NUMBER SYSTEMS OF ANALYSIS

G. Cuthbert Webber

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G. CUTHBERT WEBBER

H. Fletcher Brown Professor of Mathematics,
University of Delaware

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Preface

The system of complex numbers and its subsystems form the subject of discussion of this book. The starting point is a variant of the Peano postulate system, this variant being easier to cope with at an early stage of development than is the original. The Cantor approach through Cauchy sequences forms the bridge from rational numbers to real numbers. This approach yields a natural feeling for rational approximations of real numbers and the error involved in such approximations.

The development is carried through so that each system discussed is an actual subsystem of subsequent systems. For example, the integers are members of the rational number system, not just "up to isomorphism." This seems to increase understanding so far as the embryo mathematician is concerned.

Many individuals have difficulty in their first exposure to the "abstract"; they do not seem to understand the basic ideas of proof, and they voice the plaint, "I can follow all the steps in this proof, but how do I know where and how to start a proof of my own?" The material under discussion in this book lends itself readily to an examination of these ideas. The initial stage is so simple, seemingly, that an analysis of proof can be carried out without the added complexity of new concepts, as is the case when the structures of modern algebra are encountered first. An attempt is made, even during the course of a proof, to explain why a particular step would seem to be necessary or natural. This may well be a key in the resolution of the student plaint.

The author has used this material for classes of mathematics majors at the junior and above-average sophomore level; he intends to use it in training programs for secondary teachers of mathematics. For such groups Chapters 1 through 6 could constitute a three semester hour course, if there has been little pre-exposure to the material of Chapter 1.

The author is indebted to his colleague, Willard E. Baxter, for helpful suggestions. Thanks are also due the staff of Addison-Wesley for their cooperation and assistance.

G.C.W.

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March 1966

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CHAPTER **1**

Some Basic Concepts of Mathematics

Let us state our goal at the outset. It is to develop the number systems of elementary mathematics with due regard for rigor, and in such a fashion as to give some semblance of reasonableness to the steps taken. In the course of events, many properties of the systems will be produced through a process of proof; hence it will be advisable to discuss initially the basic ideas of logic and logical procedure. Since the discussion will center on sets of numbers, we must develop definite and precise notions concerning sets, set operations, and relationships between sets; this process leads naturally to a discussion of functions, relations, and operations.

It should be stressed that there is no intention to give herein a complete or logical development of either set theory or formal logic. That is a "story" unto itself, and the interested reader should consult other sources for it. We will be content with attempting to reach a common understanding of these tools as they are normally used.

In this first chapter, many examples involving numbers will be introduced; for those examples it will be assumed that certain properties of these numbers are known, at least on an intuitive basis. The establishment of these properties will come in due course.

1–1 SETS

The concept of set has probably been used by each of us as individuals from a very early period of awareness, from the set of fingers or set of marbles stage of development on down to the set of numbers or the set of ideas stage. It is interesting to note that the theory of sets, as it grew out of the work of the German mathematicians Georg Cantor and Ernest Zermelo about 1870 and subsequently, gave insight into many of the perplexing problems of that day and paved the way for many of the mathematical advances of this century.

The concepts of *set*, *element of a set*, and *belonging to a set* are usually taken as undefined; thus they are included among the basic "building blocks" of mathematics. In normal English parlance, when one is talking about cows the word "herd" has the same connotation as does "set" or "collection"; "herd" refers to the entity, or to the whole, rather than to the individual cows which make up the entity. Thus we speak of a set (collection) of elements (objects), these elements being physical or abstract as the situation seems to require. We may consider the set of all letters of the English alphabet, the set of persons in a room, or the set of positive integers, for example; furthermore, we may consider a set consisting of the number three, a dog, the moon, and George Washington, since no relationship need exist among members of a set except the relationship prescribed by the fact that they belong to that particular set.

Let A denote a particular set one of whose elements is x; then $x \in A$ will be used to denote that x *belongs to* the set A, or is a member of A, and it should be read "x belongs to A." A set may be denoted by listing its elements, if that is possible, as follows: $\{a, b, c\}$ or $\{\bigcirc, \square, \triangle\}$. If $S = \{a, b, c\}$, then $a \in S$, $b \in S$ and $c \in S$; if d is distinct from a, b, and c, then the fact that d does not belong to S will be denoted by $d \notin S$. It is assumed that the elements of a set are distinct, hence no element will be listed more than once; the notation $\{a, a, b, c\}$ will not be used since this is the same set as $\{a, b, c\}$. Now, the notation

$$\{\text{all even integers from 6 to 100, inclusive}\}$$

would be preferable to listing the members of the set; a better notation for such a set would be

$$\{2n \mid n \text{ an integer}, 3 \leq n \leq 50\}.$$

In this last notation a property common to all members of the set, that of being divisible by 2, has been used to indicate set membership; the inequality, $3 \leq n \leq 50$, limits the even integers to exactly those desired. Likewise,

$$\{(a, b) \mid a \text{ and } b \text{ real numbers}, a = b\}$$

could denote the set of all points on the line $x = y$; this notation implies that *all* real numbers a and b for which $a = b$ should be considered.

