot{2}\dot{8}\dot{5}\dot{7}$, $\frac{19}{6} = 3.1\dot{6}$.

The decimal system uses the ten digits $0, 1, 2, \ldots, 9$. It is possible to design number systems with fewer or more digits, e.g. the *binary system* uses only two digits 0 and 1 (see Problems 32 and 33).

GEOMETRIC REPRESENTATION of REAL NUMBERS

The geometric representation of real numbers as points on a line called the *real axis*, as in the figure below, is also well known to the student. For each real number there corresponds one and only one point on the line and conversely, i.e. there is a *one to one* (1-1) *correspondence* between the set of real numbers and the set of points on the line. Because of this we often use point and number interchangeably.

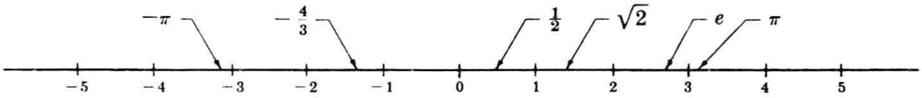

Fig. 1-1

The set of real numbers to the right of 0 is called the set of *positive numbers*; the set to the left of 0 is the set of *negative numbers*, while 0 itself is neither positive nor negative.

Between any two rational numbers (or irrational numbers) on the line there are infinitely many rational (and irrational) numbers. This leads us to call the set of rational (or irrational) numbers an *everywhere dense* set.

OPERATIONS with REAL NUMBERS

If a, b, c belong to the set R of real numbers, then:

1. $a+b$ and ab belong to R — Closure law
2. $a+b = b+a$ — Commutative law of addition
3. $a+(b+c) = (a+b)+c$ — Associative law of addition
4. $ab = ba$ — Commutative law of multiplication
5. $a(bc) = (ab)c$ — Associative law of multiplication
6. $a(b+c) = ab+ac$ — Distributive law
7. $a+0 = 0+a = a, \quad 1 \cdot a = a \cdot 1 = a$

   0 is called the *identity with respect to addition*, 1 is called the *identity with respect to multiplication*.
8. For any a there is a number x in R such that $x+a=0$.

   x is called the *inverse of a with respect to addition* and is denoted by $-a$.
9. For any $a \neq 0$ there is a number x in R such that $ax = 1$.

   x is called the *inverse of a with respect to multiplication* and is denoted by a^{-1} or $1/a$.

These enable us to operate according to the usual rules of algebra. In general any set, such as R, whose members satisfy the above is called a *field*.

INEQUALITIES

If $a-b$ is a nonnegative number we say that a is *greater than or equal to* b or b is *less than or equal to* a, and write respectively $a \geqq b$ or $b \leqq a$. If there is no possibility that $a=b$, we write $a>b$ or $b<a$. Geometrically, $a>b$ if the point on the real axis corresponding to a lies to the right of the point corresponding to b.

Examples: $3<5$ or $5>3$; $-2<-1$ or $-1>-2$; $x \leqq 3$ means that x is a real number which may be 3 or less than 3.

If a, b and c are any given real numbers, then:

1. Either $a>b$, $a=b$ or $a<b$ — Law of trichotomy
2. If $a>b$ and $b>c$, then $a>c$ — Law of transitivity
3. If $a>b$, then $a+c>b+c$
4. If $a>b$ and $c>0$, then $ac>bc$
5. If $a>b$ and $c<0$, then $ac<bc$

ABSOLUTE VALUE of REAL NUMBERS

The absolute value of a real number a, denoted by $|a|$, is defined as a if $a > 0$, $-a$ if $a < 0$, and 0 if $a = 0$.

Examples: $|-5| = 5$, $|+2| = 2$, $|-\frac{3}{4}| = \frac{3}{4}$, $|-\sqrt{2}| = \sqrt{2}$, $|0| = 0$.

1. $|ab| = |a|\,|b|$ or $|abc\ldots m| = |a|\,|b|\,|c|\ldots|m|$
2. $|a+b| \leqq |a| + |b|$ or $|a+b+c+\ldots+m| \leqq |a|+|b|+|c|+\ldots+|m|$
3. $|a-b| \geqq |a| - |b|$

The distance between any two points (real numbers) a and b on the real axis is $|a-b| = |b-a|$.

