CONTEMPORARY MATHEMATICS

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Noncommutative Rings, Group Rings, Diagram Algebras and Their Applications

International Conference
December 18–22, 2006
University of Madras, Chennai, India

S. K. Jain
S. Parvathi
Editors

with the cooperation of
Dinesh Khurana

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Preface

There has been a good deal of research activity, and organization of conferences, AMS Special Sessions etc. in the areas of Noncommutative Ring Theory, Representation Theory and Diagram Algebras throughout the world.

The 2006 - International Conference in Chennai, India on Noncommutative Rings, Group Rings, Diagram Algebras and their applications brought together experts from these fields and provided them with opportunities to share their research and learn from one another. This conference was jointly organized by the Ramanujan Institute for Advanced Study in Mathematics, University of Madras and the Center of Ring Theory and Applications, Ohio University on the campus of the University of Madras, Chennai, India during December 18-22, 2006. The main speakers in the conference included A. Facchini, T. Y. Lam, A. Leroy, J. M. Osterburg, I. B. S. Passi, D. S. Passman, Mercedes Siles, L. Small, J. B. Srivastava, V. S. Sunder, J. Szigeti, N. Vanaja, K. Varadarajan, H. Wenzl, R. Wisbauer, C. C. Xi, A. V. Yakovlev.

The social highlights of the conference featured a classical group dance by the members of the *Sanchala* School of Bharathanatyam, Chennai followed by a conference banquet. A guided tour to historical places such as Mahabalipuram and Kanchipuram was arranged.

We would like to thank all of the invited speakers as well as all of the contributors to the Proceedings. Our special thanks are due to Prof. D. S. Passman for his inaugural address. The papers that were submitted for the Proceedings have been rigorously refereed and the ones that were accepted by the referees form the contents of this volume. We would like to thank the referees for their thorough and meticulous screening of the papers.

We are grateful to the University Grants Commission, India for funding the conference through its Special Assistance Program, National Board of Higher Mathematics, Mumbai, Tamil Nadu State Council for Higher Education, Chennai, University of Madras, Ohio University, Athens and other sponsors for their financial support.

We also wish to express our appreciation to the Patrons Dr. S. Ramachandran Vice Chancellor, University of Madras and the President Roderick McDavis, Ohio University for their inspiration and support. Our thanks go to all of the students, staff and faculty at the Ramanujan institute, and the members of the organizing committee in Chennai, India, and in Athens, USA for their help in all possible ways to make this conference a success. In particular, we would like to express our appreciation to Mr. B. Sivakumar and Ms. A. Tamilselvi for their help in several ways with the running of the conference and the publication of these proceedings.

Last but not least, we are grateful to the staff of the American Mathematical Society for their outstanding work in producing these Proceedings. Special thanks are due, in particular, to Ms. Christine Thivierge for her promptness in dealing with any matter and apprising the editors about the technical details from the start to the end.

S. K. Jain
S. Parvathi
Editors

Contents

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Injective modules, spectral categories, and applications

Alberto Facchini

ABSTRACT. This paper is mainly a survey of results that appear in [**FH**], [**GaO**] and [**Goo2**] about singular modules, injective modules, spectral categories and their applications, but the paper also contains some original results. For instance, we compute the derived functors of the canonical functor P of a Grothendieck category $\mathcal{A}$ into its spectral category $\operatorname{Spec}\mathcal{A}$ (Proposition 2.2). We determine some properties of the category $\mathcal{A}'$ obtained from a Grothendieck category $\mathcal{A}$ by formally inverting all superfluous epimorphisms (Section 4). The construction of the category $\mathcal{A}'$ is the dual construction to the construction of spectral category due to Gabriel and Oberst [**GaO**]. For example, we show that the category $\mathcal{A}'$ is an additive category with cokernels, but without kernels in general.

1. Introduction and first elementary examples

In this paper, all modules are unital right modules over a fixed associative ring R with identity $1 \neq 0$. Let Mod-R be the category of all unital right R-modules.

