ISAAC

Series on Analysis, Applications and Computation – Vol. 3

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Topics in Mathematical Analysis

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Topics in Mathematical Analysis

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Vol. 2: Complex Analysis
by M W Wong

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Preface

The *Minicorsi of Mathematical Analysis* have been held at the University of Padova since 1998, and the subject of the Lectures ranges in various areas of Mathematical Analysis including Complex Variable, Differential Equations, Geometric Measure Theory, Harmonic Analysis, Potential Theory, Spectral Theory.

The purposes of the Minicorsi are:

- to provide an update on the most recent research themes in the field,
- to provide a presentation accessible also to beginners.

The Lecturers have been selected both on the basis of their outstanding scientific level, and on their clarity of exposition. Thus the Minicorsi and the present collection of Lectures are particularly indicated to young Researchers and to Graduate Students.

In this volume, the organizers have collected most of the lectures held in the years 2000–2003, and intend to provide the reader with material otherwise difficult to find and written in a way also accessible to nonexperts.

The organizers wish to express their sincere gratitude to the several participants who have contributed to the success of the Minicorsi.

The organizers are also indebted to the University of Padova, and in particular to the 'Dipartimento di Metodi e Modelli Matematici per le Scienze Applicate', and to the 'Dipartimento di Matematica Pura ed Applicata' of the University of Padova, both for the hospitality, and for the financial support. The organizers also acknowledge the financial support offered by the 'Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni', and the European Commission IHP Network "Harmonic Analysis and Related Problems".

P. Ciatti, E. Gonzalez, M. Lanza de Cristoforis, G.P. Leonardi

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PART 1

Complex variables and potential theory

Chapter 1

Integral representations in complex, hypercomplex and Clifford analysis*

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Contents

1.1. Introduction

Integral representations are one of the main tools in analysis. They are useful to determine properties of the functions represented such as smoothness, differentiability, boundary behaviour etc. They serve to reduce boundary value problems etc. for differential equations to integral equations and thus lead to existence and uniqueness proofs. Well–known representation formulas are the Cauchy formula for analytic functions and the Green representation for harmonic functions. Both these formulas are consequences from

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See also http://www.tandf.co.uk/journals/titles/10652469.asp.
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the Gauss divergence theorem where the area integral disappears because homogeneous equations (Cauchy–Riemann and Laplace, respectively) are considered. In the cases of inhomogeneous Cauchy–Riemann equations and the Poisson equation, the area integrals appearing lead to area integral operators of the Pompeiu type. They determine particular solutions to the inhomogeneous equation under consideration.

Now we make the following simple observation. Let ∂ be a linear differential operator and T be its related Pompeiu integral operator. Then ∂T is the identity mapping for a proper function space. For any power ∂^k, $k \in \mathbb{N}$, then the iteration T^k obviously is its right inverse, $\partial^k T^k$ is the identity again. More generally for two such differential operators ∂_1, ∂_2 with right inverses T_1, T_2 then the iteration $T_2 T_1$ is right inverse to $\partial_1 \partial_2$.

On this basis particular solutions for higher order differential operators can be constructed leading also to fundamental solutions. Moreover, these integral operators are useful for determining particular solutions to any higher order differential equation the leading term of which is related to them. In fact, one can solve boundary value problems to these higher order equations if, besides the particular solution for the leading term through the Pompeiu operator, the general solution to the related homogeneous leading term operator equation is taken into consideration.

This sketched procedure can be followed in complex, hypercomplex and Clifford analysis. But the resulting representation formulas of Cauchy-Pompeiu type do not automatically give solutions to related boundary value problems. This, however, is the case whenever these problems are solvable. This phenomenon is known already from the Cauchy formula. Not all functions on the boundary of a domain are boundary values of the analytic functions determined by their Cauchy integrals. In particular solvability conditions have to be observed in the theory of several complex variables where also compatibility conditions for the systems considered are important.

In these lectures the hierarchy of Pompeiu integral operators in the complex case will be presented and some higher order Cauchy–Pompeiu representation formulas given. As an application some orthogonal decompositions of the Hilbert space $L_2(G;\mathbb{C})$, $G \subset \mathbb{C}$ a regular domain, are given. For several complex variables only some results on bidomains are included. As the theory in hypercomplex analysis is analogue to the complex case only some references [2, 9] are given. The situation in Clifford analysis is shortly explained.

