ANALYTICAL ENGINEERING MECHANICS

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|| *Mangalācaranaṁ* ||

Vande Lílāvatíṁ Brahma Yoginíṁ yā Jagatprasuḥ |
Sasarynikhilaṁ viśvaṁ kríŕāt kuśalcanchalaṁ ||
Prithviṁ Bhāskaraṁ Candraṁ chandomayaṁ cakāra ca ||
Bhāskara bhostvadālokaiḣ stamomuktān vidhehi naḥ |
Ganeṡobhagvānnityaṁ sammelayatu mānasaṁ |
Vijnānbhavato jnānāt santośaṁvidadhātu naḥ ||
"Sahanāvavatu sahanaubhunaktu sahavíryaṁkarvāvahai |
Tejasvināvadhítamastu mā vidviśāvahai" ||

|| *Oṁ Śāntiḥ* ||

PREFACE

Mechanics is one of the oldest sciences, fundamental to all physical sciences and their technological applications. Naturally, it is included in the curricula for students at higher levels of school education, in elementary physico–geometrical form. In contrast, mechanics was also developed as a logically consistent, rational analytical subject by foremost mathematicians, over the last three centuries. From a somewhat combined point of view, Analytical Mechanics forms parts of curricula for advanced students of mathematics, as well as those of engineering and physical sciences.

In this book, we aim to present Analytical Mechanics with nominal reference to physical ideas. Such an approach, it must be pointed out, forms the basis of higher branches of mechanics, allows adoption of computational methods for solution of problems which can not be solved analytically and has been responsible for development of theoretical physics itself!

The theory is developed using Vector Formalism requiring basic rules of Vector Algebra and Differentiation Rules for Vector Functions of a Scalar Variable. It is fascinating to note that this formalism was discovered when the development of the subject was almost complete by the end of the nineteenth century, almost as an after thought!

In applications of the theory, resolutions are always adopted making use of elementary Trigonometry and methods of Calculus. This is true for not only illustrative examples but also those having engineering application. On occasions methods of Calculus of Variations are also used, but with informal introduction. Special attention is paid to analytical formulation procedures of the problems and demonstrate the scope of analytical solution by means of a large number of Examples and Exercises. Partial solution of all the exercises is also provided for those students who can not devote that much time towards problem solving.

Beginning with the fundamental concepts of non-relativistic mechanics, definitions and Laws in the opening chapter, the book is divided into two parts. Part I on Statics consists of eight chapters and Part II on Dynamics, of nine chapters. Statics begins with a complete theory of coplanar forces in the second chapter. It includes the theory of conditions for equilibrium of a Particle and that for body of finite size, in stages so as to arrive at the necessary conditions for an arbitrary body and establishing sufficiency when the body is assumed rigid. The chapter ends with application to Simple Trusses. The third chapter deals with special problems, mainly on friction on bars and laminas in continuous contact with planes, friction on wheels and nature of initial motion under gradually increasing forces. The next three chapters deal with Centre of Gravity (Mass), equilibrium of heavy Flexible Strings (Catenaries) and Beams which may be Elastic. The seventh chapter is on the rational theory of the Variational Principle of Virtual Work and is applied to derive the necessary and sufficient conditions for equilibrium of a rigid body under coplanar forces, the form of the Common

Catenary and the Elastic Beam Equation. The next chapter proceeds in like manner for the development of the theory of Stability of Equilibrium under Conservative Field of Forces. In the ninth chapter, a presentation of forces in three dimensions is given. Without being exhaustive, a more or less complete account of Statics is given in this part. Part II begins with development of the basic general equations of Dynamics of a Particle and that of a Body whether rigid or not, under Finite and Impulsive Forces. It ends with Elastic Impact theory and Carnot's theorems on change of Kinetic Energy. The next five chapters deal with Dynamics of a Particle or a Body in Translatory Motion. The eleventh chapter is on rectilinear motion under different laws of forces. It includes Simple Harmonic Motion, Resistive Motion under Gravity and Damped Forced Oscillations. The next three chapters on motion in a plane are on investigations in Cartesian, Polar and Intrinsic Coordinates. These are applied to Elliptic Harmonic Motion, Projectiles in Resisting Media, Central Forces, Planetary Motion and Kepler's Laws, motion of Artificial Satellites and Long Range Rockets, Constrained Motion along given Curves, the Circular and Cycloidal Pendulums and the Brachistochrone Problem. The short fifteenth chapter deals with motion of a Body of Varying Mass with application to motion of Rockets. The next two chapters are mainly devoted to Planar Dynamics of a Rigid Body under Finite and Impulsive Forces with application to the Compound and Ballistic Pendulums. An account of the motion of a rigid body, free to move about a fixed point is also provided. This section also discusses the motion of a spinning top and that of a gyroscope. The concluding eighteenth chapter touches upon Lagrange's and Hamilton's Equations and Variational Principles due to Hamilton and Maupertuis. References to motion in three dimensions is nominal in theoretical development. Effort has been made to include important engineering applications wherever possible, in the appropriate context.

