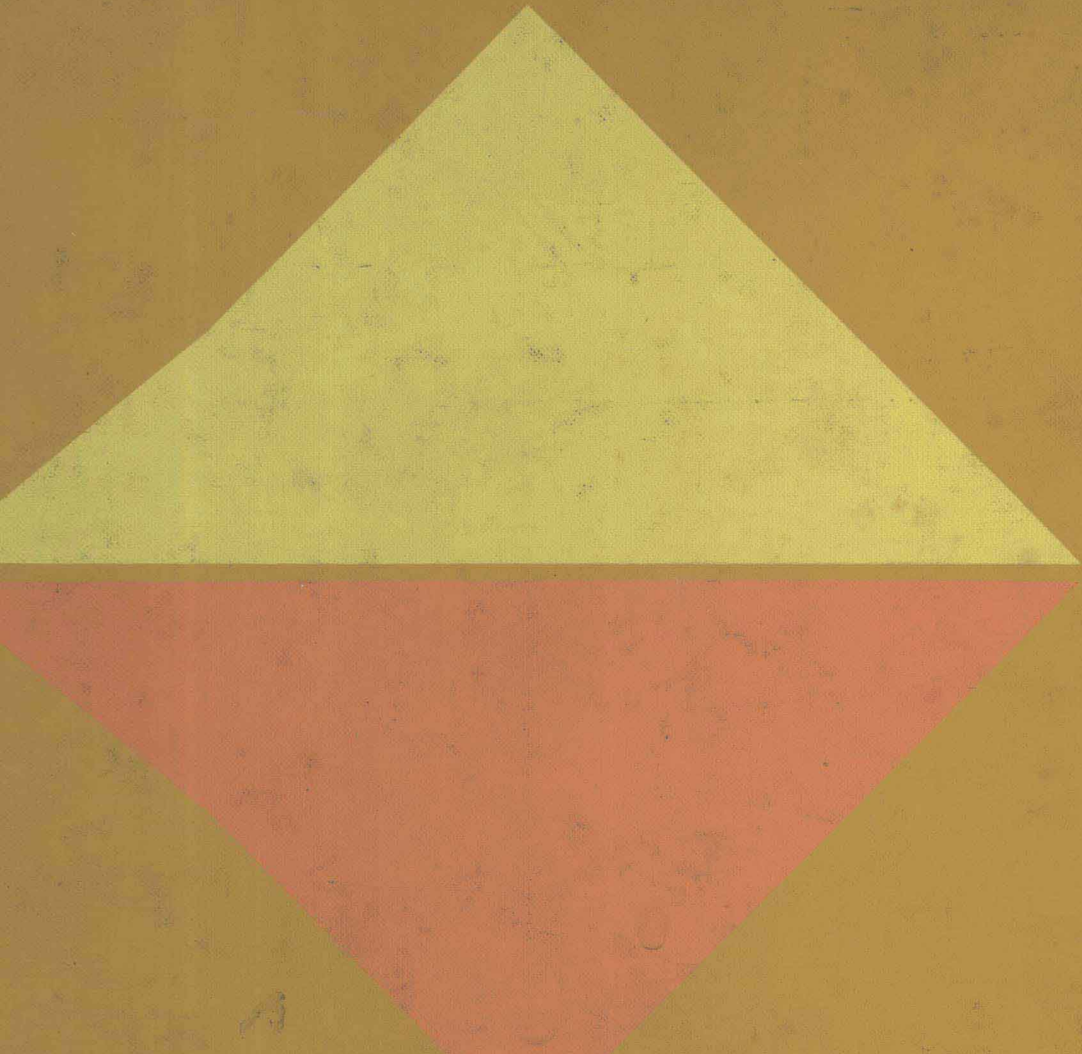

CONTEMPORARY TRIGONOMETRY

TAYLOR AND WADE

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PREFACE

This text is intended for use in an introductory trigonometry course of two to three semester hours or three to five quarter hours, or as the trigonometry portion of a combined algebra and trigonometry course.