1.2. Complex case

Gauss divergence theorem. Let $D \subset \mathbb{R}^2$ be a regular domain, $f, g \in C^1(D;\mathbb{R}) \cap C(\overline{D};\mathbb{R})$. Then

$$\int_D (f_x + g_y)\, dx\, dy = \int_{\partial D} \{f\, dy - g\, dx\}\,.$$

Complex forms: $z = x + iy$, $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)\,, \partial_{\overline{z}} = \frac{1}{2}(\partial_x + i\partial_y)$, $w = u + iv \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$. Then

$$\int_D w_{\overline{z}} dx\, dy = \frac{1}{2i}\int_{\partial D} w\, dz\,, \quad \int_D w_z dx\, dy = -\frac{1}{2i}\int_{\partial D} w\, d\overline{z}\,.$$

Cauchy–Pompeiu Representation. Let $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$. Then with $\zeta = \xi + i\eta$

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\,\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\overline{\zeta}}(\zeta)\,\frac{d\xi\, d\eta}{\zeta - z}\,,$$

$$w(z) = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\,\frac{d\overline{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int_D w_{\zeta}(\zeta)\,\frac{d\xi\, d\eta}{\overline{\zeta - z}}\,.$$

Pompeiu Operator. Let $f \in L_1(D;\mathbb{C})$. Then

$$Tf(z) = -\frac{1}{\pi}\int_D f(\zeta)\,\frac{d\xi\, d\eta}{\zeta - z}\,, \quad \overline{T}f(z) = -\frac{1}{\pi}\int_D f(\zeta)\,\frac{d\xi\, d\eta}{\overline{\zeta - z}}\,.$$

Properties of these operators are developed in [13], see also [1]. Important are

$$\partial_{\overline{z}} Tf = f\,, \quad \partial_z Tf = \Pi f\,, \quad f \in L_1(D;\mathbb{C})\,,$$

where

$$\Pi f(z) = -\frac{1}{\pi}\int_D f(\zeta)\,\frac{d\xi\, d\eta}{(\zeta - z)^2}$$

is a singular integral operator of Calderon–Zygmund type to be taken as a Cauchy principal integral. Here the derivatives are taken in the weak sense.

1.2.1. *Complex first order systems*

Theorem 1.1. *Any solution to $w_{\overline{z}} = f$ in D, $f \in L_1(D;\mathbb{C})$, is representable via $w = \varphi + Tf$ where φ is analytic in D.*

Proof. (1) Obviously $\varphi + Tf$ with $\varphi_{\overline{z}} = 0$ is a solution.
(2) If w is a solution then $(w - Tf)_{\overline{z}} = 0$ i.e. is analytic. □

Generalized Beltrami equation:

$$w_{\overline{z}} + \mu_1 w_z + \mu_2 \overline{w_z} + aw + b\overline{w} = f\,, \quad |\mu_1(z)| + |\mu_2(z)| \le q_0 < 1\,.$$

Find a particular solution in the form $w = T\rho$! Then ρ must satisfy the singular integral equation

$$\rho + \mu_1 \Pi\rho + \mu_2 \overline{\Pi\rho} + a\,T\rho + b\,\overline{T\rho} = f\,.$$

As $\mu_1\Pi\rho + \mu_2\overline{\Pi\rho}$ is contractive and $a\,T\rho + b\,\overline{T\rho}$ is compact this problem is solvable.

1.2.2. *Complex second order equations*

There are two principally different second order elliptic differential operators the main part of which is either the Laplace or the Bitsadze operator. As in the case of the generalized Beltrami equation the solutions to the inhomogeneous Laplace and Bitsadze equations can be used to solve the general equations of second order.

1.2.2.1. *Poisson equation* $w_{z\overline{z}} = f$

The Cauchy–Pompeiu formulas

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\,\frac{d\zeta}{\zeta - \tau} - \frac{1}{\pi}\int_D w_{\overline{\widetilde{\zeta}}}(\widetilde{\zeta})\,\frac{d\widetilde{\xi}\,d\widetilde{\eta}}{\widetilde{\zeta} - z}\,,$$

$$w_{\overline{\widetilde{\zeta}}}(\widetilde{\zeta}) = -\frac{1}{2\pi i}\int_{\partial D} w_{\overline{\zeta}}(\zeta)\,\frac{d\overline{\zeta}}{\overline{\zeta - \widetilde{\zeta}}} - \frac{1}{\pi}\int_D w_{\zeta\overline{\zeta}}(\zeta)\,\frac{d\xi\,d\eta}{\overline{\zeta - \widetilde{\zeta}}}$$

lead to

$$w(z) = \frac{1}{2\pi i}\int_{\partial D}\left\{\frac{w(\zeta)}{\zeta - z}\,d\zeta - \psi(\zeta, z)\,w_{\overline{\zeta}}(\zeta)\,d\overline{\zeta}\right\} - \frac{1}{\pi}\int_D \psi(\zeta, z)\,w_{\zeta\overline{\zeta}}(\zeta)\,d\xi\,d\eta \tag{1.1}$$

with

$$\psi(\zeta, z) = \frac{1}{\pi}\int_D \frac{1}{\overline{\widetilde{\zeta} - \zeta}}\,\frac{d\widetilde{\xi}\,d\widetilde{\eta}}{\widetilde{\zeta} - z}\,.$$

