MEMOIRS
of the
American Mathematical Society

Number 989

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March 2011 • Volume 210 • Number 989 (third of 4 numbers) • ISSN 0065-9266

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American Mathematical Society
Providence, Rhode Island

Library of Congress Cataloging-in-Publication Data

Limit operators, collective compactness, and the spectral theory of infinite matrices / Simon N. Chandler-Wilde, Marko Lindner.

p. cm. — (Memoirs of the American Mathematical Society, ISSN 0065-9266 ; no. 989)

"March 2011, Volume 210, number 989 (third of 4 numbers)."

Includes bibliographical references and index.

ISBN 978-0-8218-5243-9 (alk. paper)

1. Fredholm operators. 2. Compact operators. 3. Infinite matrices. 4. Spectral theory (Mathematics) I. Lindner, Marko, 1973- II. Title.

QA329.2.C53 2011

515′.7246—dc22 2010046758

Memoirs of the American Mathematical Society

This journal is devoted entirely to research in pure and applied mathematics.

Publisher Item Identifier. The Publisher Item Identifier (PII) appears as a footnote on the Abstract page of each article. This alphanumeric string of characters uniquely identifies each article and can be used for future cataloguing, searching, and electronic retrieval.

Subscription information. Beginning with the January 2010 issue, *Memoirs* is accessible from `www.ams.org/journals`. The 2011 subscription begins with volume 209 and consists of six mailings, each containing one or more numbers. Subscription prices are as follows: for paper delivery, US$741 list, US$592.80 institutional member; for electronic delivery, US$667 list, US$533.60 institutional member. Upon request, subscribers to paper delivery of this journal are also entitled to receive electronic delivery. If ordering the paper version, subscribers outside the United States and India must pay a postage surcharge of US$69; subscribers in India must pay a postage surcharge of US$95. Expedited delivery to destinations in North America US$58; elsewhere US$167. Subscription renewals are subject to late fees. See `www.ams.org/help-faq` for more journal subscription information. Each number may be ordered separately; *please specify number* when ordering an individual number.

Back number information. For back issues see `www.ams.org/bookstore`.

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Memoirs of the American Mathematical Society (ISSN 0065-9266) is published bimonthly (each volume consisting usually of more than one number) by the American Mathematical Society at 201 Charles Street, Providence, RI 02904-2294 USA. Periodicals postage paid at Providence, RI. Postmaster: Send address changes to Memoirs, American Mathematical Society, 201 Charles Street, Providence, RI 02904-2294 USA.

This publication is indexed in *Science Citation Index*®, *SciSearch*®, *Research Alert*®, *CompuMath Citation Index*®, *Current Contents*®/*Physical, Chemical & Earth Sciences.*
Printed in the United States of America.

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10 9 8 7 6 5 4 3 2 1 16 15 14 13 12 11

Dedicated to Professor Bernd Silbermann on the occasion of his 67th birthday.

Abstract

In the first half of this memoir we explore the interrelationships between the abstract theory of limit operators (see e.g. the recent monographs of Rabinovich, Roch and Silbermann (2004) and Lindner (2006)) and the concepts and results of the generalised collectively compact operator theory introduced by Chandler-Wilde and Zhang (2002). We build up to results obtained by applying this generalised collectively compact operator theory to the set of limit operators of an operator A (its operator spectrum). In the second half of this memoir we study bounded linear operators on the generalised sequence space $\ell^p(\mathbb{Z}^N, U)$, where $p \in [1, \infty]$ and U is some complex Banach space. We make what seems to be a more complete study than hitherto of the connections between Fredholmness, invertibility, invertibility at infinity, and invertibility or injectivity of the set of limit operators, with some emphasis on the case when the operator A is a locally compact perturbation of the identity. Especially, we obtain stronger results than previously known for the subtle limiting cases of $p = 1$ and ∞. Our tools in this study are the results from the first half of the memoir and an exploitation of the partial duality between ℓ^1 and ℓ^∞ and its implications for bounded linear operators which are also continuous with respect to the weaker topology (the strict topology) introduced in the first half of the memoir. Results in this second half of the memoir include a new proof that injectivity of all limit operators (the classic Favard condition) implies invertibility for a general class of almost periodic operators, and characterisations of invertibility at infinity and Fredholmness for operators in the so-called Wiener algebra. In two final chapters our results are illustrated by and applied to concrete examples. Firstly, we study the spectra and essential spectra of discrete Schrödinger operators (both self-adjoint and non-self-adjoint), including operators with almost periodic and random potentials. In the final chapter we apply our results to integral operators on $\mathbb{R}^N$.

Received by the editor April 14, 2008.

Article electronically published on September 9, 2010; S 0065-9266(2010)00626-4.

2000 *Mathematics Subject Classification.* Primary 47A53, 47B07; Secondary 46N20, 46E40, 47B37, 47L80.

Key words and phrases. Infinite matrices, limit operators, collective compactness, Fredholm operators, spectral theory.

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Contents

CHAPTER 1

Introduction

1.1. Overview

This memoir develops an abstract theory of limit operators and a generalised collectively compact operator theory which can be used separately or together to obtain information on the location in the complex plane of the spectrum, essential spectrum, and pseudospectrum for large classes of linear operators arising in applications. We have in mind here differential, integral, pseudo-differential, difference, and pseudo-difference operators, in particular operators of all these types on unbounded domains. This memoir also illustrates this general theory by developing, in a more complete form than hitherto, a theory of the limit operator method in one of its most concrete forms, as it applies to bounded linear operators on spaces of sequences, where each component of the sequence takes values in some Banach space. Finally, we apply this concrete form of the theory to the analysis of lattice Schrödinger operators and to the study of integral operators on $\mathbb{R}^N$.

