Mathematics Research Developments

Analytical and Numerical Methods for Pricing Financial Derivatives

Daniel Ševčovič
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NOVA

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Preface

The aim of this book is to acquaint the reader with basic facts and knowledge of pricing financial derivatives. It is focused on qualitative and quantitative analysis of various derivative securities. It can be used as a comprehensive textbook for graduate and undergraduate students having some background in the theory of stochastic processes, theory of partial and ordinary differential equations and their numerical approximation. It may also attract the attention of university students and their teachers with specialization on mathematical modeling in economy and finance. The book contains special topics reflecting state of the art in the pricing of derivative securities like qualitative analysis of the early exercise boundary, multi-factor interest rate models and transformation methods for pricing American style of options. These topics might be of interest to specialists in the mathematical theory of pricing financial derivatives.

Introduction

The aim of this book is to acquaint the reader with basic facts and knowledge of pricing financial derivatives. We focus our attention on qualitative analysis and practical methods of their pricing. The extensive expansion of various financial derivatives dates back to the beginning of seventies. The analysis of derivative securities was motivated by pioneering works [14, 84] due to economists Myron Scholes and Robert Merton and theoretical physicist Fisher Black. They derived and analyzed a pricing model nowadays referred to as the Black–Scholes model. The approach was indeed revolutionary as it brought the method of pricing derivative securities by means of solutions to partial differential equations. The Black–Scholes methodology enables us to price various derivatives of underlying assets as functions depending on time remaining to expiry and the underlying asset price.

The book is thematically divided into several chapters. The first ten of them can be considered as a standard introduction to pricing derivative securities by means of solutions to partial differential equations. We were deeply inspired by a comprehensive book [122] by Wilmott, Dewynne and Howison. In these introductory chapters we made an attempt to give the reader a balanced presentation of modeling issues, analytical parts as well as practical numerical realizations of derivative pricing models. We furthermore put a special attention to the comparison of theoretically computed results to financial market data. The remaining four chapters mostly represent our own research contributions to the subject of financial derivatives pricing.

In the first chapter we present a descriptive analysis of stochastic evolution processes of underlying assets and their derivative securities. We discuss basic types of derivative securities like plain vanilla options and forward contracts. Although this chapter does not have a strict mathematical character, its aim is to verbally highlight the importance of studying and analyzing financial derivatives as useful and necessary financial instruments for hedging and protecting volatile portfolios. The second chapter focuses on the standard Black–Scholes model for pricing derivative securities. Its mathematical formulation is a partial differential equation of the parabolic type, whose solution represents the price of a derivative contract. The chapter is also devoted to the presentation of the basic knowledge and facts of stochastic differential calculus, which is needed throughout the rest of the book. It is a basis for derivation of a broad class of financial derivatives pricing models. The third chapter is devoted to the classical theory of pricing European call and put options. We derive an explicit pricing formula refereed to as the Black–Scholes or Feynman-Kac formula. The content of the fourth chapter is focused on qualitative analysis of dependence of the option prices on various model parameters. We present basic concepts of analyzing financial markets including a notion of historical and implied volatilities, in particular. Then we concentrate on various sensitivity factors like Delta, Gamma, Theta, Vega and Rho of a financial derivative. These factors can be used in managing portfolios consisting of options, underlying assets and riskless money market instruments. We analyze the dependence of the sensitivity factors on underlying asset price and other model parameters. Modeling transaction costs is a main topic of the fifth chapter. In the sixth chapter we introduce basic classes of exotic derivatives and we discuss qualitative and practical aspects of their pricing. In more detail, we analyze Asian derivatives, barrier options and look-back options. In the seventh chapter, we are interested in modeling of the short interest rate. It can be considered as a nontradable underlying asset for a wide range of the so-called interest rate derivatives. In the subsequent chapter we deal with practical issues of pricing interest rate derivatives. We present the methodology of pricing bonds and other interest rate derivatives for single and multi-factor models. A special attention is put on non-arbitrage models of interest rates, such as the Vasicek or Cox-Ingersoll-Ross model and their multi-factor generalizations. The American style of financial derivatives is studied in the ninth chapter. These derivatives are characterized by a possibility of an early exercising of the financial derivative. We show that the problem of pricing American derivatives can be transformed to a mathematical problem of finding the free boundary for the Black–Scholes parabolic partial differential equations defined on a time dependent domain. In financial terminology, this free boundary is referred to as the early exercise boundary. In the tenth chapter, we present stable and robust numerical approximation methods for solving the Black–Scholes partial differential equation by means of explicit and implicit finite difference methods. We show how to solve the problem of pricing the American style of derivatives numerically by the so-called projected successive over relaxation algorithm. The eleventh chapter contains an overview of recent topics on pricing derivative securities. We present various nonlinear generalizations of the classical Black–Scholes theory. We show that, in the presence of transaction costs and risk from unprotected portfolio, the resulting pricing model is a nonlinear extension of the Black–Scholes equation in which the diffusion coefficient is no longer constant and it may depend on the option price itself. A similar nonlinear generalization of the Black–Scholes equation often arises when modeling illiquid and incomplete markets, in the presence of a dominant investor in the market, etc. We also show how to solve these nonlinear Black–Scholes models numerically. The twelfth chapter is devoted to modern transformation methods for pricing American style of derivative securities. These methods are capable of reducing the problem to construction of the early exercise position as a solution to a nonlinear integral equation. The last two chapters deal with advanced topics in modeling of interest rates. In the thirteenth chapter we focus on calibrating issues of standard one-factor interest rate models. We concentrate on estimation of the model parameters for Cox–Ingersoll–Ross model. The fourteenth chapter consists of two main parts. First, we consider an analytic approximation of bond prices in one-factor models, in which a closed explicit formula is not available yet. Then we study two-factor models, distributions of bond prices and interest rates with respect to the unobserved parameters of the model and their averaged values.