For any right R-module A_R, we shall denote by $E(A_R)$ the injective envelope of A_R. We all know that $E(-)$ cannot be seen as a functor, but we "tend to consider it" a functor of Mod-R into Mod-R, because we can associate to any object A_R of Mod-R the object $E(A_R)$ of Mod-R and to any morphism $f\colon A_R \to B_R$ an extension $E(f)\colon E(A_R) \to E(B_R)$. In trying to construct a functor in this way, we met with a number of difficulties. The first difficulty is that the injective envelope is defined only up to isomorphism. Even if we are already inside a fixed injective module E_R, a submodule A_R of E_R can have different injective envelopes in E_R.

EXAMPLE 1.1. Let R be the ring $\mathbb{Z}$ of integers and $\mathbb{Q}$ the R-module of rational numbers. Let $q\colon \mathbb{Q} \to \mathbb{Q}/2\mathbb{Z}$ denote the canonical projection and $i\colon \mathbb{Q} \to \mathbb{Q}$ the identity mapping. Consider the submodule $A_R = 2\mathbb{Z}\oplus 0$ of the R-module $\mathbb{Q}\oplus\mathbb{Q}/2\mathbb{Z}$. Since, over $R = \mathbb{Z}$, homomorphic images of injective modules are injective modules, the images of the two morphisms $(i,q)\colon \mathbb{Q} \to \mathbb{Q}\oplus\mathbb{Q}/2\mathbb{Z}$ and $(i,0)\colon \mathbb{Q} \to \mathbb{Q}\oplus\mathbb{Q}/2\mathbb{Z}$

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are two injective envelopes of the module A_R, both contained in the same module $\mathbb{Q} \oplus \mathbb{Q}/2\mathbb{Z}$.

Moreover, the extension $E(f)\colon E(A_R) \to E(B_R)$ of a given morphism $f\colon A_R \to B_R$ is not unique[1].

EXAMPLE 1.2. Let p be a prime and let R be the ring $\mathbb{Z}/p^2\mathbb{Z}$, which is a self-injective ring. Then multiplication by $1+p$ and the identity are two different endomorphisms of R_R that extend the identity $p\mathbb{Z}/p^2\mathbb{Z} \to p\mathbb{Z}/p^2\mathbb{Z}$ of the socle of R_R.

Hence, there is no guarantee that $E(g)E(f) = E(gf)$ when we have morphisms $f\colon A_R \to B_R$ and $g\colon B_R \to C_R$. A better, more complete, result is the following:

PROPOSITION 1.3. [**Goo2**, Proposition 1.12] *There does not exist a functor*

$$F\colon \text{Mod-}\mathbb{Z} \to \text{Mod-}\mathbb{Z}$$

such that $F(A) \cong E(A)$ for all abelian groups $A_{\mathbf{Z}}$.

Notice that, in this proposition, F need not even be an *additive* functor. More generally, if R is a commutative integral domain which is not a field, there does not exist a functor $F\colon \text{Mod-}R \to \text{Mod-}R$ such that $F(A_R) \cong E(A_R)$ for all right R-modules A_R [**Goo2**, Exercise 6, p. 25].

Let us recall some standard terminology and elementary properties, which also can be found in [**Goo2**]. This terminology and these properties concern singular modules and nonsingular modules, which will be necessary in the sequel. For any module A_R, the *singular submodule* of A_R is $Z(A_R) = \{\, x \in A_R \mid \operatorname{ann}_R(x)$ is essential in $R_R\,\}$. The module A_R is *singular* provided $Z(A_R) = A_R$, and is *nonsingular* provided $Z(A_R) = 0$. Then $Z\colon \text{Mod-}R \to \text{Mod-}R$ turns out to be a subfunctor of the identity functor $\text{Mod-}R \to \text{Mod-}R$. The functor $Z\colon \text{Mod-}R \to \text{Mod-}R$ is an idempotent, left exact functor, but the pair consisting of the class of all singular R-modules and the class of all nonsingular R-modules is not a torsion theory, because $Z(A_R/Z(A_R))$ can be different from 0 for suitable R-modules A_R. If we want a torsion theory, instead of Z we need Z_2, defined by $Z_2(A_R)/Z(A_R) = Z(A_R/Z(A_R))$ for every R-module A_R. Then there is a torsion theory whose torsion modules are all R-modules A_R with $Z_2(A_R) = A_R$, and whose torsion-free modules are exactly all nonsingular modules.