Let us give an idea of the methods and results that we will develop and the problems that they enable us to study. Let $Y = \ell^p = \ell^p(\mathbb{Z}, \mathbb{C})$, for $1 \le p \le \infty$, denote the usual Banach space of complex-valued bilateral sequences $x = (x(m))_{m\in\mathbb{Z}}$ for which the norm $\|x\|$ is finite; here $\|x\| := \sup_m |x(m)|$, in the case $p = \infty$, while $\|x\| := (\sum_{m\in\mathbb{Z}} |x(m)|^p)^{1/p}$ for $1 \le p < \infty$. Let $L(Y)$ denote the space of bounded linear operators on Y, and suppose $A \in L(Y)$ is given by the rule

$$Ax(m) = \sum_{n\in\mathbb{Z}} a_{mn}x(n), \quad m \in \mathbb{Z}, \tag{1.1}$$

for some coefficients $a_{mn} \in \mathbb{C}$ which we think of as elements of an infinite matrix $[A] = [a_{mn}]_{m,n\in\mathbb{Z}}$ associated with the operator A. Of course A, given by (1.1), is only a bounded operator on Y under certain constraints on the entries a_{mn}. Simple conditions that are sufficient to guarantee that $A \in L(Y)$, for $1 \le p \le \infty$, are to require that the entries are uniformly bounded, i.e.

$$\sup_{m,n} |a_{mn}| < \infty, \tag{1.2}$$

and to require that, for some $w \ge 0$, $a_{mn} = 0$ if $|m - n| > w$. If these conditions hold we say that $[A]$ is a band matrix with band-width w and that A is a band operator. We note that the tri-diagonal case $w = 1$, when A is termed a Jacobi operator, is much-studied in the mathematical physics literature (e.g. [**105, 59**]). This class includes, in particular, the one-dimensional discrete Schrödinger operator for which $a_{mn} = 1$ for $|m - n| = 1$.

It is well known (see Lemma 6.39 below and the surrounding remarks) that, under these conditions on $[A]$ (that $[A]$ is a band matrix and (1.2) holds), the spectrum of A, i.e. the set of $\lambda \in \mathbb{C}$ for which $\lambda I - A$ is not invertible as a member of the algebra $L(Y)$, is independent of p. One of our main results in Section 6.5 implies that also the essential spectrum of A (by which we mean the set of λ for which $\lambda I - A$ is not a Fredholm operator[1]) is independent of p. Moreover, we prove that the essential spectrum is determined by the behaviour of A at infinity in the following precise sense.

Let $h = (h(j))_{j\in\mathbb{N}} \subset \mathbb{Z}$ be a sequence tending to infinity for which it holds that $a_{m+h(j),n+h(j)}$ approaches a limit $\tilde{a}_{m,n}$ for every $m, n \in \mathbb{Z}$. (The existence of many such sequences is ensured by the theorem of Bolzano-Weierstrass and a diagonal argument.) Then we call the operator A_h, with matrix $[A_h] = [\tilde{a}_{mn}]$, a *limit operator* of the operator A. Moreover, following e.g. [**85**], we call the set of all limit operators of A the *operator spectrum* of A, which we denote by $\sigma^{\mathrm{op}}(A)$. In terms of these definitions our results imply that the essential spectrum of A (which is independent of $p \in [1, \infty]$) is the union of the spectra of the elements A_h of the operator spectrum of A (again, each of these spectra is independent of p). Moreover, this is also precisely the union of the point spectra (sets of eigenvalues) of the limit operators A_h in the case $p = \infty$, in symbols

$$\mathrm{spec}_{\mathrm{ess}}(A) = \cup_{A_h\in\sigma^{\mathrm{op}}(A)}\{\lambda : A_h x = \lambda x \text{ has a bounded solution } x \neq 0\}. \tag{1.3}$$

This formula and other related results have implications for the spectrum of A. In particular, if it happens that $A \in \sigma^{\mathrm{op}}(A)$ (we call A *self-similar* in that case), then it holds that

$$\mathrm{spec}(A) = \mathrm{spec}_{\mathrm{ess}}(A) = \cup_{A_h\in\sigma^{\mathrm{op}}(A)}\{\lambda : A_h x = \lambda x \text{ has a bounded solution}\}. \tag{1.4}$$

In the case $A \notin \sigma^{\mathrm{op}}(A)$ we do not have such a precise characterisation, but if we construct $B \in L(Y)$ such that $A \in \sigma^{\mathrm{op}}(B)$ (see e.g. [**63**, §3.8.2] for how to do this), then it holds that

$$\mathrm{spec}(A) \subset \mathrm{spec}_{\mathrm{ess}}(B) = \cup_{B_h\in\sigma^{\mathrm{op}}(B)}\{\lambda : B_h x = \lambda x \text{ has a bounded solution}\}. \tag{1.5}$$

A main aim of this memoir is to prove results of the above type which apply in the simple setting just outlined, but also in the more general setting where $Y = \ell^p(\mathbb{Z}^N, U)$ is a space of generalised sequences $x = (x(m))_{m\in\mathbb{Z}^N}$, for some $N \in \mathbb{N}$, taking values in some Banach space U. In this general setting the definition (1.1) makes sense if we replace $\mathbb{Z}$ by $\mathbb{Z}^N$ and understand each matrix entry a_{mn} as an element of $L(U)$. Such results are the concern of Chapter 6, and are applied to discrete Schrödinger operators and to integral operators on $\mathbb{R}^N$ in the final two chapters.

This integral operator application in Chapter 8 will illustrate how operators on $\mathbb{R}^N$ can be studied via *discretisation*. To see how this simple idea works in the case $N = 1$, let G denote the isometric isomorphism which sends $f \in L^p(\mathbb{R})$ to the sequence $x = (x(m))_{m\in\mathbb{Z}} \in \ell^p(\mathbb{Z}, L^p[0,1])$, where $x(m) \in L^p[0,1]$ is given by

$$(x(m))(t) = f(m+t), \quad m \in \mathbb{Z},\ 0 < t < 1.$$

[1]Throughout we will say that a bounded linear operator C from Banach space X to Banach space Y is: *normally solvable* if its range $C(X)$ is closed; *semi-Fredholm* if, additionally, either $\alpha(C) := \dim(\ker C)$ or $\beta(C) := \dim(Y/C(X))$ are finite; a Φ_+ operator if it is a semi-Fredholm operator with $\alpha < \infty$, and a Φ_- operator if it is semi-Fredholm with $\beta < \infty$; *Fredholm* if it is semi-Fredholm and both α and β are finite. If C is semi-Fredholm then $\alpha(C) - \beta(C)$ is called the *index* of C.