The book is designed to provide a bridge between theoretical and practical aspects of derivative securities pricing. We hope that the methodology of pricing derivative securities by means of analytical and numerical solutions to partial differential equations may attract the attention of students as well as mathematicians, engineers and practitioners having some experience with analysis and numerical solving of partial differential equations. It contains study materials which can be taught in basic and advanced courses on financial derivatives for undergraduate as well as graduate university students. The organization and presentation of the material reflects our experience with teaching the subject of financial derivatives at Comenius University and Slovak University of Technology in Bratislava, Slovakia. We thank our colleagues M. Takáč, T. Bokes and S. Kilianová for their valuable comments that helped us to improve presentation of the material contained in this book.

March 2010

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authors

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Chapter 1

The Role of Protecting Financial Portfolios

In the last decades, we have witnessed rapid expansion and development of various types of companies - starting from classical enterprises and ending with modern technological dot-com companies. One of the basic indicators of successful management and future expectations of a company is represented by the value of stock assets of the company. At the same time, it brings a defined profit in the form of dividends that are being paid to holders of stocks. From the point of view of a company development, stocks are often sources of further capitalization of the company. Although the stock price need not necessarily reflect the real value of the company, it is one of the best indicators of its present state, perspective and future development.

One of the most important problems in managing asset portfolios is the problem of effective portfolio allocation of the investment between stocks and bonds. Stocks usually bring higher returns. On the other hand, they represent a risky type of assets. Secure bonds (e.g., treasure bills) usually have lower yields, but they are less risky and volatile when compared to stocks. Investors are therefore looking for an optimal risk profile structure of their portfolios. A basic tool for protecting (hedging) an investor against risk is the so-called financial derivative. The origin of plain financial derivatives can be dated back to the 19th century. Historically first derivative security contracts were closely related to agricultural contracts for purchasing a crop. These types of derivative contracts were made during the winter season and gave the farmers a possibility of further investments and estimation of necessary amount of parcels of arable land. In modern language, these contracts can be interpreted as one of the basic type of a financial derivative called a forward. The last three decades constitute a turning point in trading financial derivatives. Derivative securities are mostly written on stocks, exchange rates or commodities. Among basic types of financial derivatives belong options and interest rate derivatives.

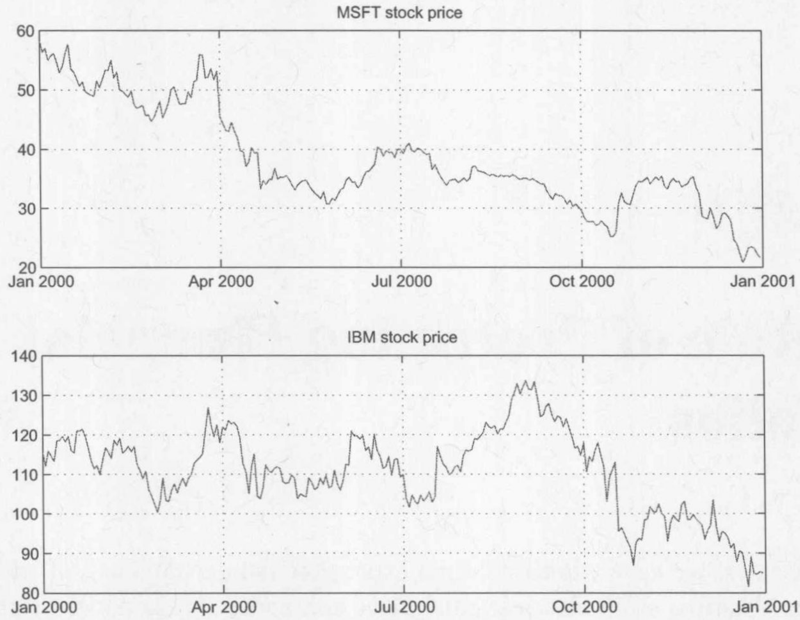

Figure 1.1. Time evolution of Microsoft (top) and IBM (bottom) stock prices in 2000.