EXPONENTS and ROOTS

The product $a \cdot a \ldots a$ of a real number a by itself p times is denoted by a^p where p is called the *exponent* and a is called the *base*. The following rules hold.

1. $a^p \cdot a^q = a^{p+q}$
2. $\dfrac{a^p}{a^q} = a^{p-q}$
3. $(a^p)^r = a^{pr}$
4. $\left(\dfrac{a}{b}\right)^p = \dfrac{a^p}{b^p}$

These and extensions to any real numbers are possible so long as division by zero is excluded. In particular by using 2, with $p = q$ and $p = 0$ respectively, we are led to the definitions $a^0 = 1$, $a^{-q} = 1/a^q$.

If $a^p = N$, where p is a positive integer, we call a a pth *root* of N, written $\sqrt[p]{N}$. There may be more than one real pth root of N. For example since $2^2 = 4$ and $(-2)^2 = 4$, there are two real square roots of 4, namely 2 and -2. It is customary to denote the positive square root by $\sqrt{4} = 2$ and the negative one by $-\sqrt{4} = -2$.

If p and q are positive integers, we define $a^{p/q} = \sqrt[q]{a^p}$.

LOGARITHMS

If $a^p = N$, p is called the *logarithm* of N to the base a, written $p = \log_a N$. If a and N are positive and $a \neq 1$, there is only one real value for p. The following rules hold.

1. $\log_a MN = \log_a M + \log_a N$
2. $\log_a \dfrac{M}{N} = \log_a M - \log_a N$
3. $\log_a M^r = r \log_a M$

In practice two bases are used, the *Briggsian system* uses base $a = 10$, the *Napierian system* uses the *natural base* $a = e = 2.71828\ldots$

AXIOMATIC FOUNDATIONS of the REAL NUMBER SYSTEM

The number system can be built up logically, starting from a basic set of *axioms* or "self evident" truths, usually taken from experience, such as statements 1-9, Page 2.

If we assume as given the natural numbers and the operations of addition and multiplication (although it is possible to start even further back with the concept of sets), we find that statements 1-6, Page 2, with R as the set of natural numbers, hold while 7-9 do not hold.

Taking 7 and 8 as additional requirements, we introduce the numbers $-1, -2, -3, \ldots$ and 0. Then by taking 9 we introduce the rational numbers.

Operations with these newly obtained numbers can be defined by adopting axioms 1-6, where R is now the set of integers. These lead to *proofs* of statements such as $(-2)(-3)=6$, $-(-4)=4$, $(0)(5)=0$, etc., which are usually taken for granted in elementary mathematics.

We can also introduce the concept of order or inequality for integers, and from these inequalities for rational numbers. For example if a, b, c, d are positive integers we define $a/b > c/d$ if and only if $ad > bc$, with similar extensions to negative integers.

Once we have the set of rational numbers and the rules of inequality concerning them, we can order them geometrically as points on the real axis, as already indicated. We can then show that there are points on the line which do not represent rational numbers (such as $\sqrt{2}, \pi$, etc.). These irrational numbers can be defined in various ways one of which uses the idea of *Dedekind cuts* (see Problem 34). From this we can show that the usual rules of algebra apply to irrational numbers and that no further real numbers are possible.

POINT SETS, INTERVALS

A set of points (real numbers) located on the real axis is called a *one-dimensional point set*.

The set of points x such that $a \leqq x \leqq b$ is called a *closed interval* and is denoted by $[a, b]$. The set $a < x < b$ is called an *open interval*, denoted by (a, b). The sets $a < x \leqq b$ and $a \leqq x < b$, denoted by $(a, b]$ and $[a, b)$ respectively, are called *half open* or *half closed* intervals.

The symbol x, which can represent any number or point of a set, is called a *variable*. The given numbers a or b are called *constants*.

> **Example:** The set of all x such that $|x| < 4$, i.e. $-4 < x < 4$, is represented by $(-4, 4)$, an open interval.

The set $x > a$ can also be represented by $a < x < \infty$. Such a set is called an *infinite* or *unbounded interval*. Similarly $-\infty < x < \infty$ represents all real numbers x.