The treatment of trigonometry is based on the fundamental concept of a two-dimensional coordinate system and the basic properties of the real numbers, together with a small amount of set notation and set algebra. The rudiments of these basic ideas are discussed in Chap. 0, with more detailed discussion given in the appendixes for use as needed.

All of the usual topics of plane trigonometry are presented, but primary emphasis is given to *analytical trigonometry*, that is, to the properties of the trigonometric functions over the real numbers and their applications to identities and equations. The trigonometric ratios of an angle are defined in Chap. 1 through the use of the idea of the circular coordinates of a point P, as well as the rectangular coordinates of P. This approach facilitates the later discussion of reduction formulas and other identities. In Chap. 2 the trigonometric ratios of a real number are defined in two ways, first by making use of the trigonometric ratios of an angle, defined in Chap. 1, and second by making use of the distance measured along the circumference of a circle (similar to, but more direct and simpler than, the "winding function" approach). The equivalence of the two definitions is discussed.

The treatment of trigonometric identities in Chap. 3 is based on a careful discussion of the meaning of an identity. The addition formulas are proved through use of the formula for the distance between two points. Solutions of trigonometric equations are given an unusually full treatment, using the concept of a solution set considered in Chap. 0. The idea of a basic solution set, that is, the members of the solution set belonging to the interval $[0;2\pi)$, is used to reduce the confusion which frequently accompanies the study of trigonometric equations. Direct and simple definitions of the inverse sine, cosine, tangent, and cotangent functions are given in Chap. 4 and their basic properties are examined.

The law of sines and the law of cosines are the chief objects of study in Chap. 5, which is concerned with the solution of triangles and finding the area of a triangle. While no extensive drill is given on the uses of logarithms, opportunity is provided in

Chaps. 5 and 7 for the student to become acquainted with logarithms and some of their uses.

The set of complex numbers is considered in Chap. 6 as the set of ordered pairs of real numbers with certain definitions of equality, addition, and multiplication. Both the rectangular form $a + bi$ and the trigonometric form $r(\cos \theta + i \sin \theta)$ of a complex number are introduced and the basic arithmetic of complex numbers is studied. De Moivre's theorem is used for finding the nth roots of any complex number.

Also in Chap. 6, two-dimensional vectors are defined, their properties studied, and their relation to the trigonometric ratios discussed. This relationship is used to give a geometric interpretation of the inner product of vectors and to solve problems concerning the resultant of forces.

Substantial portions of the material in this book, including Chaps. 4 and 5 and Tables I and II, appeared previously in the authors' book *University Freshman Mathematics*. We express our appreciation to the publishers, John Wiley and Sons, Inc., for permission to include this material in the present work.

HOWARD E. TAYLOR
THOMAS L. WADE

CONTENTS

0

BASIC CONCEPTS

0.1 **BASIC FACTS OF COORDINATE SYSTEMS** One of the most important properties of the set of real numbers* is the property known as the "completeness property," which, in its geometrical form, tells us that, given a line L, we can establish a one-to-one (order-preserving) correspondence between the points of L and the real numbers. That is, a correspondence can be established so that to each point on L there corresponds one and only one real number and, conversely, to each real number there corresponds one and only one point on L. When such a one-to-one correspondence has been established, we call the line L a number line for the real numbers or a **one-dimensional coordinate system.** Figure 0.1 illustrates such a coordinate system. On this coordinate system the point O which corresponds to the number 0 is called the origin. If the point A corresponds to the number 1, then the segment OA is called the unit of measure. A may be any point distinct from O. It is customary to take the direction to the right on the horizontal number line as the positive direction (as indicated by the arrow

one-dimensional coordinate system

* The properties of the real number system are summarized in Appendix D.