In writing a book on a subject to which so many savants have applied their mind, it is but natural that we had to go through a number of sources, some of which are given in the Bibliography. Even though most of them may be no more, our sincere reverence to them all. We are also deeply thankful to our teachers of the subject who unveiled its intricacies to us. Finally, we sincerely thank M/S Narosa Publishing House, for bringing this book to light.

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CONTENTS

CHAPTER 1

FUNDAMENTAL CONCEPTS AND LAWS OF MECHANICS

Mechanics is a science dealing with states of rest and motion of gross material bodies. The material bodies in mechanics are essentially regarded as *inert*, even though there may be occasional reference to a *living* body like man, but there can be no reference to any life mechanism. The subject is qualitatively divided into two: ***Statics*** essentially dealing with the state of rest and ***Dynamics*** with that of motion (more on this later). It is also divided into several branches depending on the geometrical and material nature of the bodies: (1) Mechanics of a ***Particle*** i.e. of a body whose dimensions can be neglected in comparison to another body influencing its state of rest or motion and having finite mass, and of a system of Particles; (2) Mechanics of a ***Rigid Body***, i.e. of a solid body whose *deformation* can be neglected; (3) Mechanics of *Solids* (theories of Elasticity and Plasticity); (4) Mechanics of *Fluids* (liquids and gases). The technological applications of these branches are immense : in the design of structures (buildings, bridges, dams, canals etc.), design, manufacture and operation of various machines, means of locomotion such as, automobiles, locomotives, ships, air-crafts and rockets.

The origins of mechanics go back to antiquity, rooted in the principles of simple machines like the *lever* and the *wheel*. The lever was used for amplifying muscular effort like lifting of heavy stones in construction and in weighing of objects. The wheel was used in chariots, for lifting water from wells and by potters. By this time the physical concepts of ***force*** as a kind of muscular effort and ***mass*** as inert material content of a body were understood by usage. The use of the word **"mechanics"** for the first time, appears in the works of Aristotle (384–322 B.C.) and is derived from the Greek word $\mu\eta\chi\alpha\nu\acute{\eta}$ which has the meaning "structure", "machine" or "device". The principles of the lever which form the fundamentals of statics appear in the works of Archimedes (287–213 B.C.). Centuries later, during which Positional Astronomy made great progress, impetus came from the three laws of planetary motion in 1609 and 1619 by Johannes Kepler (1571–1630). These laws heralded mathematically precise description of the shapes of the planetary orbits, the speed at which they moved and their periodic times of revolution. During roughly the same period Galileo (1564–1642), by experiments and mathematical calculation concluded uniform acceleration of freely falling bodies. He also concluded that in the absence of forces, the motion of a body is unaccelerated, that is, it is at rest or is in uniform motion in a straight line. Later, Sir Isaac Newton (1643–1727) propounded in fairly mathematically precise manner, three famous Laws of Motion for an arbitrary body moving under arbitrary forces, the first of which was actually due to Galileo as stated above. Kepler's discoveries also enabled Newton to formulate the theory of gravitational force, which subsequently led to the creation of the subject of *Celestial Mechanics.*