Applying the Cauchy–Pompeiu formula to $\log|\zeta - z|^2$ in the domain $D_\varepsilon = D\backslash\{z : |z-\zeta| \le \varepsilon\}$ for sufficiently small positive ε gives

$$\log|\zeta - z|^2 = \frac{1}{2\pi i}\int_{\partial D_\varepsilon} \log|\zeta - \tilde{\zeta}|^2 \frac{d\overline{\tilde{\zeta}}}{\overline{\tilde{\zeta}} - \overline{z}} - \frac{1}{\pi}\int_{D_\varepsilon} \frac{1}{\overline{\tilde{\zeta}} - \overline{\zeta}}\,\frac{d\tilde{\xi}\,d\tilde{\eta}}{\overline{\tilde{\zeta}} - \overline{z}}\,.$$

As for $\zeta \neq z$

$$\frac{1}{2\pi i}\int_{|\tilde{\zeta}-\zeta|=\varepsilon} \log|\zeta - \tilde{\zeta}|^2 \frac{d\overline{\tilde{\zeta}}}{\overline{\tilde{\zeta}} - \overline{z}} = \frac{\varepsilon\log\varepsilon}{\pi}\int_0^{2\pi} \frac{e^{it}}{\varepsilon e^{it} + \zeta - z}\,dt$$

and

$$\frac{1}{\pi}\int_{|\tilde{\zeta}-\zeta|<\varepsilon} \frac{1}{\overline{\tilde{\zeta}} - \overline{\zeta}}\,\frac{d\tilde{\xi}\,d\tilde{\eta}}{\overline{\tilde{\zeta}} - \overline{z}} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\varepsilon} e^{it}\frac{dt\,dr}{re^{it} + \zeta - z}$$

tend to zero with ε then

$$\log|\zeta - z|^2 = \overline{\tilde{\psi}(\zeta, z)} - \overline{\psi(\zeta, z)} \tag{1.2}$$

with

$$\tilde{\psi}(\zeta, z) = \frac{1}{2\pi i}\int_{\partial D} \log|\tilde{\zeta} - \zeta|^2 \frac{d\tilde{\zeta}}{\tilde{\zeta} - z}\,.$$

Because for $z, \zeta \in D$

$$\begin{aligned}\partial_\zeta\tilde{\psi}(\zeta, z) &= -\frac{1}{2\pi i}\int_{\partial D} \frac{d\tilde{\zeta}}{(\tilde{\zeta} - \zeta)(\tilde{\zeta} - z)} \\ &= -\frac{1}{2\pi i}\int_{\partial D}\left(\frac{1}{\tilde{\zeta} - \zeta} - \frac{1}{\tilde{\zeta} - z}\right)\frac{d\tilde{\zeta}}{\zeta - z} = 0\end{aligned}$$

from the Gauss theorem

$$\frac{1}{2\pi i}\int_{\partial D} \tilde{\psi}(\zeta, z)\, w_{\overline{\zeta}}(\zeta)\, d\overline{\zeta} + \frac{1}{\pi}\int_D \tilde{\psi}(\zeta, z)\, w_{\zeta\overline{\zeta}}(\zeta)\, d\xi\, d\eta = 0\,.$$

Adding this to the right–hand side of (1.1) and observing (1.2) show

$$\begin{aligned}w(z) = \frac{1}{2\pi i}\int_{\partial D} \frac{w(\zeta)}{\zeta - z}\, d\zeta &+ \frac{1}{2\pi i}\int_{\partial D} \log|\zeta - z|^2 w_{\overline{\zeta}}(\zeta)\, d\overline{\zeta} \\ &+ \frac{1}{\pi}\int_D \log|\zeta - z|^2 w_{\zeta\overline{\zeta}}(\zeta)\, d\xi\, d\eta\,.\end{aligned} \tag{1.3}$$

As is well known $2/\pi \log|\zeta - z|$ is the fundamental solution to the Laplacian $\partial_z\partial_{\overline{z}}$. The representation (1.3) has the form

$$w = \varphi + \overline{\psi} + T_{1,1}f\,, \qquad f = w_{z\overline{z}}\,,$$

with analytic functions φ and ψ.