Then the spectral properties of an integral operator K on $L^p(\mathbb{R})$, whose action is given by

$$\mathsf{K}f(t) = \int_{\mathbb{R}} k(s,t)f(s)\,ds, \quad t \in \mathbb{R},$$

for some kernel function k, can be studied by considering its *discretisation* $K := G\mathsf{K}G^{-1}$. In turn K is determined by its matrix $[K] = [\kappa_{mn}]_{m,n\in\mathbb{Z}}$, with $\kappa_{mn} \in L(L^p[0,1])$ the integral operator given by

$$\kappa_{mn}g(t) = \int_0^1 k(m+s, n+t)g(s)ds, \quad 0 \le t \le 1.$$

Let us also indicate how the results we will develop are relevant to differential operators (and other non-zero order pseudo-differential operators). Consider the first order linear differential operator L, which we can think of as an operator from $BC^1(\mathbb{R})$ to $BC(\mathbb{R})$, defined by

$$\mathsf{L}y(t) = y'(t) + a(t)y(t), \quad t \in \mathbb{R},$$

for some $a \in BC(\mathbb{R})$. (Here $BC(\mathbb{R}) \subset L^\infty(\mathbb{R})$ denotes the space of bounded continuous functions on $\mathbb{R}$ and $BC^1(\mathbb{R}) := \{x \in BC(\mathbb{R}) : x' \in BC(\mathbb{R})\}$.) In the case when $a(s) \equiv 1$ it is easy to see that L is invertible. Specifically, denoting L by L_1 in this case and defining $\mathsf{C}_1 : BC(\mathbb{R}) \to BC^1(\mathbb{R})$ by

$$\mathsf{C}_1 y(t) = \int_{\mathbb{R}} \kappa(s-t)y(s)\,ds,$$

where

$$\kappa(s) := \begin{cases} e^s, & s < 0, \\ 0, & \text{otherwise}, \end{cases}$$

it is easy to check by explicit calculation that $\mathsf{L}_1\mathsf{C}_1 = \mathsf{C}_1\mathsf{L}_1 = \mathsf{I}$ (the identity operator). Thus the study of spectral properties of the differential operator L is reduced, through the identity

$$\mathsf{L} = \mathsf{L}_1 + \mathsf{M}_{a-1} = \mathsf{L}_1(\mathsf{I} + \mathsf{K}), \tag{1.6}$$

where M_{a-1} denotes the operator of multiplication by the function $a-1$, to the study of spectral properties of the integral operator $\mathsf{K} = \mathsf{C}_1\mathsf{M}_{a-1}$.

This procedure of reduction of a differential equation to an integral equation applies much more generally; indeed the above example can be viewed as a special case of a general reduction of study of a pseudo-differential operator of non-zero order to one of zero order (see e.g. [**85**, §4.4.4]). One interesting and simple generalisation is to the case where $\underline{\mathsf{L}}$ is a matrix differential operator, a bounded operator from $(BC^1(\mathbb{R}))^M$ to $(BC(\mathbb{R}))^M$ given by

$$\underline{\mathsf{L}}x(t) = x'(t) + A(t)x(t), \quad t \in \mathbb{R},$$

where A is an $M \times M$ matrix whose entries are in $BC(\mathbb{R})$. Then, modifying the above argument, the study of $\underline{\mathsf{L}}$ can be reduced to the study of the matrix integral operator $\underline{\mathsf{K}} = \underline{\mathsf{C}}\,\underline{\mathsf{M}}_{A-I}$. Here $\underline{\mathsf{M}}_{A-I}$ is the operator of multiplication by the matrix $A - I$ (I the identity matrix) and $\underline{\mathsf{C}}$ is the diagonal matrix whose entries are the (scalar) integral operator C_1.

Large parts of the generalisation to the case when the Banach space U is infinite-dimensional apply only in the case when $A = I + K$, where I is the identity operator and the entries of $[K] = [\kappa_{mn}]_{m,n\in\mathbb{Z}}$ are collectively compact. (Where $\mathcal{I}$ is some index set, a family $\{A_i : i \in \mathcal{I}\}$ of linear operators on a Banach space U is said to be *collectively compact* if $\{A_i x : i \in \mathcal{I},\, x \in U,\, \|x\| \leq 1\}$ is relatively compact in U.) The first half of this memoir (Chapters 2-5) is devoted to developing an abstract theory of limit operators, in which Y is a general Banach space and in which the role of compactness and collective compactness ideas (in an appropriate weak sense) play a prominent role. Specifically we combine the abstract theory of limit operators as expounded recently in [**85**, Chapter 1] with the generalised collectively compact operator theory developed in [**26**], building up in Chapter 5 to general results in the theory of limit operators whose power we illustrate in the second half of the memoir, deriving results of the type (1.3).