Sets may be related to each other in various ways. For instance, $A = B$ means that sets A and B have the same elements; for every $x \in A$ it follows that $x \in B$, and for every $w \in B$ it follows that $w \in A$. A second type of relationship is expressed by the statement that A is a subset of B; A is a *subset* of B if and only if every element of A is also in B. This will be denoted by $A \subseteq B$. Thus

$$A \subseteq B \quad \text{iff} \quad x \in A \quad \text{requires} \quad x \in B, \text{ for all } x \in A$$

("iff" is an abbreviation for "if and only if", to be discussed in the next section). If $A \subseteq B$ but there are elements of B which are not in A, then A is called a *proper subset* of B; this will be denoted by $A \subset B$. Thus

$$A \subset B \quad \text{iff} \quad A \subseteq B \text{ and } A \neq B,$$

or, using different symbolism, we have

$$A \subset B \quad \text{iff} \quad A \subseteq B \ \text{ and } \ \exists x \in B \ \ni \ x \notin A.$$

The symbols $\exists$ and $\ni$ are merely abbreviations,

$$\exists \text{ for "there exists"} \qquad \text{and} \qquad \ni \text{ for "such that,"}$$

which enable us to write displays in a simple form and also to clarify relationships where the use of words might tend to cloud them. If $S = \{a, b\}$ and $T = \{a, b, c\}$, then both $S \subseteq T$ and $S \subset T$ are true, but $S \subset T$ is the "stronger" statement since it "says more." Note that $A \subseteq A$ holds for every set.

Two sets which do not have any elements in common are said to be *disjoint;* thus

$$A \text{ and } B \text{ are disjoint} \quad \text{iff} \quad \nexists x \ \ni \ x \in A \ \text{ and } \ x \in B.$$

The symbol $\nexists$ should be read "there does not exist." The sets $\{2, 4\}$ and $\{1, 3\}$ are disjoint sets, while $\{2, 4\}$ and $\{2, 3\}$ are not disjoint.

It is probable that in our very early learning years each of us took two sets of marbles, say, and put them together so as to form a single set; that is, we formed the set which consisted of each and every marble which belonged to either of the two original sets. This concept can be applied to either disjoint or nondisjoint sets; for example, from $\{1, 2, 3\}$ and $\{1, 2, 5\}$ we can form the set $\{1, 2, 3, 5\}$ by using this principle. A set formed in the above manner from A and B is called the *union* of A and B and is denoted by $A \cup B$. Thus,

$$A \cup B = \{x \mid x \in A \ \text{ or } \ x \in B\},$$

where "or" is being used in the "inclusive"* sense.

Suppose that A is the set of all persons in a particular university course denoted by $U5$ and consider the following descriptive phrases:

(1) the set of men in $U5$,
(2) the set of persons in $U5$ who are more than 20 years old,
(3) the set of persons in $U5$ whose ages lie between 10 and 25 years,
(4) the set of persons in $U5$ who are less than 3 years old.

The four phrases are alike in that each person in $U5$ is either covered or not covered by the description. For (1), (2), and (3) we would agree that the phrase describes a subset of A. For (4) there would not be any members in the set (poll the class membership, if necessary); even so, we wish to use the idea that (4) describes a set, called the *empty* set or *null* set and denoted by $\emptyset$. Since every element in $\emptyset$ is a member of B, where B is any set, then $\emptyset \subseteq B$, for any set B. As one example of the use of $\emptyset$, if B and C are disjoint sets then

$$\emptyset = \{\text{all elements which are in both } B \text{ and } C\}.$$

---

* See p. 7.

In general, the set of all elements which are in both sets X and Y is called the *intersection* of X and Y, and is denoted by $X \cap Y$. Thus

$$X \cap Y = \{x \mid x \in X \text{ and } x \in Y\}.$$

The intersection of two sets plays a significant role in set theory as a companion set operation to that of union, though little use of the concept will be made herein.