Here is a list of elementary properties of singular modules and nonsingular modules:

(1) A module A_R is singular if and only if there exists a module B_R with an essential submodule C_R such that $A_R \cong B_R/C_R$ [**Goo2**, Proposition 1.20(b)].
(2) Injective envelopes of nonsingular modules are nonsingular modules.
(3) If $f\colon A_R \to B_R$ is a homomorphism and B_R is nonsingular, then the extension of f to the injective envelopes is unique, that is, there is a

[1] We could fix an injective envelope $E(A_R)$ for each module A_R. Notice that in this case we would be making a *class* of choices. We all know that the axiom of choice is equivalent to Zorn's Lemma, to the well ordering principle, and so on. But for these equivalent statements we deal with a *set* of choices, with the existence of maximal elements in a partially ordered *set*, with well ordering a *set*. But if we chose an injective envelope for every module, we make a *class* of choices.

unique homomorphism $E(f)\colon E(A_R) \to E(B_R)$ making the diagram

$$\begin{array}{ccc} A_R & \xrightarrow{f} & B_R \\ \downarrow & & \downarrow \\ E(A_R) & \xrightarrow{E(f)} & E(B_R) \end{array}$$

commute. Here the vertical arrows are the embeddings of the modules in their injective envelope.

(Let us see the proof of (3). The module $E(A)/A$ is singular by Property (1), and $E(B)$ is nonsingular by (2). Hence $\mathrm{Hom}(E(A)/A, E(B)) = 0$, so that applying the exact functor $\mathrm{Hom}(-, E(B))$ to the exact sequence $0 \to A \to E(A) \to E(A)/A \to 0$, we get the exact sequence $0 \to \mathrm{Hom}(E(A), E(B)) \to \mathrm{Hom}(A, E(B)) \to 0$. In other words, every homomorphism $A \to E(B)$ extends to a homomorphism $E(A) \to E(B)$ in a unique way. In particular, this is true for homomorphisms $A \to B$.)

A corollary of Property (3) is the following fact. We know that if A is a submodule of an injective module E, then E contains an injective envelope of A, and this injective envelope of A is a direct summand of E. If A is a submodule of a *nonsingular* injective module E, then E contains a *unique* injective envelope of A, i.e., there is a *unique* direct summand of E in which A is an essential submodule.

Let us try to find some possible solutions to the problem that injective envelope $E(-)\colon \text{Mod-}R \to \text{Mod-}R$ is not a functor.

A first solution can be found specializing the ring, that is, restricting our attention to special rings:

PROPOSITION 1.4. [**Goo2**, Exercise 24, p. 48] *Let R be a ring such that all singular right R-modules are injective. Then there exists an additive functor F: Mod-$R \to$ Mod-R such that $F(A) \cong E(A)$ for all A_R.*

The functor F in the statement of Proposition 1.4 is defined by $F(A) = Z(A) \oplus S^\circ A$ for every right R-module A and $F(f) = f|_{Z(A)} \oplus S^\circ f$ for every homomorphism $f\colon A_R \to B_R$. Here S° is the localization functor associated with the singular torsion theory, so that $S^\circ A = E(A/Z(A))$.

Rings over which all singular right modules are injective were studied and completely characterized by Goodearl [**Goo1**]. An example of a commutative ring R over which all singular modules are injective is the ring $R := 1F + F^{(\mathbb{N})} \subseteq F^{\mathbb{N}}$. Here F is a field, $F^{\mathbb{N}}$ is the direct product of countable many copies of F, that is, the ring of all functions $\mathbb{N} \to F$, and R is the subring of $F^{\mathbb{N}}$ consisting of all functions $\mathbb{N} \to F$ that are constant almost everywhere (i.e., a function $\alpha\colon \mathbb{N} \to F$ is in R if and only if there exist a finite subset F of $\mathbb{N}$ and an element $a \in F$ such that $\alpha(x) = a$ for every $x \in \mathbb{N} \setminus F$.)

A second solution to our problem of "trying to force $E(-)$ to become a functor" is given by specializing the modules, that is, passing to a full subcategory of Mod-R. Let $\mathbf{NS}(R)$ be the full subcategory of Mod-R whose objects are all nonsingular right R-modules. Then $E(-)\colon \mathbf{NS}(R) \to \text{Mod-}R$ is a functor by Property (3) above.