1.1. Stochastic Character of Financial Assets

By examining financial data streams, we can get an idea of the stochastic evolution of the underlying asset prices, such as stocks, indices, interest rates and other assets. Their time development is often unstable and volatile, having fluctuations of larger or smaller sizes. These random changes are often caused by the influence of extensive trading in stock exchange markets. The stock prices are formed by supply and demand for those assets. When analyzing the time series, we often observe a certain trend in the stock prices and, at the same time, a fluctuating component of the price evolution. The trend part usually corresponds to a long term trend in the stock price, mostly influenced by a position and future expectations of the company. On the other hand, the fluctuating part can be due to balancing of demand and supply in the market.

In Fig.1.1 and Fig.1.2 we can see the time evolution of stock prices of Microsoft and IBM companies in the years 2000 and 2007-2008. The total trading volume of transactions is shown in bottom parts of both plots. The next Fig. 1.3 depicts evolution of the industrial Dow–Jones index.

The purpose of previous financial market data examples was to persuade ourselves about the stochastic evolution of various stock prices and indices on the market. We will deal with modeling of the stochastic behavior of stock prices in the following chapter. From the practical point of view, it should be emphasized that one of principal goals of investors is to minimize their possible losses from sudden decrease of stock prices. One of the most effective tools how to achieve this goal consists in usage of modern hedging instruments such as various derivative securities.

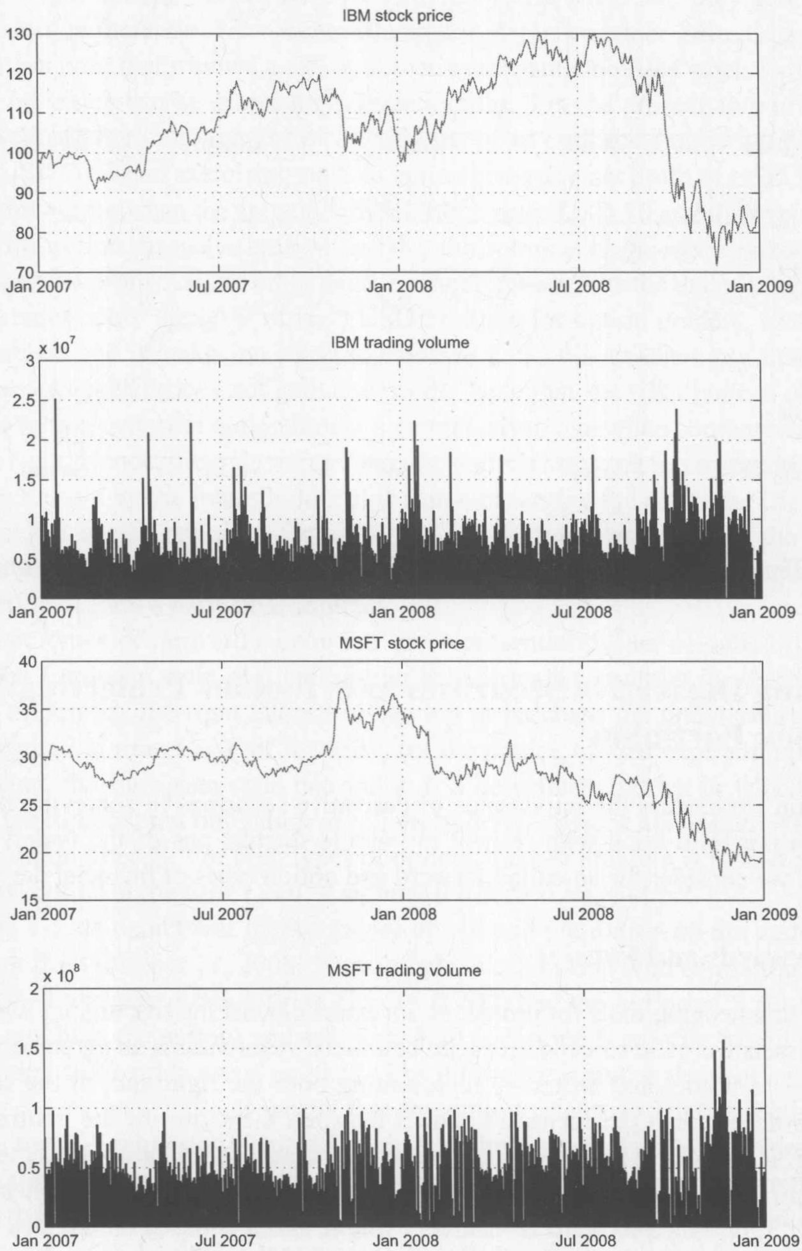

Figure 1.2. Time evolution of Microsoft (top) and IBM (bottom) stock prices in 2007 and 2008 and their trading volumes.