COUNTABILITY

A set is called *countable* or *denumerable* if its elements can be placed in 1-1 correspondence with the natural numbers.

> **Example:** The even natural numbers $2, 4, 6, 8, \ldots$ is a countable set because of the 1-1 correspondence shown.

Given set	2	4	6	8	...
	↕	↕	↕	↕	
Natural numbers	1	2	3	4	...

A set is *infinite* if it can be placed in 1-1 correspondence with a subset of itself. An infinite set which is countable is called *countably infinite*.

The set of rational numbers is countably infinite while the set of irrational numbers or all real numbers is non-countably infinite (see Problems 17-20).

The number of elements in a set is called its *cardinal number*. A set which is countably infinite is assigned the cardinal number $\aleph_0$ (the Hebrew letter *aleph-null*). The set of real numbers (or any sets which can be placed into 1-1 correspondence with this set) is given the cardinal number C, called the *cardinality of the continuum*.

NEIGHBORHOODS

The set of all points x such that $|x-a|<\delta$ where $\delta>0$, is called a δ *neighborhood* of the point a. The set of all points x such that $0<|x-a|<\delta$ in which $x=a$ is excluded, is called a *deleted* δ *neighborhood* of a.

LIMIT POINTS

A *limit point, point of accumulation* or *cluster point* of a set of numbers is a number l such that every deleted δ neighborhood of l contains members of the set. In other words for any $\delta>0$, however small, we can always find a member x of the set which is not equal to l but which is such that $|x-l|<\delta$. By considering smaller and smaller values of δ we see that there must be infinitely many such values of x.

A finite set cannot have a limit point. An infinite set may or may not have a limit point. Thus the natural numbers have no limit point while the set of rational numbers has infinitely many limit points.

A set containing all its limit points is called a *closed set*. The set of rational numbers is not a closed set since, for example, the limit point $\sqrt{2}$ is not a member of the set (Problem 5). However, the set $0 \leqq x \leqq 1$ is a closed set.

BOUNDS

If for all numbers x of a set there is a number M such that $x \leqq M$, the set is *bounded above* and M is called an *upper bound.* Similarly if $x \geqq m$, the set is *bounded below* and m is called a *lower bound.* If for all x we have $m \leqq x \leqq M$, the set is called *bounded.*

If $\underline{M}$ is a number such that no member of the set is greater than $\underline{M}$ but there is at least one member which exceeds $\underline{M}-\epsilon$ for every $\epsilon>0$, then $\underline{M}$ is called the *least upper bound* (l.u.b.) of the set. Similarly if no member of the set is smaller than $\bar{m}$ but at least one member is smaller than $\bar{m}+\epsilon$ for every $\epsilon>0$, then $\bar{m}$ is called the *greatest lower bound* (g.l.b.) of the set.

WEIERSTRASS-BOLZANO THEOREM

The Weierstrass-Bolzano theorem states that every bounded infinite set has at least one limit point. A proof of this is given in Problem 23, Chapter 3.

ALGEBRAIC and TRANSCENDENTAL NUMBERS

A number x which is a solution to the *polynomial equation*

$$a_0x^n + a_1x^{n-1} + a_2x^{n-2} + \cdots + a_{n-1}x + a_n = 0 \tag{1}$$

where $a_0 \neq 0,\ a_1, a_2, \ldots, a_n$ are integers and n is a positive integer, called the *degree* of the equation, is called an *algebraic number.* A number which cannot be expressed as a solution of any polynomial equation with integer coefficients is called a *transcendental number.*

Examples: $\frac{2}{3}$ and $\sqrt{2}$ which are solutions of $3x-2=0$ and $x^2-2=0$ respectively, are algebraic numbers.

The numbers π and e can be shown to be transcendental numbers. We still cannot determine whether some numbers such as $e\pi$ or $e+\pi$ are algebraic or not.

The set of algebraic numbers is a countably infinite set (see Problem 23) but the set of transcendental numbers is non-countably infinite.

The COMPLEX NUMBER SYSTEM

Since there is no real number x which satisfies the polynomial equation $x^2+1=0$ or similar equations, the set of complex numbers is introduced.