Let us give a flavour of the general theory we expound in the first half of the memoir. To do this it is helpful to first put the example we have introduced above in more abstract notation. In the case $Y = \ell^p = \ell^p(\mathbb{Z}, \mathbb{C})$, let $V_k \in L(Y)$, for $k \in \mathbb{Z}$, denote the translation operator defined by

$$V_k x(m) = x(m-k), \quad m \in \mathbb{Z}. \tag{1.7}$$

Then it follows from our definition above that A_h is a limit operator of the operator A defined by (1.1) if $[V_{-h(j)} A V_{h(j)}]$ (the matrix representation of $V_{-h(j)} A V_{h(j)}$) converges elementwise to $[A_h]$ as $j \to \infty$. Let us introduce, moreover, $P_n \in L(Y)$ defined by

$$P_n x(m) = \begin{cases} x(m), & |m| \leq n, \\ 0, & \text{otherwise.} \end{cases}$$

Given sequences $(y_n) \subset Y$ and $(B_n) \subset L(Y)$ and elements $y \in Y$ and $B \in L(Y)$ let us write $y_n \stackrel{s}{\to} y$ and say that (y_n) converges strictly to y if the sequence (y_n) is bounded and

$$\|P_m(x_n - x)\| \to 0 \text{ as } n \to \infty, \tag{1.8}$$

for every m, and write $B_n \stackrel{\mathcal{P}}{\to} B$ if the sequence (B_n) is bounded and

$$\|P_m(B_n - B)\| \to 0 \text{ and } \|(B_n - B)P_m\| \to 0 \text{ as } n \to \infty, \tag{1.9}$$

for every m. Then A_h is a limit operator of A if

$$V_{-h(n)} A V_{h(n)} \stackrel{\mathcal{P}}{\to} A_h. \tag{1.10}$$

Defining, moreover, for $b = (b(m))_{m \in Z} \in \ell^\infty$, the multiplication operator $M_b \in L(Y)$ by

$$M_b x(m) = b(m) x(m), \quad m \in \mathbb{Z}, \tag{1.11}$$

we note that A is a band operator with band width w if and only if A has a representation in the form

$$A = \sum_{|k| \leq w} M_{b_k} V_k, \tag{1.12}$$

for some $b_k \in \ell^\infty$. The set $BO(Y)$ of band operators on Y is an algebra. The Banach subalgebra of $L(Y)$ that is the closure of $BO(Y)$ in operator norm will be called the algebra of *band-dominated operators*, will be denoted by $BDO(Y)$, and will play a main role in the second half of the memoir, from Chapter 6 onwards.

In the general theory we present in the first five chapters, following [**85**] and [**26**], Y becomes an arbitrary Banach space, the specific operators P_n are replaced by a a sequence $\mathcal{P} = (P_n)_{n=0}^{\infty}$ of bounded linear operators on Y, satisfying constraints specified at the beginning of Chapter 2, the specific translation operators

V_n are replaced by a more general discrete group of isometric isomorphisms, and then the definitions (1.8), (1.9), and (1.10) are retained in essentially the same form. The notion of compactness that proves important is with respect to what we term (adapting the definition of Buck [**12**]) the *strict topology* on Y, a topology in which $\stackrel{s}{\to}$ is the sequential convergence. Moreover, when we study operators of the form $A = I + K$ it is not compactness of K with respect to the strict topology that we require (that K maps a neighbourhood of zero to a relatively compact set), but a weaker notion, that K maps bounded sets to relatively compact sets, operators having this property sometimes denoted *Montel* in the topological vector space literature. (The notions 'compact' and 'Montel' coincide in normed spaces; indeed this is also the case in metrisable topological vector spaces.)

In the remainder of this introductory chapter, building on the above short overview and flavour of the memoir, we detail a history of the limit operator method and compactness ideas applied in this context, with the aim of putting the current memoir in the context of extensive previous developments in the study of differential and pseudo-differential equations on unbounded domains; in this history, as we shall see, a prominent role and motivating force has been the development of theories for operators with almost periodic coefficients. In the last section we make a short, but slightly more detailed summary of the contents of the chapters to come.

1.2. A Brief History

The work reported in this memoir has a number of historical roots. One we have already mentioned is the paper by Buck [**12**] whose strict topology we adapt and use throughout this memoir. A main thread is the development of limit operator ideas. The historical development of this thread of research, which commences with the study of differential equations with almost periodic solutions, can be traced through the papers of Favard [**37**], Muhamadiev [**71, 72, 73, 74**], Lange and Rabinovich [**55, 56, 57**], culminating in more recent work of Rabinovich, Roch and Silbermann [**83, 84, 85**]. The other main historical thread, which has developed rather independently but overlaps strongly, is the development of collectively compact operator theory and generalisations of this theory, and its use to study well-posedness and stability of approximation methods for integral and other operator equations.

Limit Operators. To our knowledge, the first appearance of limit operator ideas is in a 1927 paper of Favard [**37**], who studied linear ordinary differential equations with almost periodic coefficients. His paper deals with systems of ODEs on the real line with almost periodic coefficients, taking the form

$$x'(t) + A(t)x(t) = f(t), \quad t \in \mathbb{R}, \tag{1.13}$$

where the $M \times M$ matrix $A(t)$ has entries that are almost periodic functions of t and the function f is almost periodic. A standard characterisation of almost periodicity is the following. Let $\mathcal{T}(A) := \{V_s A : s \in \mathbb{R}\}$ denote the set of translates of A (here $(V_s A)(t) = A(t-s)$). Then the coefficients of A are almost periodic if and only if $\mathcal{T}(A)$ is relatively compact in the norm topology on $BC(\mathbb{R})$. If A is almost periodic, the compact set that is the closure of $\mathcal{T}(A)$ is often denoted $\mathcal{H}(A)$ and called the *hull* of A. A main result in [**37**] is the following: if

$$x'(t) + \tilde{A}(t)x(t) = 0, \quad t \in \mathbb{R}, \tag{1.14}$$

has only the trivial solution in $BC^1(\mathbb{R})$, for all $\tilde{A} \in \mathcal{H}(A)$, and (1.13) has a solution in $BC^1(\mathbb{R})$, then (1.13) has a solution that is almost periodic. (Since $A \in \mathcal{H}(A)$, this is the unique solution in $BC^1(\mathbb{R})$.)

Certain of the ideas and concepts that we use in this memoir are present already in this first paper, for example the role in this concrete setting of the strict convergence $\stackrel{s}{\to}$ and of compactness arguments. In particular, conditions analogous to the requirement that (1.14) have no non-trivial bounded solutions for all $A \in \mathcal{H}(A)$ will play a strong role in this memoir. Conditions of this type are sometimes referred to as *Favard conditions* (e.g. Shubin [**102, 103**], Kurbatov [**52, 54**], Chandler-Wilde & Lindner [**20**]).