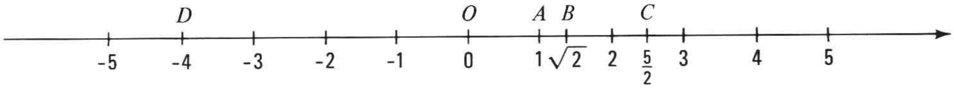


FIG. 0.1 A one-dimensional coordinate system.

in Fig. 0.1), and the direction to the left as the negative direction. This is an arbitrary choice; the positive direction can be taken to the left if we wish.

coordinate of a point

graph of a number

The real number x_1 that corresponds to a point P on a one-dimensional coordinate system is called the **coordinate of** P and P is called the **graph of** x_1. We write $P(x_1)$ to mean "the point P with coordinate x_1." Because there is a one-to-one correspondence between the points on the line and the real numbers, we frequently speak of $P(x_1)$ simply as "the point x_1." Referring to Fig. 0.1 we write $O(0)$, $A(1)$, $B(\sqrt{2})$, $C(\frac{5}{2})$, and $D(-4)$ to indicate that the point O has coordinate 0, the point A has coordinate 1, B has coordinate $\sqrt{2}$, C has coordinate $\frac{5}{2}$, and D has coordinate -4.

distance between points

If $P_1(x_1)$ and $P_2(x_2)$ are points on a coordinate line, the **distance between** P_1 and P_2 is denoted by $|P_1P_2|$ and is defined by the statement*

$$|P_1P_2| = |x_2 - x_1|$$

directed distance

The **directed distance** *from* $P_1(x_1)$ *to* $P_2(x_2)$ is denoted by $\overline{P_1P_2}$ and is defined by the statement

$$\overline{P_1P_2} = x_2 - x_1$$

Thus, the distance between P_1 and P_2 is the absolute value of the directed distance from P_1 to P_2. Referring to Fig. 0.1, we find that the directed distance from $C(\frac{5}{2})$ to $D(-4)$ is equal to $-4 - \frac{5}{2}$, which is the negative number $-\frac{13}{2}$ and D lies to the left of C. The distance between $C(\frac{5}{2})$ and $D(-4)$ is given by $|-4 - \frac{5}{2}| = |-\frac{13}{2}| = \frac{13}{2}$. The distance $|P_1P_2|$ between points P_1 and P_2 is frequently called the **length of the line segment** P_1P_2.

length of a segment

The concept of a one-dimensional coordinate system can be used to establish a one-to-one correspondence between the points in a plane and the ordered pairs of real numbers. On a plane we construct

* Recall that $|a|$ denotes the *absolute value* of a and is defined by

$$|a| = a \text{ if } a \geq 0$$
$$|a| = -a \text{ if } a < 0$$

two one-dimensional coordinate systems OX and OY perpendicular to each other and so that their origins coincide, as shown in Fig. 0.2. The two number lines so constructed form a **two-dimensional (rectangular) coordinate system.** The line OX is called the x axis and the line OY is called the y axis.