We can consider a complex number as having the form $a+bi$ where a and b are real numbers called the *real* and *imaginary parts,* and $i=\sqrt{-1}$ is called the *imaginary unit.* Two complex numbers $a+bi$ and $c+di$ are *equal* if and only if $a=c$ and $b=d$. We can consider real numbers as a subset of the set of complex numbers with $b=0$. The complex number $0+0i$ corresponds to the real number 0.

The *absolute value* or *modulus* of $a+bi$ is defined as $|a+bi| = \sqrt{a^2+b^2}$. The *complex conjugate* of $a+bi$ is defined as $a-bi$. The complex conjugate of the complex number z is often indicated by $\bar{z}$ or z^*.

The set of complex numbers obeys rules 1-9 of Page 2, and thus constitutes a field. In performing operations with complex numbers we can operate as in the algebra of real numbers, replacing i^2 by -1 when it occurs. Inequalities for complex numbers are not defined.

From the point of view of an axiomatic foundation of complex numbers, it is desirable to treat a complex number as an ordered pair (a,b) of real numbers a and b subject to certain operational rules which turn out to be equivalent to those above. For example, we define $(a,b)+(c,d) = (a+c,\ b+d)$, $(a,b)(c,d) = (ac-bd,\ ad+bc)$, $m(a,b) = (ma,mb)$, etc. We then find that $(a,b) = a(1,0)+b(0,1)$ and we associate this with $a+bi$, where i is the symbol for $(0,1)$.

POLAR FORM of COMPLEX NUMBERS

If real scales are chosen on two mutually perpendicular axes $X'OX$ and $Y'OY$ (the x and y axes) as in Fig. 1-2 below, we can locate any point in the plane determined by these lines by the ordered pair of numbers (x,y) called *rectangular coordinates* of the point. Examples of the location of such points are indicated by P,Q,R,S and T in Fig. 1-2.

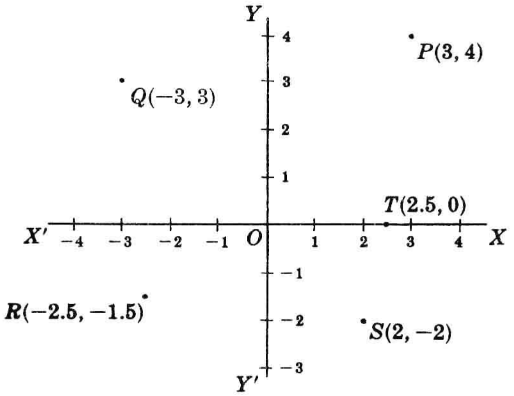

Fig. 1-2

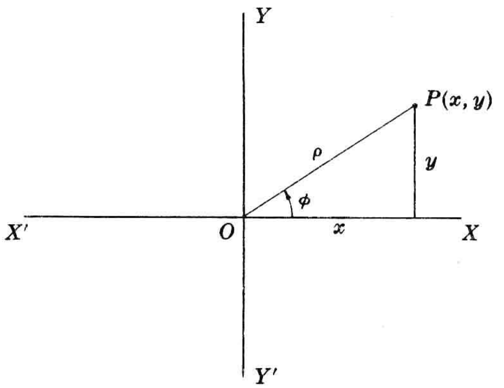

Fig. 1-3

Since a complex number $x+iy$ can be considered as an ordered pair (x,y), we can represent such numbers by points in an xy plane called the *complex plane* or *Argand diagram.* Referring to Fig. 1-3 above we see that $x = \rho\cos\phi$, $y = \rho\sin\phi$ where $\rho = \sqrt{x^2+y^2} = |x+iy|$ and ϕ, called the *amplitude* or *argument,* is the angle which line OP makes with the positive x axis OX. It follows that

$$z \;=\; x+iy \;=\; \rho(\cos\phi + i\sin\phi) \tag{2}$$

called the *polar form* of the complex number, where ρ and ϕ are called *polar coordinates.* It is sometimes convenient to write cis ϕ instead of $\cos\phi + i\sin\phi$.