The first appearance of limit operators per se would seem to be in the work of Muhamadiev [**71, 72**]. In [**71**] Muhamadiev develops Favard's theory as follows. In terms of the differential operator $\mathsf{L} : (BC^1(\mathbb{R}))^M \to (BC(\mathbb{R}))^M$ given by (1.6), equation (1.13) is

$$\mathsf{L}x = f.$$

Under the same assumptions as Favard (that A is almost periodic and the Favard condition holds) Muhamadiev proves that $\mathsf{L} : (BC^1(\mathbb{R}))^M \to (BC(\mathbb{R}))^M$ is a bijection. Combining this result with that of Favard, it follows that L is also a bijection from $(AP^1(\mathbb{R}))^M$ to $(AP(\mathbb{R}))^M$. (Here $AP(\mathbb{R}) \subset BC(\mathbb{R})$ is the set of almost periodic functions and $AP^1(\mathbb{R}) = AP(\mathbb{R}) \cap BC^1(\mathbb{R})$.) New ideas which play an important role in the proof of these results include a method of approximating almost periodic by periodic functions and the fact that, if A is a periodic function, then injectivity of L implies invertibility. (These ideas are taken up in the proofs of Theorems 6.7 and 6.38 in Chapter 6.)

Muhamadiev also considers in the same paper the more general situation when the entries of A are in the much larger set $BUC(\mathbb{R}) \subset BC(\mathbb{R})$ of bounded uniformly continuous functions. A key property here (which follows from the Arzela-Ascoli theorem and a diagonal argument) is that, if the sequence $(t_n) \subset \mathbb{R}$ tends to infinity, then $A(\cdot - t_n)$ has a subsequence which is convergent to a limit $\tilde{A}$, uniformly on every finite interval. (Cf. the concept of a *rich* operator introduced in §5.3.) Denoting by $\mathrm{Lim}(A)$ the set of limit functions $\tilde{A}$ obtained in this way, the following theorem is stated: if (1.14) only has the trivial solution in $BC^1(\mathbb{R})$ for every $\tilde{A} \in \mathrm{Lim}(A)$ then $\tilde{\mathsf{L}} : (BC^1(\mathbb{R}))^M \to (BC(\mathbb{R}))^M$ is a bijection for every $\tilde{A} \in \mathrm{Lim}(A)$ (here $\tilde{\mathsf{L}}$ denotes the operator defined by (1.6) with A replaced by $\tilde{A}$).

This is a key result in the development of limit operator theory and it is a shame that [**71**] does not sketch what must be an interesting proof (we are told only that it 'is complicated'). Denoting by M_A the operator of multiplication by A, the set $\{\mathsf{M}_{\tilde{A}} : \tilde{A} \in \mathrm{Lim}(A)\}$ is a set of limit operators of the operator M_A, and so the set $\{\tilde{\mathsf{L}} : \tilde{A} \in \mathrm{Lim}(A)\}$ is a set of limit operators of the operator L. Thus this result takes the form: if each limit operator $\tilde{\mathsf{L}}$ is injective, specifically $\tilde{\mathsf{L}}x = 0$ has no non-trivial bounded solution, then each limit operator is invertible. A result of this form is a component in the proof of (1.3) and similar results in this memoir (and see [**20**]). In the case that A is almost periodic it is an easy exercise to show that $\mathcal{H}(A) = \mathrm{Lim}(A)$, i.e. the hull of A coincides with the set of limit functions of A (cf. Theorem 6.10). Thus this second theorem of Muhamadiev includes his result for the case when A is almost periodic.

The first extension of results of this type to multidimensional problems is the study of systems of partial differential equations in $\mathbb{R}^N$ in [**72**]. Muhamadiev studies differential operators elliptic in the sense of Petrovskii with bounded uniformly Hölder continuous coefficients, specifically those operators $\underline{\mathsf{L}}$ that are what he terms *recurrent*, by which he means that $\sigma^{\mathsf{op}}(\underline{\mathsf{L}}) = \sigma^{\mathsf{op}}(\tilde{\underline{\mathsf{L}}})$, for all $\tilde{\underline{\mathsf{L}}} \in \sigma^{\mathsf{op}}(\underline{\mathsf{L}})$. Here $\sigma^{\mathsf{op}}(\underline{\mathsf{L}})$ is an appropriate version of the operator spectrum of $\underline{\mathsf{L}}$. Precisely, where $A_p(t)$, for $t \in \mathbb{R}^N$ and for multi-indices p with $|p| \leq r$, is the family of coefficients of the operator $\underline{\mathsf{L}}$ (here r is the order of the operator), the differential operator of the same form $\tilde{\underline{\mathsf{L}}}$ with coefficients $\tilde{A}_p(t)$ is a member of $\sigma^{\mathsf{op}}(\underline{\mathsf{L}})$ if there exists a sequence $t_k \to \infty$ such that, for every p,

$$A_p(t - t_k) \to \tilde{A}_p(t) \tag{1.15}$$

uniformly on compact subsets of $\mathbb{R}^N$ as $k \to \infty$.

The main result he states is for the case where $\underline{\mathsf{L}}$ is recurrent and is also, roughly speaking, almost periodic with respect to the first $N-1$ variables. His result takes the form that if a Favard condition is satisfied ($\tilde{\underline{\mathsf{L}}}x = 0$ has no non-trivial bounded solutions for all $\tilde{\underline{\mathsf{L}}} \in \sigma^{\mathsf{op}}(\underline{\mathsf{L}})$) and if supplementary conditions are satisfied which ensure that approximations to $\underline{\mathsf{L}}$ with periodic coefficients have index zero as a mapping between appropriate spaces of periodic functions, then $\underline{\mathsf{L}}$ is invertible as an operator between appropriate spaces of bounded Hölder continuous functions.