A third possible solution is changing the morphisms, that is, changing the category. In the next Section we will analyze this situation.

2. The spectral category of a Grothendieck category

Let R be a ring. Assume that we have an additive functor "injective envelope"

$$E(-)\colon \text{Mod-}R \to \mathcal{C}$$

for a suitable category $\mathcal{C}$. Then:

(1) in order that our notation may have a meaning, $\mathcal{C}$ must be an (additive) category whose class of objects contains all injective right R-modules,
(2) for every injective R-module A_R, we want that $E(A_R) = A_R$,
(3) for every R-module A_R, the embedding $A_R \to E(A_R)$ in Mod-R must go to its natural extension $E(A_R) \to E(A_R)$, which is the identity morphism $E(A_R) \to E(A_R)$.

Therefore, let $P\colon \text{Mod-}R \to \mathcal{C}$ be an additive functor and assume that

(1) $\mathcal{C}$ is an additive category whose class of objects contains all injective right R-modules,
(2) for every right R-module A_R, $P(A_R) = E(A_R)$ (in particular, $P(A_R) = A_R$ for every injective right R-module A_R.)
(3) For every R-module A_R, the functor P sends the embedding $A_R \to E(A_R)$ to the identity morphism $1_{P(A_R)}$ of $P(A_R)$.

Under these hypotheses, let $f\colon A_R \to B_R$ be an essential monomorphism in Mod-R. Then there is a commutative square

$$\begin{array}{ccc} A_R & \xrightarrow{f} & B_R \\ \downarrow & & \downarrow \\ E(A_R) & \xrightarrow{g} & E(B_R) \end{array}$$

in Mod-R, where the extension g of f is an isomorphism, so that when we apply the functor P we get a commutative square

$$\begin{array}{ccc} P(A_R) = E(A_R) & \xrightarrow{P(f)} & P(B_R) = E(B_R) \\ {\scriptstyle 1_{P(A_R)}}\downarrow & & \downarrow {\scriptstyle 1_{P(B_R)}} \\ P(A_R) = E(A_R) & \xrightarrow{P(g)} & P(B_R) = E(B_R) \end{array}$$

in $\mathcal{C}$, and $P(g)$ is an isomorphism in $\mathcal{C}$, so that $P(f)$ also must be an isomorphism in $\mathcal{C}$. Therefore any functor $P\colon \text{Mod-}R \to \mathcal{C}$ with the three properties (1), (2) and (3) must send every essential monomorphism of Mod-R to an isomorphism of $\mathcal{C}$. Hence the idea in order to force injective envelope to become a functor is the following: we must invert essential monomorphisms. And, luckily, this is sufficient, as we shall see in the rest of this section.

Let $\mathcal{C}$ be the category obtained from Mod-R formally inverting the essential monomorphisms of Mod-R. When we say "formally inverting the essential monomorphisms of Mod-R", we mean a construction very similar to the construction of the ring of fractions $RS^{-1} = \{\, fs^{-1} \mid f \in R, s \in S \,\}$ of a commutative ring R with respect to a multiplicative closed subset S of R. In that case, we have a commutative ring R, a multiplicatively closed subset S of R, we consider the cartesian product $R \times S = \{\, (f,s) \mid f \in R, s \in S \,\}$ and the equivalence relation $\sim$ on $R \times S$ defined by $(f,s) \sim (f',s')$ if there exists a $t \in S$ with $fs't = f'st$, we construct the factor set $RS^{-1} := R \times S/\sim$, denote its elements, that is, the equivalence classes of the pairs (f,s), as fs^{-1}, and give RS^{-1} a ring structure, in which the elements of S have become invertible. Thus we have formally inverted the elements of S in the sense that we have constructed from R the "smallest ring RS^{-1} containing R" in which the elements of S are now invertible. (In fact, the new ring RS^{-1} does not necessarily contain R, there is only a ring morphism $R \to RS^{-1}$, and it is the smallest one in the sense that it has the suitable universal property.) Now we follow the same construction, but instead of considering the ring R and its multiplicatively closed subset S, we consider the class of all morphisms in Mod-R and its multiplicatively closed subclass of all essential monomorphisms. We thus construct the category whose objects are all right R-modules and whose morphisms $A_R \to B_R$ are the "fractions" fs^{-1}, where $f\colon A'_R \to B_R$ is an arbitrary morphism in Mod-R, and $s\colon A'_R \to A_R$ is an essential monomorphism.