If $z_1 = x_1 + iy_1 = \rho_1(\cos\phi_1 + i\sin\phi_1)$ and $z_2 = x_2 + iy_2 = \rho_2(\cos\phi_2 + i\sin\phi_2)$ we can show that

$$z_1 z_2 = \rho_1\rho_2\{\cos(\phi_1+\phi_2) + i\sin(\phi_1+\phi_2)\} \tag{3}$$

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2}\{\cos(\phi_1-\phi_2) + i\sin(\phi_1-\phi_2)\} \tag{4}$$

$$z^n = \{\rho(\cos\phi + i\sin\phi)\}^n = \rho^n(\cos n\phi + i\sin n\phi) \tag{5}$$

where n is any real number. Equation (5) is sometimes called *De Moivre's theorem.* We can use this to determine roots of complex numbers. For example if n is a positive integer,

$$z^{1/n} = \{\rho(\cos\phi + i\sin\phi)\}^{1/n} \tag{6}$$

$$= \rho^{1/n}\left\{\cos\left(\frac{\phi+2k\pi}{n}\right) + i\sin\left(\frac{\phi+2k\pi}{n}\right)\right\} \qquad k = 0, 1, 2, 3, \ldots, n-1$$

from which it follows that there are in general n different values for $z^{1/n}$. Later (Chap. 11) we will show that $e^{i\phi} = \cos\phi + i\sin\phi$ where $e = 2.71828\ldots$. This is called *Euler's formula.*

MATHEMATICAL INDUCTION

The principle of *mathematical induction* is an important property of the positive integers. It is especially useful in proving statements involving all positive integers when it is known for example that the statements are valid for $n = 1, 2, 3$ but it is *suspected* or *conjectured* that they hold for all positive integers. The method of proof consists of the following steps.

1. Prove the statement for $n = 1$ (or some other positive integer).
2. Assume the statement true for $n = k$ where k is any positive integer.
3. From the assumption in 2 prove that the statement must be true for $n = k+1$. This is the part of the proof establishing the induction and may be difficult or impossible.
4. Since the statement is true for $n = 1$ [from step 1] it must [from step 3] be true for $n = 1+1 = 2$ and from this for $n = 2+1 = 3$, etc., and so must be true for all positive integers.

Solved Problems

OPERATIONS with NUMBERS

1. If $x = 4$, $y = 15$, $z = -3$, $p = \frac{2}{3}$, $q = -\frac{1}{6}$, and $r = \frac{3}{4}$, evaluate (a) $x + (y+z)$, (b) $(x+y) + z$, (c) $p(qr)$, (d) $(pq)r$, (e) $x(p+q)$.

(a) $x + (y+z) = 4 + [15 + (-3)] = 4 + 12 = 16$

(b) $(x+y) + z = (4+15) + (-3) = 19 - 3 = 16$

The fact that (a) and (b) are equal illustrates the *associative law of addition.*

(c) $p(qr) = \frac{2}{3}\{(-\frac{1}{6})(\frac{3}{4})\} = (\frac{2}{3})(-\frac{3}{24}) = (\frac{2}{3})(-\frac{1}{8}) = -\frac{2}{24} = -\frac{1}{12}$

(d) $(pq)r = \{(\frac{2}{3})(-\frac{1}{6})\}(\frac{3}{4}) = (-\frac{2}{18})(\frac{3}{4}) = (-\frac{1}{9})(\frac{3}{4}) = -\frac{3}{36} = -\frac{1}{12}$

The fact that (c) and (d) are equal illustrates the *associative law of multiplication.*

(e) $x(p+q) = 4(\frac{2}{3} - \frac{1}{6}) = 4(\frac{4}{6} - \frac{1}{6}) = 4(\frac{3}{6}) = \frac{12}{6} = 2$

Another method: $x(p+q) = xp + xq = (4)(\frac{2}{3}) + (4)(-\frac{1}{6}) = \frac{8}{3} - \frac{4}{6} = \frac{8}{3} - \frac{2}{3} = \frac{6}{3} = 2$ using the *distributive law.*

2. Explain why we do not consider (*a*) $\dfrac{0}{0}$ (*b*) $\dfrac{1}{0}$ as numbers.

(*a*) If we define a/b as that number (if it exists) such that $bx = a$, then $0/0$ is that number x such that $0x = 0$. However, this is true for all numbers. Since there is no unique number which $0/0$ can represent, we consider it undefined.