Muhamadiev's results apply in particular in the case when the coefficients of the differential operator are almost periodic (an almost periodic function is recurrent and its set of limit functions is its hull). Shubin, as part of a review of differential (and pseudo-differential) operators with almost periodic solutions [**103**], gives a detailed account of Muhamadiev's theory, in the almost periodic scalar case (one case where Muhamadiev's supplementary conditions are satisfied), and of results which relate invertibility in spaces of bounded functions to invertibility in $L^2(\mathbb{R}^N)$. Specifically, his paper includes a proof, for a scalar elliptic differential operator L with C^∞ almost periodic coefficients, that the following are equivalent: (i) that the Favard condition holds; (ii) that L is invertible as an operator on $BC^\infty(\mathbb{R}^N)$; (iii) that L is invertible as an operator on $L^2(\mathbb{R}^N)$ in an appropriate sense.

In [**73**] Muhamadiev continues the study of the same class of differential operators L on $\mathbb{R}^N$, elliptic in the sense of Petrovskii, but now, for some of his results, with no constraints on behaviour of coefficients at infinity beyond boundedness, though his main results require also uniform Hölder continuity of all his coefficients. With this constraint (which, inter alia, is a *richness* requirement in the sense of §5.3), he studies Fredholmness (or Noethericity) of $\underline{\mathsf{L}}$ considered as a bounded operator between appropriate spaces of bounded Hölder continuous functions. It is in this paper that a connection is first made between Fredholmness of an operator and invertibility of its limit operators. The identical Favard condition to that in [**72**] plays a key role. His main results are the following: (i) that $\underline{\mathsf{L}}$ is Φ_+ iff the Favard condition holds; (ii) that if $\underline{\mathsf{L}}$ is Φ_- then all the limit operators of $\underline{\mathsf{L}}$ are surjective; (iii) (his Theorem 2.5 and his remark on p. 899) that $\underline{\mathsf{L}}$ is Fredholm iff all the limit operators of $\underline{\mathsf{L}}$ are invertible. We note further that his methods of argument in the proof of his Theorem 2.1 show moreover that if $\underline{\mathsf{L}}$ is Fredholm then the limit operators of $\underline{\mathsf{L}}$ are not only invertible but the inverses are also uniformly bounded, i.e.

$$\sup_{\tilde{\underline{\mathsf{L}}} \in \sigma^{\mathsf{op}}(\underline{\mathsf{L}})} \|\tilde{\underline{\mathsf{L}}}^{-1}\| < \infty.$$

Extensions of these results to give criteria for normal solvability and Fredholmness of $\underline{\mathsf{L}}$ as an operator on Sobolev spaces are made in [**74**].

In [**73**] Muhamadiev also, briefly, introduces what we can term a *weak limit operator*. Uniform continuity of the coefficients $A_p(t)$ is required to ensure that every sequence $t_k \to \infty$ has a subsequence, which we denote again by t_k, such that the limits (1.15) exist uniformly on compact subsets (cf. the definition of *richness* in §5.3). The set of all limit operators defined by (1.15) where the convergence is uniform on compact sets we have denoted by $\sigma^{\mathsf{op}}(\underline{\mathsf{L}})$. Muhamadiev notes that it is enough to require that the coefficients A_p be bounded (and measurable) for the same *richness* property to hold but with convergence uniformly on compact sets replaced[2] by weak convergence in $L^2(\mathbb{R}^N)$. In the case when the coefficients A_p are bounded, the set of limit operators defined by (1.15) where the convergence is weak convergence in $L^2(\mathbb{R}^N)$ we will term the set of *weak limit operators* of $\underline{\mathsf{L}}$. We note that this set coincides with $\sigma^{\mathsf{op}}(\underline{\mathsf{L}})$ in the case when each A_p is uniformly continuous. In [**74**] Muhamadiev gives criteria for Fredholmness of $\underline{\mathsf{L}}$ on certain function spaces in terms of invertibility of each of the weak limit operators of $\underline{\mathsf{L}}$.

Muhamadiev's work has been a source of inspiration for the decades that followed. For example, similar to his main results in [**73**] but much more recently, A. and V. Volpert show that, for a rather general class of scalar elliptic partial differential operators L on rather general unbounded domains and also for systems of such, a Favard condition is equivalent to the Φ_+ property of L on appropriate Hölder [**109, 110, 111**] or Sobolev [**108, 110, 111**] spaces.

Lange and Rabinovich [**55**], inspired by and building on Muhamadiev's paper [**73**], carry the idea of (semi-)Fredholm studies by means of limit operators over to the setting of operators on the discrete domain $\mathbb{Z}^N$. They give sufficient and necessary Fredholm criteria for the class $BDO(Y)$ of band-dominated operators (as defined after (1.12) and studied in more detail below in §6.3) acting on $Y = \ell^p(\mathbb{Z}^N, \mathbb{C})$ spaces. For $1 < p < \infty$, they show that such an operator is Fredholm iff all its limit operators are invertible and if their inverses are uniformly bounded. Their proof combines the limit operator arguments of Muhamadiev [**73**] with ideas of Simonenko and Kozak [**49, 100, 101**] for the construction of a Fredholm regulariser of A by a clever assembly of local regularisers. Lange and Rabinovich are thereby the first to completely characterise Fredholmness in terms of invertibility of limit operators for the general class of band-dominated operators on $\ell^p(\mathbb{Z}^N, \mathbb{C})$. Before, Simonenko [**100, 101**] was able to deal with the subclass of those operators whose coefficients (i.e. matrix diagonals) converge along rays at infinity; later Shteinberg [**104**] was able to relax this requirement to a condition of slow oscillation at infinity. Lange and Rabinovich require nothing but boundedness of the operator coefficients.

