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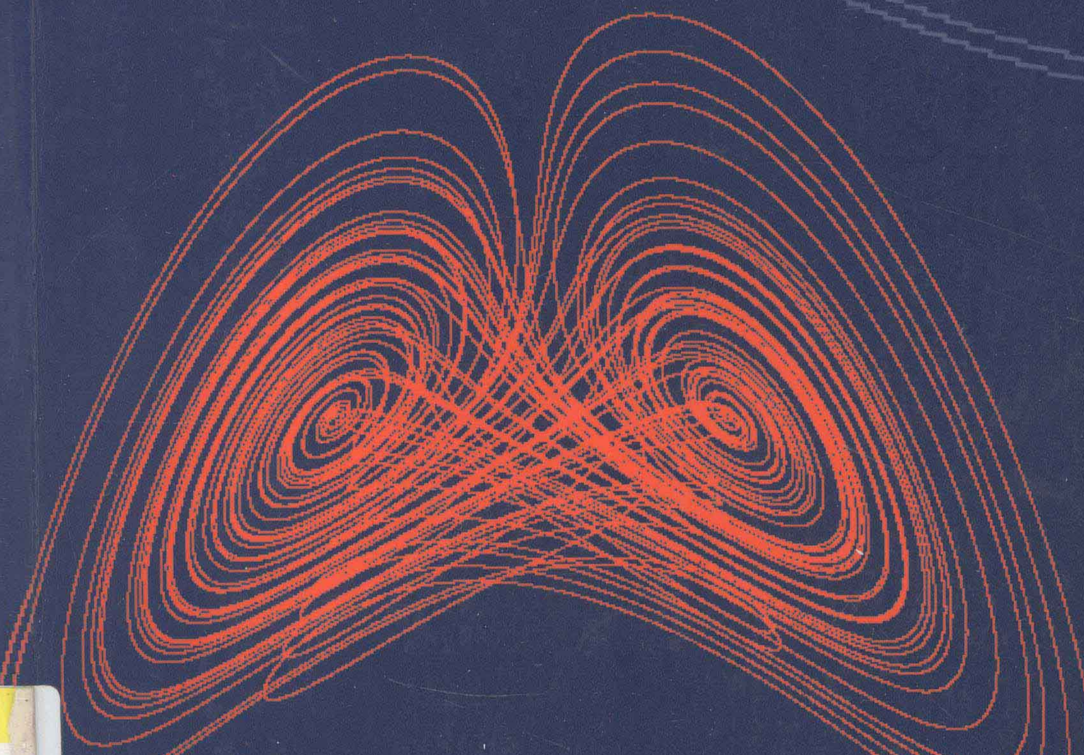
Chaos and Time-Series Analysis



Julien Clinton Sprott

Chaos and Time-Series Analysis

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Preface

This book grew out of a survey course by the same title that I developed and teach at the University of Wisconsin–Madison. The course consists of fifteen 100-minute weekly lectures and a weekly programming project. These lectures form the fifteen chapters of the book, although I have included considerably more material in the book than I am able to cover in the lectures. The students are about half undergraduates and half graduates and other researchers, representing a wide variety of fields in science and engineering. More details about the course can be found on the World Wide Web at http://sprott.physics.wisc.edu/phys505/.

The book is an introduction to the exciting developments in chaos and related topics in nonlinear dynamics, including the detection and quantification of chaos in experimental data, fractals, and complex systems. I have tried to mention, however briefly, most of the important concepts in nonlinear dynamics. Most of the basic ideas are encountered several times with increasing sophistication. This is the way most people learn, and it emphasizes the interconnectedness of the various topics. Emphasis is on the physical concepts and useful results rather than mathematical proofs and derivations. The book is aimed at the student or researcher who wants to learn how to use the ideas in a practical setting, rather than the mathematically inclined reader who wants a deep theoretical understanding.

While many books on chaos are purely qualitative and many others are highly mathematical, I have tried to minimize the mathematics while still giving the essential equations in their simplest possible form. I assume only an elementary knowledge of calculus. Complex numbers, differential equations, matrices, and vector calculus are used in places, but those tools are described as required. The level should thus be suitable for graduate and advanced undergraduate students in all fields of science and engineering as well as professional scientists in most disciplines.

I feel that chaos is best learned by a hands-on approach, and because of its nature, this means writing simple computer programs. Thus, in addition to the usual algebraic exercises, I have included at the end of each chapter the computer project that my students turn in each week. The projects are open-ended and have an optional part meant to challenge the more ambitious students. I have found that the required part usually takes about one to four hours, depending on the student's computational skill.

Programming advice

I do not recommend any particular computer platform or programming language, and I do not provide much formal help with programming. I feel programming is a skill, like mathematics, that all science and engineering students should acquire somewhere during their training. Students who have never programmed can usually learn to do so while taking the course. In fact, chaos offers an enjoyable way to develop and hone these skills.