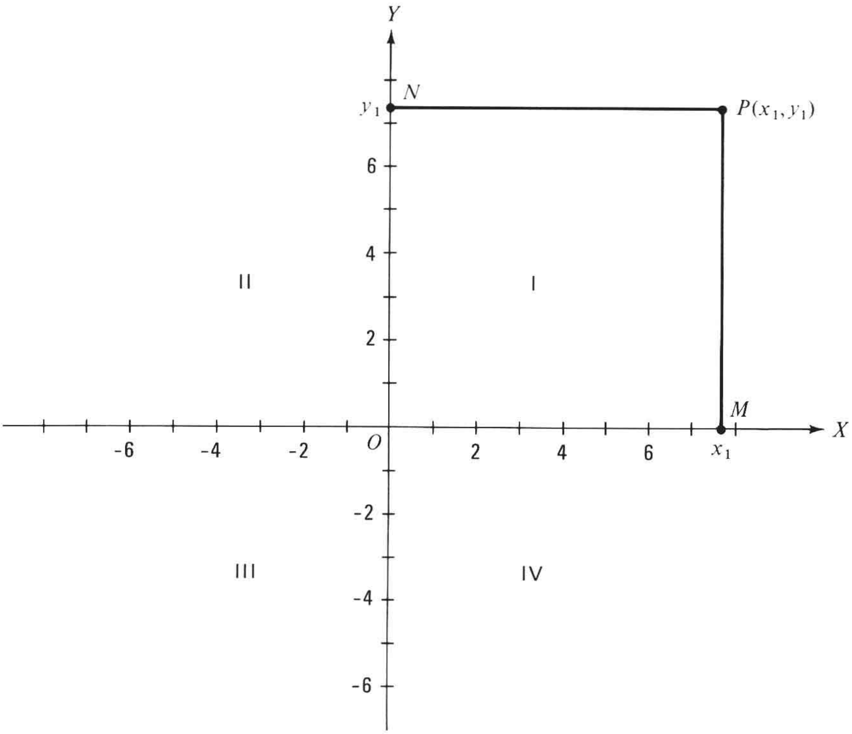
two-dimensional coordinate system

If P is any point in a plane on which a two-dimensional coordinate system has been constructed, a unique ordered pair of real numbers is associated with P in the following way. Construct two lines through P, one line perpendicular to each coordinate axis (Fig. 0.2). Let M be the point at which the line perpendicular to the x axis intersects that axis, and let N be the point at which the line perpendicular to the y axis intersects that axis. Let x_1 be the coordinate of M on the x axis, and let y_1 be the coordinate of N on the y axis. The ordered pair (x_1, y_1) is thus associated with the point P. The numbers x_1 and y_1 are the rectangular plane coordinates, or simply the **coordinates** of the point P; x_1 is called the *abscissa*, or x coordinate, of P; y_1 is called the *ordinate*, or y coordinate, of P.

abscissa

ordinate

FIG. 0.2 A two-dimensional coordinate system.



Conversely, if (x_1, y_1) is any ordered pair of real numbers, a unique point in the plane can be associated with (x_1, y_1) as follows. Construct a line perpendicular to the x axis at the point $M(x_1)$ and a line perpendicular to the y axis at the point $N(y_1)$. The point P of intersection of these two lines is the unique point associated with the ordered pair (x_1, y_1) of real numbers. This point P is called the **graph** of the ordered pair (x_1, y_1).

graph of an ordered pair

Thus, a one-to-one correspondence between the points in a plane and the ordered pairs of real numbers is established. We write $P(x_1, y_1)$ to denote "the point P with coordinates x_1 and y_1," and the point is called simply "the point (x_1, y_1)."

quadrants

The coordinate axes divide the plane into four regions, called *quadrants*. These quadrants are denoted by Q_{I}, Q_{II}, Q_{III}, and Q_{IV}. Customarily, Q_{I} is that region in which each point has both coordinates positive; Q_{II} is that region in which each point has a negative abscissa and a positive ordinate; and so on, as shown in Fig. 0.2.

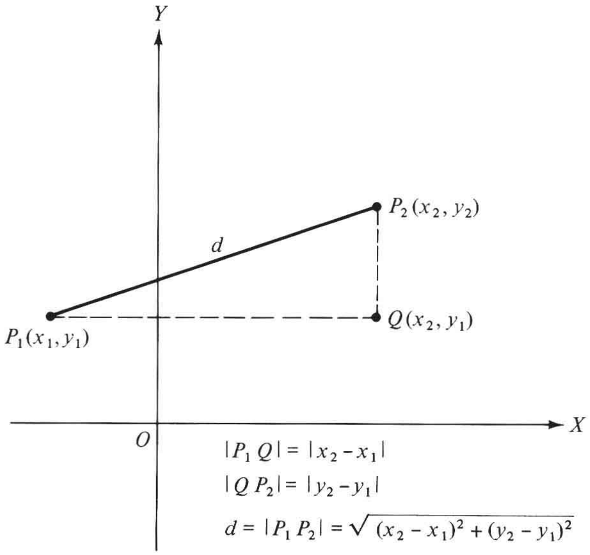
coordinate plane

A plane in which a two-dimensional coordinate system has been constructed is called a **coordinate plane.**