In analytic geometry each point in the plane is represented by a pair of real numbers, called its coordinates. In general the sets $\{1, 2\}$ and $\{2, 1\}$ are the same set, but in coordinate notation the "points" $(1, 2)$ and $(2, 1)$ would be different points. Here the order in which the numbers are named is of importance; such a set is called an *ordered pair*. The notation $\langle a, b\rangle$ will denote an ordered pair; likewise, $\langle a, b, c\rangle$ will denote an ordered triple. The equality of ordered pairs is defined as follows:

$$\langle a, b\rangle = \langle c, d\rangle \quad \text{iff} \quad a = c \quad \text{and} \quad b = d.$$

This concept of ordered pair can be defined formally in terms of the set concept but such a definition will not be introduced here.

Let A and B be sets and $a_i \in A$, $b_i \in B$. A set called the *cross product* of A and B, denoted by $A \times B$, can be formed from A and B as follows:

$$A \times B = \{\langle a_i, b_j\rangle \mid a_i \in A,\ b_j \in B\},$$

where $\langle c, d\rangle$ denotes an ordered pair of elements. Thus $A \times B$ consists of all ordered pairs of elements of A and B, where the first element is any element of A and the second is any element of B. For example, if $A = \{1, 2, 3\}$ and $B = \{u, v\}$, then

$$A \times B = \{\langle 1, u\rangle, \langle 2, u\rangle, \langle 3, u\rangle, \langle 1, v\rangle, \langle 2, v\rangle, \langle 3, v\rangle\}.$$

It should be noted that, in general, $A \times B \neq B \times A$.

If A and B are sets, then $A - B$ is called the *complement* of B in A and consists of all elements of A which are not in B. Thus, for $A = \{2, 3, 6, 7, 9\}$, $B = \{2, 3, 7\}$, and $C = \{1, 2, 3, 7\}$ both $A - B$ and $A - C$ are $\{6, 9\}$. In general,

$$A - B = \{\alpha \mid \alpha \in A \text{ and } \alpha \notin B\}.$$

PROBLEM SET 1–1

1. Write all the subsets of $\{a, b, c\}$.
2. Use the notation $\{\ |\ \}$ to denote each of the following sets.
   (a) The set of all integers between -3 and 7, including -3 but excluding 7.
   (b) The set of all presidents of the United States.
   (c) The set of all integers divisible by 5.

(d) The set of all ordered pairs of elements of $\{1, 3, 5\}$.
(e) The set of all ordered pairs, the first element being in B, the second in A.
(f) The set of all sets which are disjoint from A.
(g) The set of all sums of an element of A and an element of B.

3. In the following, answer each question or follow the direction given. In parts (a) through (c) the question or direction pertains to the sentence stated in symbolic form.

   (a) $\exists x \ni x \in A$ and $x \in B$. Are A and B disjoint?
   (b) $\nexists x \ni x \in A$ and $x \in B$. Name the set of all elements common to A and B.
   (c) $\exists n \ni 3n > 7$. Is this statement true of false? If true, name a number which satisfies the statement.
   (d) Is $\{\emptyset\}$ the same as $\emptyset$? Why?

4. Let $A = \{1, 2, 3\}$, $B = \{3, 5, 6\}$, and $C = \{7, 9\}$.

   (a) Find $A \cup B$ and $A \cup C$.
   (b) Write a notation for $A \times B$ by listing the elements.
   (c) The same as (b) for $B \times A$.
   (d) Find $A - B$.

1–2 ELEMENTS OF LOGIC

We usually communicate by means of sentences, sometimes using simple sentences but sometimes quite complex ones. When a compound sentence is broken down into its component parts it is easier to understand exactly what is being said. Moreover, when several sentences are used in the course of drawing a conclusion, whether or not that conclusion is valid will be easier to determine if the sentences are analyzed and their interrelationships noted. Hence we will now turn our attention to some of the basic ideas of logic, using some of the "apparatus" of formal logic so as to gain both simplification and clarification.

The equality symbol, as used in mathematics, when translated as "is equal to" becomes the verb phrase in a sentence; accordingly, $a = b$ is actually an English sentence. But what is the mathematical significance of this sentence? Quite simply, a and b are two symbols for the same "thing"; this "thing" may be a number, as in the example $6 = 4 + 2$, or it may be a set, as $A = B$ was used in the last section. Likewise, if a and b represent numbers, then $(a + b)^2 = a^2 + 2ab + b^2$ means that $(a + b)^2$ and $a^2 + 2ab + b^2$ represent the same number.