I recommend that students acquire a personal computer and a modern compiler in a language of their choice. Just as the best language for speaking is the one most familiar to you, the best computer language is the one you are most comfortable using. If you are skilled in a language such as BASIC, C, Java, Pascal, or FORTRAN, get a modern interactive compiler for that language and use it on your PC. Any language will suffice, and modern compilers in the various languages are so good that there is little reason to prefer one over another. If you have never done any serious programming, you might start by learning BASIC. It is easy to learn and more than adequate for the projects in this book. My personal favorite is PowerBASIC (http://www.powerbasic.com/) because it is easy to learn, powerful, and as fast as any C compiler I have encountered. I do most of my programming in DOS, but Windows versions of the PowerBASIC compiler are available.

Another possibility is one of the math packages such as Mathematica, Maple, Matlab, MathCAD, Derive, or Theorist, or even a modern spreadsheet such as Excel, Quattro Pro, or Lotus 1-2-3. This option would be most sensible if you are already highly skilled in its use. You should be able to complete most if not all of the projects in this way. In the long run, you will probably find a conventional programming language more versatile and useful, however.

In any case, I would advise you to develop your programs as modular subroutines and to document them so that they can be reused. There will be occasions while working through the book where you will need something you did several chapters before. Especially in Chapters 9–13, dealing with time-series analysis, you will develop routines that may be of use in analyzing data from your own research.

Web resources

A Web page for the book at http://sprott.physics.wisc.edu/chaostsa/ contains supplementary materials, computer programs, color versions of some of the figures, animations, errata, answers to the exercises, and links to Web resources. I will keep this updated as links change and as I become aware of new resources that may be of interest. You will also find information there on how to contact me in case you find errors in the book, want to comment on it, or make suggestions for future editions.

Acknowledgments

I am indebted to many people in the preparation of this work. First and foremost is George Rowlands of the University of Warwick who introduced me to the subject and continues to be a valued mentor and colleague. Cliff Pickover of IBM–Watson has been a source of inspiration and advice. I have had productive collaborations with Wajdi Ahmad of the University of Sharjah in the United Arab Emirates (electronics), Debbie Aks of the University of Wisconsin–Whitewater (psychology), Janine Bolliger of the Swiss Federal Research Institute (landscape ecology), Robin Chapman of the University of Wisconsin–Madison (communicative disorders), Dorina Creanga of Al. I. Cuza University in Romania (biophysics), Dee Dechert of the University of Houston (economics), Hans Gottlieb of Griffith University in Australia (physics), Wendell Horton of the University of Texas (physics), Ken Kiers of Taylor University (physics), Stefan Linz of the University of Münster in Germany (physics), Karl Lonngren of the University of Iowa (electrical and computer engineering), and Keith Warren of Ohio State University (social work).