If $P_1(x_1, y_1)$ and $P_2(x_2, y_2)$ are two points in a coordinate plane, the (undirected) distance between P_1 and P_2 is denoted by $|P_1P_2|$ and is given by the formula (see Fig. 0.3)

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2} \tag{1}$$

FIG. 0.3 The distance formula.



Formula (1) is called the **distance formula** (for plane analytic geometry). The distance formula is a consequence of the theorem of Pythagoras: In any right triangle the square of the length of the hypotenuse is equal to the sum of the squares of the lengths of the other two sides.

Further use of two-dimensional coordinate systems in graphing relations and functions is discussed in Appendix C.

EXAMPLE 1 Find the distance between the points $P_1(9,14)$ and $P_2(-7,2)$.

Solution. Using formula (1), with $x_1 = 9$, $y_1 = 14$, $x_2 = -7$, and $y_2 = 2$, we have

$$|P_1P_2| = \sqrt{(-7-9)^2 + (2-14)^2} = \sqrt{(16)^2 + (-12)^2} = \sqrt{400} = 20$$

EXAMPLE 2 Find the distance between the points $P_3(-4,-3)$ and $P_4(2,7)$.

Solution. Using formula (1) we find

$$|P_3P_4| = \sqrt{[2-(-4)]^2 + [7-(-3)]^2} = \sqrt{36+100} = \sqrt{136} = 2\sqrt{34}$$

EXERCISES In each of Exercises 1 to 4 points P_1 and P_2 on a one-dimensional coordinate system have the given coordinates x_1 and x_2, respectively. Find $\overline{P_1P_2}$ and $|P_1P_2|$.

1. $x_1 = 9$, $x_2 = 13$ **2.** $x_1 = -6$, $x_2 = -1$
3. $x_1 = 7$, $x_2 = -4$ **4.** $x_1 = 4\sqrt{3}$, $x_2 = 3\sqrt{5}$

5. Graph each of the following points on the same coordinate system, and label each point: $P_1(3,4)$, $P_2(-2,-5)$, $P_3(5,-7)$, $P_4(0,7)$, $P_5(-4,-6)$, $P_6(6,0)$.

6. Find the distance between each of the following pairs of points:

(*a*) $(5,8)$ and $(-3,2)$ (*b*) $(2,-3)$ and $(3,3)$
(*c*) $(10,2)$ and $(-2,-2)$ (*d*) $(1,3)$ and $(-5,5)$

7. Determine whether each triangle whose vertexes are given is scalene, isosceles, or equilateral:

(*a*) $(4,4)$, $(4,-2)$, $(-4,-2)$ (*b*) $(2,6)$, $(-3,-3)$, $(6,2)$
(*c*) $(-2,-2)$, $(2,2)$, $(2\sqrt{3},-2\sqrt{3})$ (*d*) $(-3,-2)$, $(7,4)$, $(1,14)$.

8. Which of the triangles in Exercise 7 are right triangles?

0.2 SET NOTATION

We regard "set" (or "collection," or "aggregate") as an undefined concept. We consider a set as determined when we are able to decide whether or not a given object is in the set. We say that the objects that are in the set are *members* or elements of the set. To indicate that an object a is a member of the set A we write

$$a \in A$$

and we read these symbols as "a is a member of A" or as "a belongs to A." If b is not a member of A, we write

$$b \notin A$$

and read these symbols as "b is not a member of A," or as "b does not belong to A." To illustrate, if $A = \{2,4,6,8\}$, then $4 \in A$ and $5 \notin A$.