We will be dealing with declarative sentences, that is, sentences which make a statement. The sentences

(a) Jack Smith is wearing a hat,
(b) The number 7 is greater than 3,
(c) $6^2 + 5 = 2 \cdot 6 + 8$,

are all declarative sentences, and in each instance the sentence is either true or false. The logic which we are considering is usually called Aristotelian logic, after the Greek philosopher Aristotle. Two of the basic tenets of that system of logic are:

(1) A statement is either true or false (*Law of the Excluded Middle*).

(2) A statement cannot be both true and false (*Law of Contradiction*).

It will be assumed that these basic principles hold for declarative sentences.

Are the following sentences true or false?

(d) He is wearing a hat.

(e) The number x is greater than 3.

(f) $x^2 + 5 = 2 \cdot x + 8$.

In all three cases it is impossible to say whether the sentence is true or false, since it is not known to what person or to what number reference is being made. Suppose, however, that in case (d) "he" refers to some member of

$$S = \{\text{Jack Smith, Henry Tufts, Charles King}\};$$

then, when a particular element of S is used in place of "he," sentence (d) is similar to sentence (a), and its truth or falsity is determinable. Likewise, if for sentences (e) and (f) the permissible substitutions for x are restricted to $T = \{1, 4, 6, 7, 10\}$, then the truth or falsity is determinable; for example, (e) is true if and only if $x = 4$, 6, 7, or 10, and (f) is false for all members of T. Sentences such as (d), (e), and (f) are called *open sentences;* in each instance the sentence is open for substitution from some appropriate set, after which substitution the sentence is either true or false.

In the following, letters such as p and q will be used to denote sentences; compound sentences will be expressed in terms of basic sentences, p and q, by using certain connectives. It is our intention to discuss these connectives and a modifier in the subsections which follow immediately.

Negation. If p represents "Jack Smith is wearing a hat," to be denoted henceforth by

$$p\text{:} \quad \text{Jack Smith is wearing a hat,}$$

then by $\sim p$ (read: "negation of p," or "not p") is meant "Jack Smith is not wearing a hat" or "it is not true that Jack Smith is wearing a hat." Here, p and $\sim p$ have the opposite truth value in the sense that if p is true, then $\sim p$ is false. Likewise,

$$\text{if } p\text{:} \quad 2 \cdot 3 + 4 = 10, \qquad \text{then } \sim p\text{:} \quad 2 \cdot 3 + 4 \neq 10.$$

Since $\sim p$ is a modification of p, then "negation" is a "modifier."

In general, p and $\sim p$ have the opposite truth value as indicated in the table to the right. This table is called the *truth table* for negation; each possible truth value for p is shown in the first column and the resulting truth value for $\sim p$ appears in the second column. In a formal development of logic, this truth table would be used as the definition of negation.

p	$\sim p$
T	F
F	T

Conjunction. The sentence "$2 \cdot 3 + 4 = 12$ and $7^2 = 49$" can be broken into simpler sentences in an obvious manner. Let

$$p\colon\ 2 \cdot 3 + 4 = 12 \qquad \text{and} \qquad q\colon\ 7^2 = 49;$$

then this sentence can be written p and q, or in the usual notation of logic $p \wedge q$. As illustrated, the symbol $\wedge$, like others to follow, is a connective. In the above example p is false and q is true, and we would feel that $p \wedge q$ should be false. Let us agree that

$$p \wedge q \text{ is true} \qquad \text{iff} \qquad p \text{ and } q \text{ are both true.}$$

The truth table for $p \wedge q$, as given in Table 1–1(a), is in agreement with this last statement; note that there are exactly four possible combinations of T and F for the truth values of p and q.

TABLE 1–1

(a)

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

(b)

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Disjunction. In terms of the sentences p and q of the preceding paragraph,

$$2 \cdot 3 + 4 = 12 \quad \text{or} \quad 7^2 = 49$$

would be written p or q, and denoted by $p \vee q$. Now, we will probably agree that common usage indicates that $p \vee q$ is true in the preceding example, since q is true; that p is false does not affect this conclusion. Consider the following sentences:

(1) John had steak or turkey for dinner.
(2) John had steak or coffee for dinner.

Let us suppose that John had, among other things, *both* steak and coffee for dinner. We will probably agree that (1) is true, but some of us may wonder about the truth of (2). There are two kinds of "or," the exclusive "or" exemplified by (1) [an individual does not have two entrees at the same meal, in general] and the inclusive "or" exemplified by (2):

Exclusive: p or q is true **if** one of p and q is true, but not both.

Inclusive: p or q is true **if** one of p and q is true, possibly both.

Hence it is necessary that we specify which of these is meant by "or."