1.2.2.2. *Bitsadze equation* $w_{\overline{z}\,\overline{z}} = f$

Similarly to the preceding subsection the Cauchy–Pompeiu formulas

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\,\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\overline{\widetilde{\zeta}}}(\widetilde{\zeta})\,\frac{d\widetilde{\xi}\,d\widetilde{\eta}}{\widetilde{\zeta} - z}\,,$$

$$w_{\overline{\widetilde{\zeta}}}(\widetilde{\zeta}) = \frac{1}{2\pi i}\int_{\partial D} w_{\overline{\zeta}}(\zeta)\,\frac{d\zeta}{\zeta - \widetilde{\zeta}} - \frac{1}{\pi}\int_D w_{\overline{\zeta}\,\overline{\zeta}}(\zeta)\,\frac{d\xi\,d\eta}{\zeta - \widetilde{\zeta}}$$

imply

$$w(z) = \frac{1}{2\pi i}\int_{\partial D}\left\{\frac{w(\zeta)}{\zeta - z} + \psi(\zeta, z)\,w_{\overline{\zeta}}(\zeta)\right\}d\zeta - \frac{1}{\pi}\int_D \psi(\zeta, z)\,w_{\overline{\zeta}\,\overline{\zeta}}(\zeta)\,d\xi\,d\eta \tag{1.4}$$

with

$$\psi(\zeta, z) = \frac{1}{\pi}\int_D \frac{1}{\widetilde{\zeta} - \zeta}\,\frac{d\widetilde{\xi}\,d\widetilde{\eta}}{\widetilde{\zeta} - z}\,.$$

Applying the Cauchy–Pompeiu formula to $\frac{\overline{\zeta - z}}{\zeta - z}$ in the domain $D_\varepsilon = D\backslash\{z : |z - \zeta| \le \varepsilon\}$ for sufficiently small positive ε gives

$$\frac{\overline{\zeta - z}}{\zeta - z} = \frac{1}{2\pi i}\int_{\partial D_\varepsilon} \frac{\overline{\zeta - \widetilde{\zeta}}}{\zeta - \widetilde{\zeta}}\,\frac{d\widetilde{\zeta}}{\widetilde{\zeta} - z} + \frac{1}{\pi}\int_{D_\varepsilon}\frac{1}{\zeta - \widetilde{\zeta}}\,\frac{d\widetilde{\xi}\,d\widetilde{\eta}}{\widetilde{\zeta} - z}\,.$$

Observing that when ε tends to zero

$$\frac{1}{2\pi i}\int_{|\widetilde{\zeta} - \zeta| = \varepsilon}\frac{\overline{\widetilde{\zeta} - \zeta}}{\widetilde{\zeta} - \zeta}\,\frac{d\widetilde{\zeta}}{\widetilde{\zeta} - z} = \frac{\varepsilon}{2\pi}\int_0^{2\pi} e^{-it}\,\frac{dt}{\varepsilon e^{it} + \zeta - z}$$

and

$$\frac{1}{\pi}\int_{|\widetilde{\zeta} - \zeta| < \varepsilon}\frac{1}{\widetilde{\zeta} - \zeta}\,\frac{d\widetilde{\zeta}\,d\widetilde{\eta}}{\widetilde{\zeta} - z} = \frac{1}{\pi}\int_0^{2\pi}\int_0^{\varepsilon} e^{-it}\,\frac{dt\,dr}{re^{it} + \zeta - z}$$

tend to zero then

$$\frac{\overline{\zeta - z}}{\zeta - z} = \widetilde{\psi}(\zeta, z) - \psi(\zeta, z) \tag{1.5}$$

with

$$\widetilde{\psi}(\zeta, z) = \frac{1}{2\pi i}\int_{\partial D}\frac{\overline{\widetilde{\zeta} - \zeta}}{\widetilde{\zeta} - \zeta}\,\frac{d\widetilde{\zeta}}{\widetilde{\zeta} - z}\,.$$

As for $z, \zeta \in D$

$$\partial_{\overline{\zeta}}\widetilde{\psi}(\zeta, z) = -\frac{1}{2\pi i}\int_{\partial D}\frac{1}{\widetilde{\zeta} - \zeta}\,\frac{d\widetilde{\zeta}}{\widetilde{\zeta} - z} = 0$$