If sets A and B have precisely the same members, we say that A and B are equal sets and write $A = B$. If sets A and B are not equal, we write $A \neq B$. Thus, $\{1,2\} = \{2,1\}$, but $\{1,2\} \neq \{1,2,3\}$.

variable
universe
value of a variable

A *variable* is a symbol that represents an unspecified member of a set. The given set is called the **universal set** or the **universe** of the variable, and each member of the universe is a **value** of the variable. A useful way of determining or designating a set is by the use of a sentence S_x containing a variable. If the sentence S_x has the property that whenever x is replaced by a member of the universe U a statement is obtained which is either true or false but not both, we call S_x a condition in U. We use the symbols (see Appendix B)

$$\{x \in U \mid S_x\}$$

to denote "the set of all those members of the universe U which satisfy the condition S_x." To illustrate, if U is the set of positive integers, then

$$\{x \in U \mid x < 10\} = \{1,2,3,4,5,6,7,8,9\}$$
$$\{x \in U \mid x^2 - 7x + 10 = 0\} = \{2,5\}$$

If the universe U of the variable x is clearly indicated or understood, we abbreviate $\{x \in U \mid S_x\}$ to

$$\{x \mid S_x\}$$

empty set

The set with no members is called the **null set** or **empty set** and is denoted by $\varnothing$, which we read simply as "the empty set." We may indicate that a set A has no members by writing $A = \varnothing$, and we may indicate that a set B has at least one member by writing $B \neq \varnothing$. For example, if $U = \{0,1,2,3,4,5\}$, then

$$\{x \in U \mid x^2 - 3x + 2 = 0\} = \{1,2\}$$
$$\{x \in U \mid x^2 = 0\} = \{0\}$$
$$\{x \in U \mid x^2 + 1 = 0\} = \varnothing$$

The intersection of sets A and B, denoted by $A \cap B$, is the set of all objects that are *common to both* A and B. The union of sets A and B, denoted by $A \cup B$, is the set of all objects that are members of *at least one* of A and B. To illustrate, for $A = \{2,4,6,8\}$ and $B = \{1,2,3,4\}$, we have

$$A \cap B = \{2,4\} \qquad A \cup B = \{1,2,3,4,6,8\}$$

EXERCISES

1. Determine which of the following statements are true and which are false.

   (*a*) $2 \in \{1,2,3\}$ (*b*) $2 \in \{5,6,7\}$
   (*c*) $5 \notin \{1,2,3\}$ (*d*) $3 \notin \{1,2,3\}$

2. Describe two sets, each of which is an illustration of the empty set.
3. Find each of the following:

   (*a*) $\{1,2,3,4,5\} \cap \{3,4,5,6,7\}$
   (*b*) $\{1,2,3,4,5\} \cup \{3,4,5,6,7\}$

4. For the sets $A = \{a,b,c,d\}$ and $B = \{b,d,e\}$, find $A \cap B$, $A \cup B$, $A \cap A$, $B \cup B$.

In each of Exercises 5 to 8 list the members of the set if U is the set of positive integers less than 24.

5. $\{x \mid x^2 \in U\}$ 6. $\{x \mid x^2 \text{ is less than } 17\}$
7. $\{x \mid x^2 - 6x = 0\}$ 8. $\{x \mid x^2 - 5x + 6 = 0\}$

0.3 INEQUALITIES AND INTERVALS

$a < b$

If a and b are real numbers, we are familiar with the notation $a < b$, which we read as "a is less than b" and which means that $b - a$ is a *positive* number. In this case we also say that "b is greater than a" and we write $b > a$. We recall the fact that if c and d are any given real numbers, then one and only one of the following is true:

$$c < d \qquad d < c \qquad c = d$$

trichotomy property

This fact is known as the *trichotomy property* of the "less than" or "order" concept on the set of real numbers.