During the 18th and 19th century, succeeding leading mathematicians developed and wrote on mechanics. Leonhard Euler (1707–1783) systematically introduced analytical methods for solving the problems of Particle and Rigid Dynamics by the methods of calculus. Before him, Gottfreid Wilhelm Leibniz (1646-1716) solved the problem of the catenary (shape acquired by a heavy string suspended between two points) and Jakob Bernoulli (1655–1705) that of bending of elastic beams which he applied to the design of bridges. His brother Johann Bernoulli (1667-1748) studied the minimum time problem of fall along a curve and gave formalism of the laws of statics as a *Variational Principle* and the implied mathematics was formalised later by Euler as the *Calculus of Variations.* Gustav Peter Lejeune Dirichlet (1805–1859) developed the theory of equilibrium and stability in conservative field of force. Pierre-Siméon Laplace (1749–1827) and Jules-Henri Poincaré (1854–1912) made outstanding contributions in celestial mechanics. Simon-Denis Poisson (1781–1840), William Thomson (Lord Kelvin) (1824–1907), Ernst Mach (1838–1916) wrote on the subject including its conceptual foundations. The laws of dynamics were given a brilliant formalism based on the *calculus of variations* by Joseph Luis Lagrange (1736–1813) and as an alternative formalism was later given by William Rowan Hamilton (1805–1865), which later in the 20th century provided the foundation of *Quantum Mechanics* to describe motions on atomic scale. Hamilton also developed the cumbersome mathematics of now less known *Quaternions,* which was simplified by Josiah Willard Gibbs (1839–1903)– founder of *Statistical Mechanics,* to the familiar analysis of *Vectors.* The vector formalism since then has been found most suitable for developing the theory of mechanics. Gustav Robert Kirchoff (1824–1887), Georg Karl Wilhelm Hamel (1877–1954), Paul Emile Appell (1855–1930), David Hilbert (1862–1943), Charles Joseph de la Vallé Poussin (1866–1962), Stefan Banach (1892–1945), and Stephen P. Timoshenko (1878–1972) also wrote on the subject, providing it a firm analytical foundation, that has made the subject so useful for a host of applications.

To cope with the immenseness of the subject we shall treat here mostly one and two dimensional cases of Particle and Rigid Mechanics touching upon three dimensions and deformable bodies in statics only–in order just to provide flavour. Solid and Fluid Mechanics are vast subjects.

Newton's first law of motion demarcates the scopes of statics and dynamics more precisely. The law does not distinguish between states of rest and that of uniform motion in a straight line. It is a statement on *inertness* of a body. This law can be readily appreciated : when we fly in an airplane cruising uniformly we feel to be at rest; similarly when we rotate with the earth slowly in a huge circle–so that over a limited period of time the motion is nearly uniform in a straight line–we again feel at rest. The inter-relation among the causes (forces) which lead to such states really constitutes the subject of ***Statics***, while the subject matter of other types of (accelerated) motion constitutes ***Dynamics***. In dynamics, purely geometrical aspects of motion in space with increasing time is an important ingredient called ***Kinematics***. Its separate recognition was due to applications in studying motions of ***linked mechanisms*** that took place in the first half of the 19th century.

The laws of mechanics are physical in nature because they deal with physical objects. By idealised modelling of material bodies, the laws can be expressed precisely in mathematical terms. This enables development of the subject purely by mathematical reasoning, but at the same time, schematic diagrams of the actual physical system under consideration are always drawn for physical perception. Any investigation of an actual mechanical problem even though mathematical, has this dual character. Usually it ends in a physically interpretable result, just as it begins in a physical formulation.

A variety of mathematical methods are employed for the purpose of analysis. At the elementary level these are : methods of Calculus, Vectors and Calculus of Variations. A method is adopted according to the nature of formulation.

Mechanics is founded on the abstract concepts of Euclidean space, Time, Particles and Rigid Bodies as (extreme !) models of material bodies, and mass and Force. On account of the dual nature mentioned earlier, these concepts are also physically perceived and rendered measurable by physical means. We discuss these fundamental concepts in this Chapter as well as the Newton's Laws of Motion with some examples of forces. The basic concepts are really interdependent, but in understanding any one of them, the others are kept in the background.

1. Space and Time

From our sense perceptions we conclude that material bodies are contained in space–which is the subject matter of study of *Euclidean Geometry*. Space by itself is an abstract mathematical concept which is often visualised by means of physical drawings. The mathematical attribute of space is that it is densely packed by *points*, often marked by a dot on paper, which has only a *position* and no size. The relative position of two points has the attribute of *distance*. By choosing the distance between two arbitrarily *chosen* points as 1 (unity), the distance between any two other points may be expressed as integral and fractional multiple of the unit distance. Physically we can use a scale (a mete00r scale in practice) for the unit distance and read off integral and fractional *lengths of the scale* between the two points. Mathematically we admit the fractional part to be commensurate or incommensurate; so the distance between two points is a *scalar* which is a positive *real number* (rational or irrational). The position of any point P can now be determined uniquely in the following way (fig. 1.1). Select a convenient point O and three mutually perpendicular straight lines through O, two of them say in the horizontal plane and the third vertical. Introduce co-ordinate axes along them according to convenience, for example OZ vertically upwards and let (x, y, z) be the coordinates of P, where x, y and z are the distances OL, OM and ON signed appropriately (in the figure all +). The position of P is thus uniquely determined by three *algebraic numbers*–the coordinates (x, y, z) and space is said to have three *dimensions*. A more compact representation of the position of P is by means of the *position vector* **r**:

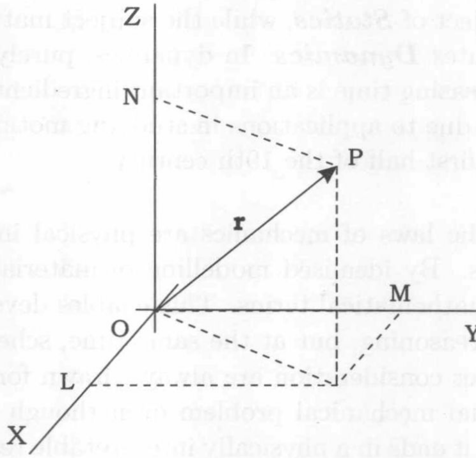

Fig. 1.1

$$\mathbf{r} := \overrightarrow{OP} = \left\{ \overline{OP}, \text{ direction of } P \text{ relative to } O \right\}$$

where $\overline{OP}$ = distance between O and P. If $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are unit vectors at O, along OX, OY, OZ, respectively, then projecting $\overrightarrow{OP}$ on the coordinate axes, we have the well known relation

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{1}$$

where according to Pythagorous Theorem, $r := |\mathbf{r}|$ (length of $\mathbf{r}$) $= \sqrt{x^2 + y^2 + z^2}$. Given a material body, how do we specify its *location physically* ? Imagine three very long *rods* along OX, OY and OZ called ***frame of reference*** with respect to which specification of location of the body is sought. We can at once say that it is the set of all position vectors **r** of the points occupied by the body.

Remark 1. The direction of **r** viz. the direction of P relative to O is determined by three algebraic quantities–the direction cosines $(x/r, y/r, z/r)$.

Remark 2. OX, OY, OZ can be selected to be oblique instead of being orthogonal. Equation (1) will still hold, but the expression for r in terms of x, y, z will be more complicated involving these non-orthogonal angles. For this reason such systems are rarely used in practice.

Remark 3. Often in simple cases we can restrict to consideration in a *plane*, so that $\mathbf{r} = x\mathbf{i} + y\mathbf{j}$, $r = \sqrt{x^2 + y^2}$. Here the dimension is two. There can even be restriction to a *straight* line with $\mathbf{r} = x\mathbf{i}, r = |x|$ where the dimension is one.

Remark 4. It has been mentioned that the unit of distance or the length of the line segment joining the two points is the *meter*. It is briefly denoted by m.

The concept of time does not arise for a body at rest or in uniform motion in a straight line (imagine cruising in an air-plane) when nothing *seems* to change. On the contrary, often our senses perceive happenings something more than ordinary. Generally speaking these are called *events*. Contained explosion of three nuclear devices on May 11, 1998 at 3:45 PM (at Pokhran) and birth of Gautam Buddha two thousand five hundred and sixty one years before on the same day, are examples of events. The orderly successive occurrence of events gives us the sense of flow of time which continually increases. Some events occur repeatedly such as beginning of the day with sunrise. The interval of repetition is however not fixed (except at the equator) and fluctuates in an interval (dependent on the latitude of the place). Had it been fixed, it could have been used as a unit for expressing time, like a meter scale for measuring distance between two points. All that is needed is to choose a suitable instant as the starting point and start measuring time as multiples and fractions (commensurate or non-commensurate) of the fixed periodic time (the unit) assigning negative values to events of the past. Earth's annual time of revolution round the sun is on the other hand fixed, but this motion being unrelated to our daily lives, is inconvenient except for astronomical purposes. In Astronomy, these difficulties are taken care of by introducing a fictitious Mean Sun, related to the actual sun, such that, its daily motion is of fixed duration of 24 hours. The clocks give fractions of this time every day. In this way, we have a physical means of expressing time at any instant by an algebraic scalar quantity t. Reverting back to events in general, they can occur simultaneously also and this helps us in ascertaining the time of occurrence of an event. For instance, at the time of occurrence of explosion of nuclear devices, the clocks showed 3:45 PM.

Remark 5. We have thought of events to occur here instantaneously as an idealisation for the purpose of treating time t as a real variable. In common parlance, it implies a limited duration of time in a limited amount of space (place). We do not highlight the later in order to understand the former. In the description of an event as something more than ordinary is a somewhat subjective statement. For instance time shown by a clock, such as, 3:45 PM may not be seen as more than ordinary by some–but it does serve our purpose for expressing time at that instant.

Remark 6. In dynamics the unit of time is taken as second and constitutes of $1/24 \times 3600 = 1/86{,}400$th of the mean solar day. It is briefly written as sec.