The final section of [**55**] studies (semi-)Fredholmness of operators in the so-called *Wiener algebra* $\mathcal{W}$ (see our §6.5) consisting of all operators

$$A = \sum_{k \in \mathbb{Z}^N} M_{b_k} V_k \qquad \text{with} \qquad \sum_k \|b_k\|_\infty < \infty, \tag{1.16}$$

where $b_k \in \ell^\infty(\mathbb{Z}^N, \mathbb{C})$ for every $k \in \mathbb{Z}^N$ are the coefficients (or diagonals) of the operator A and V_k and M_{b_k} are the shift and multiplication operators defined in

[2] We note that, since the coefficients A_p are bounded so that the sequence $A_p(\cdot - t_k)$ is bounded, requiring that the limits (1.15) exist uniformly on compact subsets is equivalent to requiring convergence $\overset{s}{\to}$ in the strict topology, while weak convergence in $L^2(\mathbb{R}^N)$ is equivalent to weak$*$ convergence in $L^\infty(\mathbb{R}^N)$

(1.7) and (1.11). Operators $A \in \mathcal{W}$ belong to $BDO(Y)$ for all spaces $Y = \ell^p(\mathbb{Z}^N, \mathbb{C})$, $p \in [1, \infty]$. For $p = \infty$, an analogue of the main result of [**73**] is formulated (in fact, the proof in [**55**] literally consists of the sentence 'The proofs of Theorems 4.1 and 4.2 repeat the proofs of Theorems 2.1 and 2.2 in [**73**], with obvious amendments.'): A is Φ_+ iff all its limit operators are injective, i.e. Favard's condition holds; if A is Φ_- then all its limit operators are surjective. The paper concludes with a first, simplified version of our Theorem 6.40 below, with a somewhat abbreviated proof: that $A \in \mathcal{W}$ is either Fredholm on all spaces $Y = \ell^p(\mathbb{Z}^N, \mathbb{C})$, $p \in [1, \infty]$, or on none of them. Moreover, the uniform boundedness condition on the inverses of its limit operators is redundant. The latter implies that

$$\text{spec}_{\text{ess}}(A) = \bigcup_{A_h \in \sigma^{\text{op}}(A)} \text{spec}(A_h) \tag{1.17}$$

if $A \in \mathcal{W}$, with all expressions independent of $p \in [1, \infty]$.

From here on we mainly follow the discrete branch of the limit operator story since this is the focus of our memoir, noting that the further generalisation from scalar-valued to vector-valued ℓ^p spaces $Y = \ell^p(\mathbb{Z}^N, U)$ with an arbitrary complex Banach space U enables us to emulate differential, integral and pseudo-differential operators on $L^p(\mathbb{R}^N)$ (e.g. [**56**]) by operators on the discrete space Y with $U = L^p([0, 1]^N)$ (see e.g. [**54, 84**], the discussion in the paragraphs after equation (1.5) above, and Chapter 8 below).

In the last 10 years, the limit operators of band-dominated operators on the discrete spaces $Y = \ell^p(\mathbb{Z}^N, U)$ with $p \in (1, \infty)$ have been extensively studied by Rabinovich, Roch, Silbermann and a small number of their coauthors. The first work of this troika was [**83**], where the results of [**55**] for $p \in (1, \infty)$ are picked up, this time with full proofs, and are extended, utilised and illuminated in connection with other problems and concepts such as the applicability of the so-called finite section method (a truncation method for the approximate solution of corresponding operator equations) and the idea of two different symbol calculi in the factor algebra of $BDO(Y)$ modulus compact operators. Another important result of [**83**] is the observation that the limit operator idea is compatible with the local principle of Allan [**2**] and Douglas [**36**] for the study of invertibility in non-commutative Banach algebras. The latter result was used to slightly relax the uniform boundedness condition on the inverses of the limit operators in the general Fredholm criterion [**83**, Corollary 5] and to completely remove this condition in the case of slowly oscillating coefficients [**83**, Theorem 9].

In [**84**], building on results of [**80, 83**], the same authors tackle the case when U is an arbitrary Hilbert space under the additional condition that $p = 2$ so that $Y = \ell^2(\mathbb{Z}^N, U)$ is a Hilbert space too and the set of band-dominated operators on it is a C*-algebra. In this C* setting, which makes life slightly easier than the more general case when $BDO(Y)$ is merely a Banach algebra, the serious obstacle of an infinite dimensional space U is overcome. The matrix $[A]$ that corresponds to an operator $A \in BDO(Y)$ now has operator entries $a_{ij} \in L(U)$ which are infinite dimensional operators themselves. This changes the Fredholm theory completely: An operator A with only finitely many nonzero entries a_{ij} is in general no longer of finite rank – not even compact. That is why Rabinovich, Roch and Silbermann replace the ideal $K(Y)$ of compact operators by another set, later on denoted by $K(Y, \mathcal{P})$, which is the norm closure of the set of all operators A with finitely many nonzero matrix entries. Also this set is contained in $BDO(Y)$, it is an ideal there and it is shown that if for $A \in BDO(Y)$ there exists a $K(Y,\mathcal{P})$-regulariser $B \in L(Y)$, i.e. $AB-I$ and $BA-I$ are in $K(Y,\mathcal{P})$, then automatically $B \in BDO(Y)$. If U is finite dimensional and $p \in (1,\infty)$, which was the setting of [**83**], then $K(Y,\mathcal{P})$ is the same as $K(Y)$ and invertibility modulo $K(Y,\mathcal{P})$, termed *invertibility at infinity* in [**84**], coincides with invertibility modulo $K(Y)$ alias Fredholmness. So one could argue that in [**83**] the subject already was invertibility at infinity which, as a coincidence, turned out to be Fredholmness too. In fact, the major milestone in [**84**] was to understand that the limit operator method studies invertibility at infinity and not Fredholmness, and therefore the new ideal $K(Y,\mathcal{P})$ was the right one to work with. Fortunately, invertibility at infinity and Fredholmness are closely related properties so that knowledge about one of them already says a lot about the other and so the limit operator method can still be used to make statements about Fredholmness – via invertibility at infinity.