(*b*) As in (*a*), if we define $1/0$ as that number x (if it exists) such that $0x = 1$, we conclude that there is no such number.

Because of these facts we must look upon division by zero as meaningless.

3. Simplify $\dfrac{x^2 - 5x + 6}{x^2 - 2x - 3}$.

$\dfrac{x^2 - 5x + 6}{x^2 - 2x - 3} = \dfrac{(x-3)(x-2)}{(x-3)(x+1)} = \dfrac{x-2}{x+1}$ provided that the cancelled factor $(x-3)$ is not zero, i.e. $x \neq 3$. For $x = 3$ the given fraction is undefined.

RATIONAL and IRRATIONAL NUMBERS

4. Prove that the square of any odd integer is odd.

Any odd integer has the form $2m + 1$. Since $(2m+1)^2 = 4m^2 + 4m + 1$ is 1 more than the even integer $4m^2 + 4m = 2(2m^2 + 2m)$, the result follows.

5. Prove that there is no rational number whose square is 2.

Let p/q be a rational number whose square is 2, where we assume that p/q is in lowest terms, i.e. p and q have no common integer factors except ± 1 (we sometimes call such integers *relatively prime*).

Then $(p/q)^2 = 2$, $p^2 = 2q^2$ and p^2 is even. From Problem 4, p is even since if p were odd, p^2 would be odd. Thus $p = 2m$.

Substituting $p = 2m$ in $p^2 = 2q^2$ yields $q^2 = 2m^2$, so that q^2 is even and q is even.

Thus p and q have the common factor 2, contradicting the original assumption that they had no common factors other than ± 1. By virtue of this contradiction there can be no rational number whose square is 2.

6. Show how to find rational numbers whose squares can be made arbitrarily close to 2.

We restrict ourselves to positive rational numbers. Since $(1)^2 = 1$ and $(2)^2 = 4$, we are led to choose rational numbers between 1 and 2, e.g. $1.1, 1.2, 1.3, \ldots, 1.9$.

Since $(1.4)^2 = 1.96$ and $(1.5)^2 = 2.25$, we consider rational numbers between 1.4 and 1.5, e.g. 1.41, $1.42, \ldots, 1.49$.

Continuing in this manner we can obtain closer and closer rational approximations, e.g. $(1.414213562)^2$ is less than 2 while $(1.414213563)^2$ is greater than 2.

7. Given the equation $a_0 x^n + a_1 x^{n-1} + \cdots + a_n = 0$ where $a_0, a_1, \ldots, a_n$ are integers and a_0 and $a_n \neq 0$. Show that if the equation is to have a rational root p/q, then p must divide a_n and q must divide a_0 exactly.

Since p/q is a root we have, on substituting in the given equation and multiplying by q^n, the result

$$a_0 p^n + a_1 p^{n-1} q + a_2 p^{n-2} q^2 + \cdots + a_{n-1} p q^{n-1} + a_n q^n = 0 \qquad (1)$$

or dividing by p,

$$a_0 p^{n-1} + a_1 p^{n-2} q + \cdots + a_{n-1} q^{n-1} = -\frac{a_n q^n}{p} \qquad (2)$$

Since the left side of (*2*) is an integer the right side must also be an integer. Then since p and q are relatively prime, p does not divide q^n exactly and so must divide a_n.

In a similar manner, by transposing the first term of (*1*) and dividing by q, we can show that q must divide a_0.

8. Prove that $\sqrt{2} + \sqrt{3}$ cannot be a rational number.

If $x = \sqrt{2} + \sqrt{3}$ then $x^2 = 5 + 2\sqrt{6}$, $x^2 - 5 = 2\sqrt{6}$ and squaring, $x^4 - 10x^2 + 1 = 0$. The only possible rational roots of this equation are ± 1 by Problem 7, and these do not satisfy the equation. It follows that $\sqrt{2} + \sqrt{3}$, which satisfies the equation, cannot be a rational number.

9. Prove that between any two rational numbers there is another rational number.

If a and b are rational numbers, then $\frac{a+b}{2}$ is a rational number between a and b.