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Julien Clinton Sprott
Madison, Wisconsin
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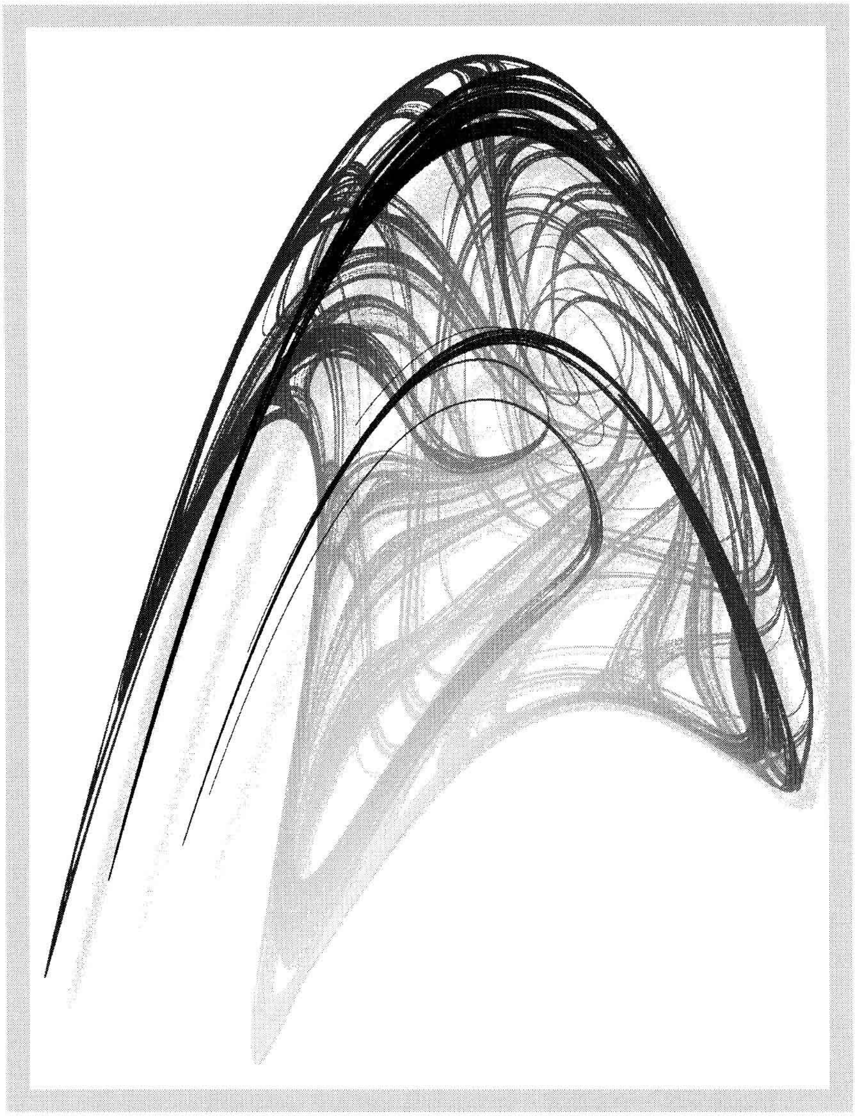
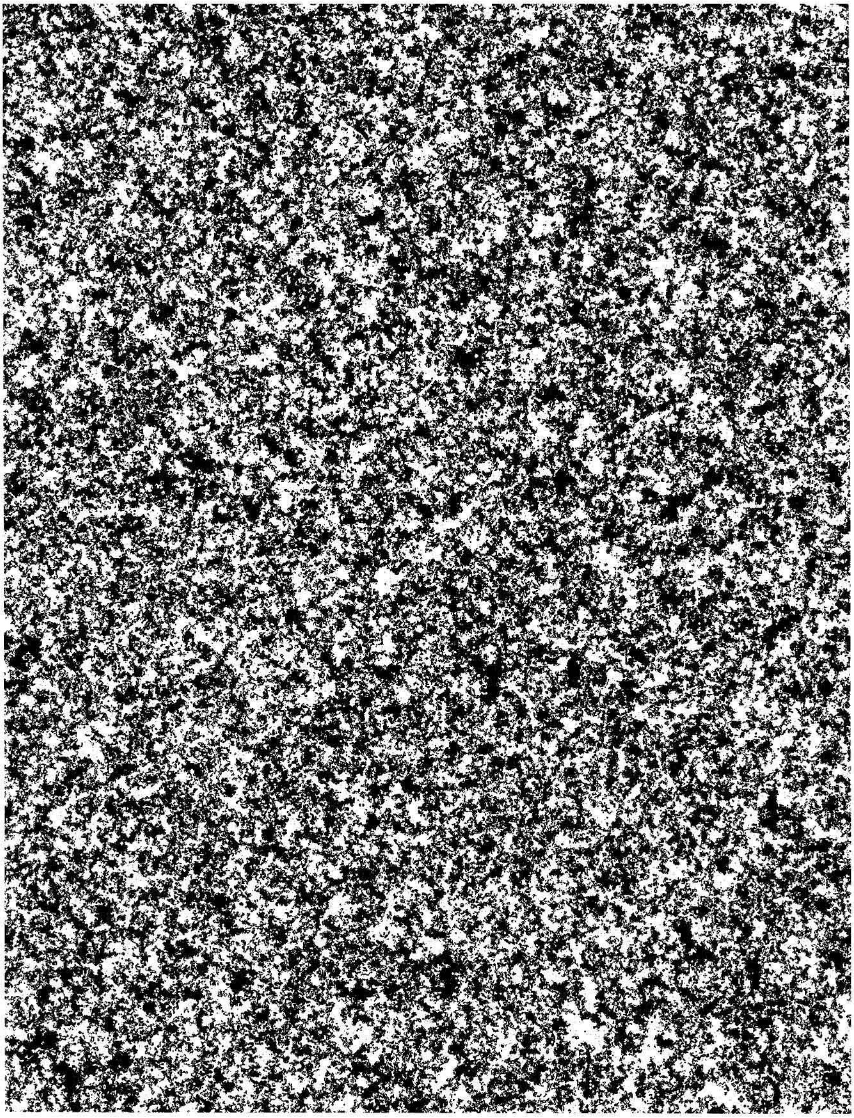


Strange attractor for the three-dimensional map $X_{n+1} = X_n^2 - 0.2X_n - 0.9X_{n-1} + 0.6X_{n-2}$ (see §6.10.2).



Complex two-dimensional pattern from the hundred-thousandth generation of a deterministic cellular automaton in which a dead cell remains dead if exactly six of its eight nearest neighbors are alive and otherwise gives birth, and a live cell remains alive if one, two, or six of its eight nearest neighbors are alive and otherwise dies (see §15.1.4).

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