Another problem that occurs when passing to an infinite dimensional space U is that the simple Bolzano-Weierstrass argument (coupled with a diagonal construction) previously showing that, for $A \in BDO(Y)$, every sequence $h = (h(k))_{k\in\mathbb{N}} \subset \mathbb{Z}^N$ with $|h(k)| \to \infty$ has a subsequence g such that the matrix of the translates $[V_{-g(k)}AV_{g(k)}] = [a_{i+g(k),j+g(k)}]_{i,j\in\mathbb{Z}^N}$ converges entry-wise as $k \to \infty$, is no longer applicable as the matrix diagonals are bounded sequences in the infinite dimensional space $L(U)$ now. So the class of all operators $A \in BDO(Y)$ for which every such sequence h has a subsequence g with this convergence property (the limiting operator being the limit operator A_g) had to be singled out in [**84**]. Operators of this class were later on termed *rich operators*.

There is one more technical subtlety when passing to an infinite dimensional space U: The so-called $\mathcal{P}-$convergence (1.9) that is used in (1.10) is equivalent to strong convergence $B_n \to B$ and $B_n^* \to B^*$ if $p \in (1,\infty)$ and U is finite dimensional; in fact, this is how it was treated in [**83**]. So this was another difference to [**83**] although nothing new since $\mathcal{P}-$convergence was de facto introduced for exactly this purpose by Muhamadiev [**73**] already.

The next two works in this story were the very comprehensive monograph [**85**] by the troika Rabinovich, Roch and Silbermann, which summarised the state of the art to which it largely contributed itself, and the PhD thesis [**61**] of the second author of this memoir. Both grew at roughly the same time and under mutual inspiration and support. In [**85**], besides many other things that cannot be discussed here, the case $Y = \ell^p(\mathbb{Z}^N, U)$ was successfully treated for arbitrary Banach spaces U and $p \in (1,\infty)$. The gaps at $p \in \{1,\infty\}$ are filled in [**68**] and finally in [**61**]. The challenge about $p = \infty$ is that duality, which is a frequent instrument in the arguments of [**83, 84, 85**], is more problematic since the dual space of Y is no longer one of the Y-spaces at hand. Instead one works with the predual and imposes the existence of a preadjoint operator acting on it. Note that some of these ideas have been picked up and are significantly extended and improved in Section 6.2 below.

Another important thread that should be mentioned here is the determination not only of Fredholmness but also of the Fredholm index by means of limit operators. The key paper in this respect is [**82**] by Rabinovich, Roch and Roe, where the case $N = 1$, $p = 2$, $U = \mathbb{C}$ has been studied using C^*-algebra techniques combined with $K-$theory. The idea is to decompose $Y = \ell^2(\mathbb{Z},\mathbb{C})$ into the subspaces Y_- and Y_+ that correspond to the negative and the non-negative half axis, respectively, thereby splitting the twosided infinite matrix $[A]$ of $A \in BDO(Y)$ into the four onesided infinite submatrices $[A_{--}], [A_{-+}], [A_{+-}]$ and $[A_{++}]$. Since $[A_{-+}]$ and $[A_{+-}]$ are compact (note that U has finite dimension), these two blocks can be removed without changing Fredholmness or the index. By a similar argument, for every $m \in \mathbb{N}$, the first m rows and columns of both $[A_{--}]$ and $[A_{++}]$ can be removed without losing any information about Fredholmness and the index. So it is not really surprising that also the index of A is exclusively stored in the asymptotic behaviour of the matrix entries of $[A_{--}]$ and $[A_{++}]$ at infinity, i.e. in the limit operators of A. Indeed, calling the index of $A_{\pm\pm}$, understood as an operator on $Y_{\pm}$, the $\pm-$index of A, respectively, it is shown in [**82**] that all limit operators of A with respect to sequences tending to $\pm\infty$ have the same $\pm-$index as A has, respectively. Since the index of A is the sum of its plus- and its minus-index, this gives a formula for the index of A in terms of plus- and minus-indices of two of its limit operators. The index formula of [**82**] was later carried over to the case $N = 1$, $p \in (1, \infty)$, $U = \mathbb{C}$ in [**91**] (where it was shown that the index of A does not depend on p – see [**64**] for the same result in the setting of a more general Banach space U and $p \in [1, \infty]$), re-proved by completely different techniques (using the sequence of the finite sections of A) in [**86**] and generalised to the case of an arbitrary Banach space U in case $A = I + K$ with a locally compact operator K (i.e. all entries of $[K]$ are compact operators on U) in [**81**].

The most recent extended account of the limit operator method is the monograph [**63**] by the second author. Besides a unification of techniques and results of [**61**] and [**85**], an exposition of the topic of infinite matrices, in particular band-dominated operators, that is accessible for a wide audience and a number of additions and clarifications to the theory, it also contains the first fruits of the work with the other author of this memoir. For example, it contains a treatment of boundary integral equations on unbounded surfaces (also see [**18, 19**]), their Fredholmness and finite sections, as well as more complete results on the interplay of Fredholmness and invertibility at infinity and on different aspects of the finite section method.

The above is an account of the main development of limit operator ideas and the theory of limit operators, starting with the work of Favard [**37**]. However, there are many other branches of this story (such as the "frequency limit operators" discussed in [**53**] or the "zoom limit operators" discussed in [**11**] and briefly in Section 3.6 of [**63**]) that we have not mentioned explicitly. We have also omitted mention of a number of instances where limit-operator-type ideas have been discovered and applied independently. In particular, limit-operator-type ideas have been applied recently to great effect in the spectral theory of discrete Schrödinger and Jacobi operators as well as more general bounded linear operators on Hilbert spaces. One instance is the recent work of Davies [**29, 30**], where the spectrum of a random Jacobi operator A is studied by looking at strong limits of sequences $U_n A U_n^*$, where U_n denotes a sequence of unitary operators. (This idea in the work of Davies dates back to an earlier paper of Davies and Simon [**32**], where the idea of the *limit class* of an operator is introduced, which has some similarity to the idea of an operator spectrum.) In [**29**] the notion of a pseudo-ergodic operator is introduced (we take up their study as a significant example in Chapter 7 below), this idea capturing many aspects of the spectral behaviour of random operators while eliminating probabilistic arguments. In limit operator terminology, a Jacobi