This can be done, more generally, for any Grothendieck category $\mathcal{A}$ [**GaO**], so that we will present the construction for any such category and not only for the category Mod-R. Recall that a *Grothendieck category* is an abelian category with exact direct limits and a generator. Let $\mathcal{A}$ be a Grothendieck category. For any object A of $\mathcal{A}$, the set of the essential subobjects of A is downwards directed, because $A' \leq_e A$ and $A'' \leq_e A$ imply $A' \cap A'' \leq_e A$. Here we have written $A' \leq_e A$ to denote the fact that A' is an essential subobject of A. Therefore if we fix another object B of $\mathcal{A}$ and apply the contravariant functor $\mathrm{Hom}_{\mathcal{A}}(-, B)$ to the inclusions $A' \to A''$, where $A' \leq_e A$, $A'' \leq_e A$ and $A' \subseteq A''$, we get an upwards directed set of abelian group morphisms $\mathrm{Hom}_{\mathcal{A}}(A'', B) \to \mathrm{Hom}_{\mathcal{A}}(A', B)$. Thus we can construct the direct limit $\varinjlim \mathrm{Hom}_{\mathcal{A}}(A', B)$, where A' ranges in the set of all essential subobjects of A. Let $\mathrm{Spec}\,\mathcal{A}$ be the category with the same objects as $\mathcal{A}$ and, for objects A and B of $\mathcal{A}$, with $\mathrm{Hom}_{\mathrm{Spec}\,\mathcal{A}}(A, B) = \varinjlim \mathrm{Hom}_{\mathcal{A}}(A', B)$ $(A' \leq_e A)$. Then $\mathrm{Spec}\,\mathcal{A}$ turns out to be a Grothendieck category is which every object is injective [**GaO**, Satz 1.3]. It is called the *spectral category* of $\mathcal{A}$. More generally, a category is *spectral* if it is a Grothendieck category in which every object is injective. See [**GaO**] or [**St**, Ch. V, §7]. The next Proposition characterizes spectral categories.

PROPOSITION 2.1. ([**GaO**, Satz 2.1], [**GooB**, Theorem 1.14]) *For any ring R, the full subcategory* $\mathbf{NSI}(R)$ *of* Mod-R *whose objects are all nonsingular injective right R-modules is a spectral category.*

Conversely, if $\mathcal{A}$ is a spectral category, let U be a generator of $\mathcal{A}$ and $R := \mathrm{Hom}_{\mathcal{A}}(U, U)$ the endomorphism ring of U. Then R is a Von Neumann regular, right self-injective ring, and $\mathcal{A}$ is equivalent to $\mathbf{NSI}(R)$.

For any Grothendieck category $\mathcal{A}$, there is a canonical functor $P\colon \mathcal{A} \to \mathrm{Spec}\,\mathcal{A}$ which is the identity on objects, takes $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$ to its canonical image in $\mathrm{Hom}_{\mathrm{Spec}\,\mathcal{A}}(A, B)$, and has the three properties (1), (2) and (3). In other words, changing the category, $E(-)$ has become a functor Mod-$R \to$ Spec Mod-R.

Spectral categories go back to Gabriel and Oberst [**GaO**], that is, to the sixties. Though their construction appears in some books, it does not seem to have had many applications. Searching MathSciNet for "spectral categories", we find that spectral categories are presently cited in less than twenty papers. The next Section will be devoted to describing further applications of spectral categories.