To prove this assume $a < b$. Then by adding a to both sides, $2a < a+b$ and $a < \frac{a+b}{2}$. Similarly adding b to both sides, $a+b < 2b$ and $\frac{a+b}{2} < b$. Thus $a < \frac{a+b}{2} < b$.

To prove that $\frac{a+b}{2}$ is a rational number, let $a = \frac{p}{q}$ and $b = \frac{r}{s}$ where p, q, r, s are integers and $q \neq 0$, $s \neq 0$.

Then $\frac{a+b}{2} = \frac{1}{2}\left(\frac{p}{q}+\frac{r}{s}\right) = \frac{1}{2}\left(\frac{ps}{qs}+\frac{qr}{qs}\right) = \frac{ps+qr}{2qs}$ is a rational number.

INEQUALITIES

10. For what values of x is $x + 3(2-x) \geqq 4 - x$?

$x+3(2-x) \geqq 4-x$ when $x+6-3x \geqq 4-x$, $6-2x \geqq 4-x$, $6-4 \geqq 2x-x$, $2 \geqq x$, i.e. $x \leqq 2$.

11. For what values of x is $x^2 - 3x - 2 < 10 - 2x$?

The required inequality holds when

$$x^2-3x-2-10+2x < 0, \quad x^2-x-12 < 0 \quad \text{or} \quad (x-4)(x+3) < 0$$

This last inequality holds only in the following cases.

Case 1: $x-4 > 0$ *and* $x+3 < 0$, i.e. $x > 4$ and $x < -3$. This is *impossible* since x cannot be both greater than 4 and less than -3.

Case 2: $x-4 < 0$ *and* $x+3 > 0$, i.e. $x < 4$ and $x > -3$. This is possible when $-3 < x < 4$.

Thus the inequality holds for the set of all x such that $-3 < x < 4$.

12. If $a \geqq 0$ and $b \geqq 0$, prove that $\frac{1}{2}(a+b) \geqq \sqrt{ab}$.

A method of proof is often arrived at by *assuming* the required result to be true and performing valid operations until a result is obtained which is *known* to be true. By reversing the steps (assuming this possible) the proof follows.

In this problem we start with the required result to obtain successively $a+b \geqq 2\sqrt{ab}$, $(a+b)^2 \geqq 4ab$ or $a^2-2ab+b^2 \geqq 0$, i.e. $(a-b)^2 \geqq 0$, which is known to be true. Retracing the steps, the result follows.

Another method: Since $(\sqrt{a}-\sqrt{b})^2 \geqq 0$ we have $a-2\sqrt{ab}+b \geqq 0$ or $\frac{1}{2}(a+b) \geqq \sqrt{ab}$.

This result can be generalized to $\frac{a_1+a_2+\cdots+a_n}{n} \geqq \sqrt[n]{a_1a_2\cdots a_n}$ where $a_1, \ldots, a_n$ are non-negative. The left and right sides are called respectively the *arithmetic mean* and *geometric mean* of the numbers $a_1, \ldots, a_n$.

13. If $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ are any real numbers, prove *Schwarz's inequality*

$$(a_1b_1+a_2b_2+\cdots+a_nb_n)^2 \leqq (a_1^2+a_2^2+\cdots+a_n^2)(b_1^2+b_2^2+\cdots+b_n^2)$$

For all real numbers λ, we have

$$(a_1\lambda+b_1)^2 + (a_2\lambda+b_2)^2 + \cdots + (a_n\lambda+b_n)^2 \geqq 0$$

Expanding and collecting terms yields

$$A^2\lambda^2 + 2C\lambda + B^2 \geqq 0 \tag{1}$$

where

$$A^2 = a_1^2+a_2^2+\cdots+a_n^2, \quad B^2 = b_1^2+b_2^2+\cdots+b_n^2, \quad C = a_1b_1+a_2b_2+\cdots+a_nb_n \tag{2}$$

Now *(1)* can be written

$$\lambda^2 + \frac{2C}{A^2}\lambda + \frac{B^2}{A^2} \geqq 0 \quad \text{or} \quad \left(\lambda+\frac{C}{A^2}\right)^2 + \frac{B^2}{A^2} - \frac{C^2}{A^4} \geqq 0 \tag{3}$$

But this last inequality is true for all real λ if and only if $\frac{B^2}{A^2}-\frac{C^2}{A^4} \geqq 0$ or $C^2 \leqq A^2B^2$ which gives the required inequality upon using *(2)*.