Subsequently, the universe of any variable used is understood to be the set of real numbers unless specifically stated to the contrary.

A sentence in any one of the forms*

* $a \leq b$ is read "a is less than or equal to b" and means that $b - a$ is either positive or zero.

$$a < b \qquad a \leqslant b \qquad a > b \qquad a \geqslant b \tag{2}$$

inequality

is an **inequality.** Inequalities have many useful properties, and we state here several theorems which give some of these properties.

The symbols a, b, and c used in (2) and in the theorems can be variables whose universe is the set of real numbers, or they can be expressions involving one or more variables, provided these expressions become real numbers when the variables are replaced by numbers from a given set. To illustrate,

$$3 \leqslant 5 \qquad -2 \leqslant \sqrt{2} \qquad -8 < 4$$

$$x + 2 < 5 \qquad 7x^2 - 4x \geqslant 15$$

$$x + 2y \geqslant 4x - 2y \qquad \frac{x+2}{2x-3} < \frac{17x}{5-x}$$

are all inequalities within the meaning of the word as used in this book.

Theorem 0.1 *If $a < b$ and $b < c$, then $a < c$.*

To illustrate,

If $-2 < 5$ and $5 < 10$, then $-2 < 10$
If $x + 3 < 2x - 1$ and $2x - 1 < 5$, then $x + 3 < 5$

Theorem 0.2 *$a < b$ if and only if $a + c < b + c$.*

Thus

$2 < 4$ if and only if $2 + 5 < 4 + 5$
$2x < 3y$ if and only if $2x + 4 < 3y + 4$

Since "if, then" statements and "if and only if" statements play such a prominent role in mathematics, we make use of the customary "arrow" notation. The symbols $p \Rightarrow q$ will denote the statement "if p then q," and the symbols $p \Leftrightarrow q$ will denote the statement "p if and only if q." Thus the statement of Theorem 0.1 can be written

$$(a < b \text{ and } b < c) \;\Rightarrow\; a < c$$

and the statement of Theorem 0.2 can be written

$$a < b \;\Leftrightarrow\; a + c < b + c$$

A brief discussion of the meaning and use of the "if, then" and "if and only if" symbols and other logical connectives is given in Appendix A.

Theorem 0.3 *If c is positive, then*

$$a < b \iff ca < cb$$

To illustrate,

$$4 < 7 \iff 2 \cdot 4 < 2 \cdot 7$$
$$(x-2) < 3 \iff 6(x-2) < 18$$

Theorem 0.4 *If c is negative, then*

$$a < b \iff ca > cb$$

Thus,

$$-3 < 5 \iff (-2)(-3) > (-2)(5)$$
$$(x+1) < 2 \iff -2(x+1) > -4$$

When applying Theorems 0.3 and 0.4 great care must be taken to observe the hypotheses. For the result in Theorem 0.3 to be true, *c must be positive;* for the result in Theorem 0.4 to be true, *c must be negative.*

Theorem 0.5

$$a \cdot b > 0 \iff [(a > 0 \text{ and } b > 0) \text{ or } (a < 0 \text{ and } b < 0)]$$
$$a \cdot b < 0 \iff [(a > 0 \text{ and } b < 0) \text{ or } (a < 0 \text{ and } b > 0)]$$

This theorem is the symbolic form of the statements: The product of two real numbers is positive if and only if the numbers have the same sign; the product of two numbers is negative if and only if the numbers have opposite signs.

Theorems 0.1 to 0.5 remain true if the symbols $<$ and $>$ are interchanged throughout, or if they are replaced by $\leqslant$ and $\geqslant$, respectively. Subsequently when reference is made to any one of these theorems we will mean whichever of the forms is applicable to the discussion.

If an inequality contains a variable we can speak of the solution set, or simply the **solution of an inequality** (see Appendix B). For example, if the universe of the variables is $U = \{1,2,3,4,5\}$, the solution