Let us compute the right derived functors of the functor $P\colon \mathcal{A} \to \mathrm{Spec}\,\mathcal{A}$. Let $\mathcal{A}$ be a Grothendieck category. The functor $P\colon \mathcal{A} \to \mathrm{Spec}\,\mathcal{A}$ is a left exact, covariant, additive functor and every object in the Grothendieck category $\mathrm{Spec}\,\mathcal{A}$ is injective and projective. Notice that every Grothendieck category is a category with injective envelopes (see, for instance, [**P**, Theorem 3.10.10]). Thus it is possible to define the right derived functors of P (see, for instance, [**Gr**, p. 143]). Notice that every object A of $\mathcal{A}$ has a minimal injective resolution $0 \to A \to E_0 \to E_1 \to E_2 \to \dots$.

PROPOSITION 2.2. *Let $P^{(n)}$ denote the n-th right derived functor of P. Then $P^{(n)}(A) = P(E_n)$ for every $n \geq 0$.*

PROOF. The minimal injective resolution

$$0 \to A \xrightarrow{\delta_{-1}} E_0 \xrightarrow{\delta_0} E_1 \xrightarrow{\delta_1} E_2 \xrightarrow{\delta_2} \dots \tag{2.1}$$

is the exact sequence defined inductively in the following way. Let $E_{-1} = A$ and let $\delta_{-1}\colon E_{-1} \to E_0$ be an injective envelope of E_{-1}. Assume that $\delta_{n-1}\colon E_{n-1} \to E_n$ has been defined for some $n \geq 0$. The morphism $\delta_n\colon E_n \to E_{n+1}$ is the composite morphism of the canonical epimorphism $E_n \to \operatorname{coker} \delta_{n-1}$ and an injective envelope $\operatorname{coker} \delta_{n-1} \to E_{n+1}$ of $\operatorname{coker} \delta_{n-1}$. In particular, the image of δ_n is essential in E_{n+1} for every $n \geq -1$. As (2.1) is exact, we get that the kernel of $\delta_{n+1}\colon E_{n+1} \to E_{n+2}$ is essential, so that $P(\delta_{n+1})\colon P(E_{n+1}) \to P(E_{n+2})$ is the zero morphism for every $n \geq -1$. Hence, the $P^{(n)}(A)$'s are the cohomology groups of the complex $0 \longrightarrow P(E_0) \xrightarrow{0} P(E_1) \xrightarrow{0} P(E_2) \xrightarrow{0} \dots$, so that $P^{(n)}(A) = P(E_n)$ for every $n \geq 0$. □

COROLLARY 2.3. *For a module A, $P^{(n+1)}(A) = 0$ if and only if A has injective dimension $\leq n$.*

PROOF. This follows from the fact that, for a module E, $P(E) = 0$ if and only if $E = 0$. □

3. Applications of spectral categories

Let R and S be rings. A ring morphism $\varphi\colon R \to S$ is said to be *local* if, for every $r \in R$, $\varphi(r)$ invertible in S implies r invertible in R. We will denote by $J(R)$ the Jacobson radical of the ring R. A ring R is called *semilocal* if $R/J(R)$ is a semisimple artinian ring. Semilocal rings can be characterized in a number of ways. In the next two results we recall some of these characterizations.

THEOREM 3.1 (Camps and Dicks [**CD**]). *A ring R is semilocal if and only if there exists a local morphism of R into a semilocal ring, if and only if there exists a local morphism of R into a semisimple artinian ring.*

Now recall that the notion of Goldie dimension is a notion of lattice theory. Let $(L, \vee, \wedge)$ be a modular lattice with 0 and 1, that is, a lattice with a smallest element 0 and a greatest element 1 such that $a \wedge (b \vee c) = (a \wedge b) \vee c$ for every $a, b, c \in L$ with $c \leq a$. A finite subset $\{\, a_i \mid i \in I \,\}$ of $L \setminus \{0\}$ is said to be *join-independent* if $a_i \wedge (\bigvee_{j \neq i} a_j) = 0$ for every $i \in I$. The empty subset of $L \setminus \{0\}$ is join-independent. A modular lattice L with 0 and 1 is said to be of *Goldie dimension* n, where $n \geq 0$ is an integer, if n is the greatest of the cardinalities of the finite subsets of L that are join-independent [**F1**, Section 2.6]. A modular lattice is said to be *of finite Goldie dimension* if it has Goldie dimension n for some $n \geq 0$, otherwise it is said to be *of infinite Goldie dimension.*