14. Prove that $\dfrac{1}{2}+\dfrac{1}{4}+\dfrac{1}{8}+\cdots+\dfrac{1}{2^{n-1}} < 1$ for all positive integers $n>1$.

Let $\quad S_n = \frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n-1}}$

Then $\quad \frac{1}{2}S_n = \frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^{n-1}}+\frac{1}{2^n}$

Subtracting, $\quad \frac{1}{2}S_n = \frac{1}{2}-\frac{1}{2^n}$. Thus $S_n = 1-\frac{1}{2^{n-1}} < 1$ for all n.

EXPONENTS, ROOTS and LOGARITHMS

15. Evaluate each of the following.

(a) $\dfrac{3^4\cdot 3^8}{3^{14}} = \dfrac{3^{4+8}}{3^{14}} = 3^{4+8-14} = 3^{-2} = \dfrac{1}{3^2} = \dfrac{1}{9}$

(b) $\sqrt{\dfrac{(5\cdot 10^{-6})(4\cdot 10^2)}{8\cdot 10^5}} = \sqrt{\dfrac{5\cdot 4}{8}\cdot\dfrac{10^{-6}\cdot 10^2}{10^5}} = \sqrt{2.5\cdot 10^{-9}} = \sqrt{25\cdot 10^{-10}} = 5\cdot 10^{-5}$ or 0.00005

(c) $\log_{2/3}(\frac{27}{8}) = x$. Then $(\frac{2}{3})^x = \frac{27}{8} = (\frac{3}{2})^3 = (\frac{2}{3})^{-3}$ or $x=-3$.

(d) $(\log_a b)(\log_b a) = u$. Let $\log_a b = x$, $\log_b a = y$ assuming $a, b > 0$ and $a, b \neq 1$.

Then $a^x = b$, $b^y = a$ and $u = xy$.
Since $(a^x)^y = a^{xy} = b^y = a$ we have $a^{xy} = a^1$ or $xy = 1$ the required value.

16. If $M>0$, $N>0$ and $a>0$ but $a\neq 1$, prove that $\log_a \dfrac{M}{N} = \log_a M - \log_a N$.

Let $\log_a M = x$, $\log_a N = y$. Then $a^x = M$, $a^y = N$ and so

$$\frac{M}{N} = \frac{a^x}{a^y} = a^{x-y} \qquad \text{or} \qquad \log_a\frac{M}{N} = x - y = \log_a M - \log_a N$$

COUNTABILITY

17. Prove that the set of all rational numbers between 0 and 1 inclusive is countable.

Write all fractions with denominator 2, then 3, ... considering equivalent fractions such as $\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \ldots$ no more than once. Then the 1-1 correspondence with the natural numbers can be accomplished as follows.

Rational numbers	0	1	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{2}{3}$	$\frac{1}{4}$	$\frac{3}{4}$	$\frac{1}{5}$	$\frac{2}{5}$	...
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
Natural numbers	1	2	3	4	5	6	7	8	9	...

Thus the set of all rational numbers between 0 and 1 inclusive is countable and has cardinal number $\aleph_0$ (see Page 4).

18. If A and B are two countable sets, prove that the set consisting of all elements from A or B (or both) is also countable.

Since A is countable, there is a 1-1 correspondence between elements of A and the natural numbers so that we can denote these elements by $a_1, a_2, a_3, \ldots$.

Similarly we can denote the elements of B by $b_1, b_2, b_3, \ldots$.

Case 1: Suppose elements of A are all distinct from elements of B. Then the set consisting of elements from A or B is countable since we can establish the following 1-1 correspondence.

A or B	a_1	b_1	a_2	b_2	a_3	b_3	...
	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	$\updownarrow$	
Natural numbers	1	2	3	4	5	6	...

Case 2: If some elements of A and B are the same, we count them only once as in Problem 17. Then the set of elements belonging to A or B (or both) is countable.