We can apply these notions to the lattice $\mathcal{L}(M_R)$ of all submodules of a module M_R. If the lattice $\mathcal{L}(M_R)$ has finite Goldie dimension n, then n will be said to be the *Goldie dimension* $\dim(M_R)$ of the module M_R, otherwise M_R will be said to be of *infinite Goldie dimension.*

If $(L, \vee, \wedge)$ is a modular lattice, then its dual lattice $(L, \wedge, \vee)$ is also a modular lattice. The Goldie dimension of the dual lattice of the lattice $\mathcal{L}(M_R)$ of all submodules of a module M_R is called the *dual Goldie dimension* of M_R. It is denoted by $\operatorname{codim}(M_R)$.

PROPOSITION 3.2. *The following conditions are equivalent for a ring R.*

(1) *The ring R is semilocal.*
(2) *The right R-module R_R has finite dual Goldie dimension.*
(3) *The left R-module ${}_RR$ has finite dual Goldie dimension.*

Moreover, if these equivalent conditions hold,

$$\operatorname{codim}(R_R) = \operatorname{codim}({}_RR) = \dim(R/J(R)).$$

Modules whose endomorphism ring is semilocal have good decomposition properties, for instance they cancel from direct sums (if A_R, B_R, C_R are modules over a ring R, $\operatorname{End}(A_R)$ is semilocal and $A_R \oplus B_R \cong A_R \oplus C_R$, then $B_R \cong C_R$), have the n-th root property (if A_R, B_R are modules over a ring R, $\operatorname{End}(A_R)$ is semilocal, n is a positive integer and $A_R^n \cong B_R^n$, then $A_R \cong B_R$), have only finitely many direct summands up to isomorphism, etc. [**F1**, Section 4.2]. For instance, let's prove, as an exercise, the following Proposition.

PROPOSITION 3.3. *Let $R \to S$ be a ring morphism, and let M_S be an S-module with* $\operatorname{End}(M_R)$ *semilocal. Then* $\operatorname{End}(M_S)$ *is semilocal.*

PROOF. The module M_S is an R-module M_R in a natural way via the ring morphism $R \to S$, and there is an embedding $\operatorname{End}(M_S) \to \operatorname{End}(M_R)$, because every S-endomorphism of M is an R-endomorphism a fortiori. This embedding $\operatorname{End}(M_S) \to \operatorname{End}(M_R)$ is a local morphism, because every S-endomorphism of M_S which is an R-automorphism of M_R (i.e., is bijective) is also an S-automorphism of M_S. □

Being semilocal is a finiteness condition on rings, and having a semilocal endomorphism ring is a finiteness condition on modules. Modules with a semilocal endomorphism ring have a regular geometric behavior as far as their direct summands are concerned [**F2**].

The rest of this Section will be devoted to presenting some applications of the properties of the functor P to the study of objects A of a Grothendieck category $\mathcal{A}$ with a semilocal endomorphism ring $\operatorname{End}_{\mathcal{A}}(A)$. The applications we present are taken from [**FH**].

Any additive functor F of a preadditive category $\mathcal{A}$ into a preadditive category $\mathcal{B}$ induces a ring morphism $\operatorname{End}_{\mathcal{A}}(A) \to \operatorname{End}_{\mathcal{B}}(F(A))$ for every object A of $\mathcal{A}$. This happens, in particular, when $\mathcal{A}$ is a Grothendieck category, $\mathcal{B}$ is the spectral category of $\mathcal{A}$ and F is the canonical functor P of $\mathcal{A}$ into $\operatorname{Spec}\mathcal{A}$. Thus, for every object A of a Grothendieck category $\mathcal{A}$, there is a ring morphism $\varphi_A\colon \operatorname{End}_{\mathcal{A}}(A) \to \operatorname{End}_{\operatorname{Spec}\mathcal{A}}(A)$ induced by the functor $P\colon \mathcal{A} \to \operatorname{Spec}\mathcal{A}$.

An object A of a Grothendieck category $\mathcal{A}$ is said to be *directly finite* if it is not isomorphic to a proper direct summand of itself, that is, if $A \oplus B \cong A$ implies $B = 0$ for any object B of $\mathcal{A}$.

Proposition 3.4. [**FH**, Proposition 4.3] *Let A be an object in a Grothendieck category $\mathcal{A}$ such that every monomorphism $A \to A$ is an isomorphism. Then $\varphi_A \colon \mathrm{End}_{\mathcal{A}}(A) \to \mathrm{End}_{\mathrm{Spec}\,\mathcal{A}}(A)$ is a local morphism. Conversely, if φ_A is a local morphism and $E(A)$ is directly finite, then every monomorphism $A \to A$ is an isomorphism.*

Lemma 3.5. *If an object A of a Grothendieck category $\mathcal{A}$ has finite Goldie dimension n, then $P(A)$ is a semisimple object of composition length n in the spectral category* $\mathrm{Spec}\,\mathcal{A}$, *and* $\mathrm{End}_{\mathrm{Spec}\,\mathcal{A}}(A)$ *is a semisimple artinian ring.*

From Theorem 3.1 and Proposition 3.4 we get the following result, which was first proved by Rosa Camps and Warren Dicks [**CD**]:

Corollary 3.6. *Endomorphism rings of artinian modules are semilocal.*

More generally, if A is an object of finite Goldie dimension in a Grothendieck category $\mathcal{A}$ and every monomorphism $A \to A$ is an isomorphism, then the endomorphism ring $\mathrm{End}_{\mathcal{A}}(A)$ is semilocal.

We say that an object A of a Grothendieck category $\mathcal{A}$ is *finitely copresented* if there exists an exact sequence $0 \to A \to L_0 \to L_1 \to 0$ where L_0 is injective and both L_0 and L_1 are of finite Goldie dimension. It is possible to prove [**FH**, Section 5] that for every finitely copresented object A, there is a local morphism $\chi \colon \mathrm{End}_{\mathcal{A}}(A) \to \mathrm{End}_{\mathrm{Spec}\,\mathcal{A}}(A) \times \mathrm{End}_{\mathrm{Spec}\,\mathcal{A}}(L_1)$. It is defined as follows: given $f \in \mathrm{End}_{\mathcal{A}}(A)$, χ maps f to $(P(f), P(f_1))$, where $f_1 \colon L_1 \to L_1$ is any morphism making the following diagram commute:

$$\begin{array}{ccccccccc} 0 & \to & A & \to & L_0 & \to & L_1 & \to & 0 \\ & & \downarrow f & & \downarrow f_0 & & \downarrow f_1 & & \\ 0 & \to & A & \to & L_0 & \to & L_1 & \to & 0. \end{array}$$

Here $f_0 \colon L_0 \to L_0$ is any extension of $f \colon A \to A$ to the injective object L_0 of $\mathcal{A}$ containing A.

Corollary 3.7. *The endomorphism ring of a finitely copresented object A of a Grothendieck category $\mathcal{A}$ is semilocal. More precisely, if A is finitely copresented and L_0 is its injective envelope, then* $\mathrm{codim}(\mathrm{End}_{\mathcal{A}}(A)) \leq \dim(A) + \dim(L_0/A)$.

The last two results of this Section generalize two theorems proved by Warfield. Warfield proved in [**W**, Theorem 5.2] that if R is a semilocal commutative principal ideal domain and S is an R-algebra that is torsion-free and of finite rank as an R-module, then S is a semilocal ring. This can be extended to the following:

Corollary 3.8. [**FH**, Corollary 5.9] *Let R be a commutative noetherian semilocal domain of Krull dimension* 1 *and let S be an R-algebra. Let A_S be an S-module that is torsion-free of finite rank as an R-module. Then* $\mathrm{End}(A_S)$ *is semilocal.*

Recall that an R-module A is *uniserial* if, for any submodules B and C of A, we have $B \subseteq C$ or $C \subseteq B$, and a module is *serial* if it is a direct sum of uniserial modules. Thus a module is serial of finite Goldie dimension if and only if it is a direct sum of finitely many uniserial modules, and a commutative integral domain R is a valuation domain if and only if R_R is a uniserial module. Warfield [**W**, Theorem 5.4] proved that if R is a commutative valuation domain and S is an R-algebra that is torsion-free and of finite rank as an R-module, then S is a semilocal ring. This can be extended to the following: