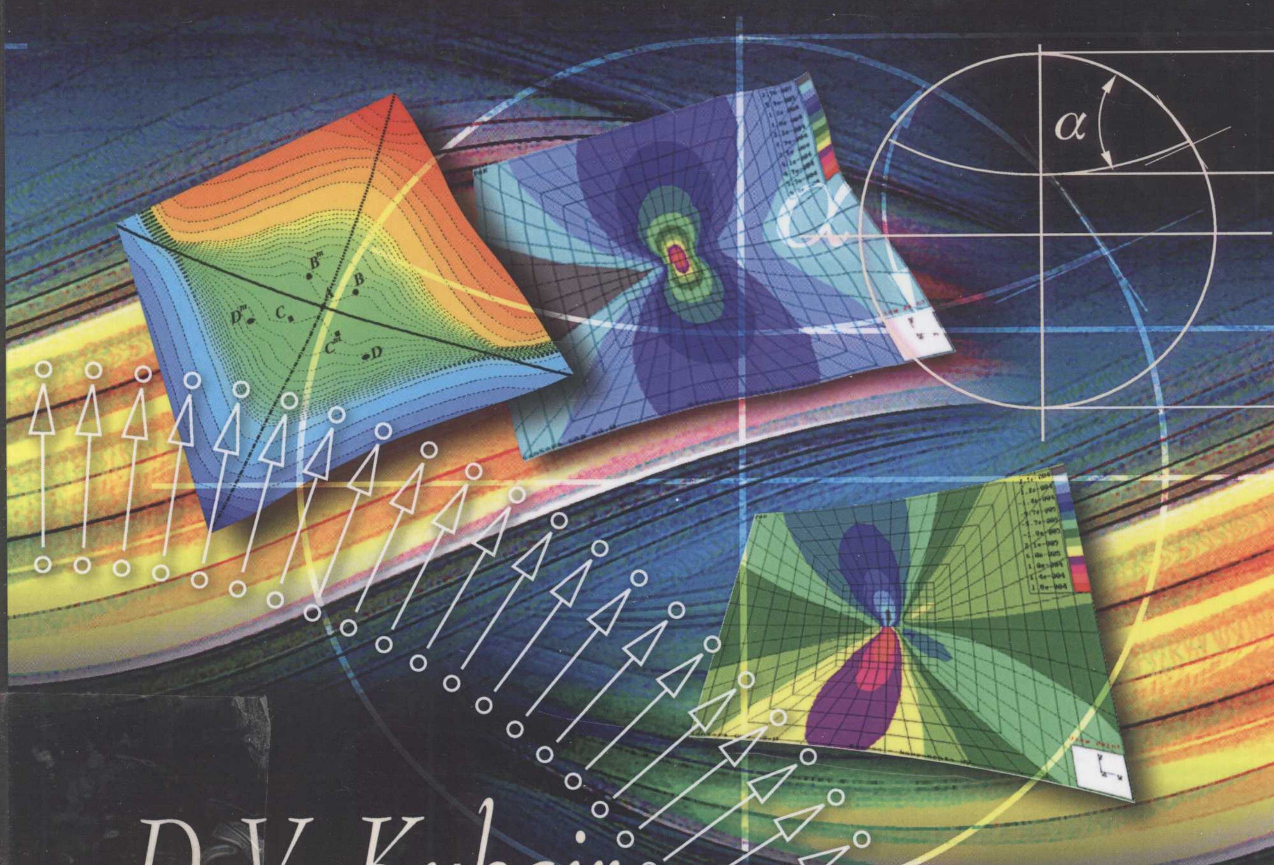
Crack Growth

Rates, Prediction and Prevention



D. V. Kubair
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MECHANICAL ENGINEERING THEORY AND APPLICATIONS

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RATES, PREDICTION AND PREVENTION

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EDITOR

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Preface

The engineering necessity of fracture mechanics is to improve the empirical data-handbook style and mechanics of materials based design. This book examines the phenomenon of "crack growth". The contributed chapters in this book have been derived from various research groups' results from their experiments and/or simulations of crack growth in various media. The contributed chapters discuss the effect of inhomogeneity, plasticity effects in single crystals, interfaces and welded joints. The book gives a snapshot of the research effort of various groups around the globe in the past decade or more on the important topic of crack growth.

Chapter 1 - Nature has evolved inhomogeneous materials in several biological systems such as bones, soft-tissues, cartilages and wood. Material properties such as hardness, Young's modulus or density vary as a function of spatial position. Inhomogeneous variation of the mechanical properties of these materials allows them to effectively respond to external loads enhancing their usability. Mimicking nature humans have adopted the concept of tailoring the mechanical properties in engineering components. A classic example is the case hardening (eg., carburizing or nitriding) of components, in which the surface of the component is hardened and the interior remains tough and ductile. Case hardening treatments are usually performed to increase the wear resistance properties of components such as gears, shafts, bearings where contact is desired and wear is expected. The formal introduction of the nomenclature of *functionally graded materials* (FGMs), however began in the 1980's during the quest for suitable alternative materials to be used as thermal barrier coatings in gas turbine engines. Thermal barrier coatings are systems that are exposed to high temperatures on one hand and on the other require to carry a load without cracking. Ceramics are good at resisting high temperature, however are very brittle and crack easily. Metals on the other hand are tough and ductile but their relatively lower melting point compared to ceramics excludes them to be used in very-high temperature applications. The best of the two is achieved by a functionally graded material, which is a physical mixture of both ceramics and metals and does not contain any physically distinct interface between the constituent phases. From its inception as alternative materials to be used in thermal barrier coatings, functionally graded materials have found numerous engineering applications in wear resistant coatings, MEMs and electronic devices. Combination of two physical constituents other than ceramics and metals are also seen in recent graded materials. Novel processing techniques are also in place to obtain bulk graded structures, ranging from plasma sintering, selective UV radiation to centrifugal casting and selective laser sintering. The graded components once manufactured needs to verified whether the desired mechanical properties are attained as designed and classical mechanical testing methods such as indentation, nano-indentation, tensile testing and ultrasonic wave propagation experiments are in place. Solid mechanics analysis of the test methods and experiments are also available to further modify and optimize the graded material systems. Modern machine design methodologies include the concept of fracture mechanics owing to the complex nature of the materials and loading scenarios. Functionally graded materials being relatively newer materials have received some focus in terms of understanding their fracture response and crack propagation characteristics. Experimental, theoretical and numerical fracture mechanics studies are available in the literature dealing with graded and inhomogeneous materials. There is also a lot of ongoing interest in the solid mechanics community to further the understanding of fracture of functionally graded materials. In this chapter, the authors have summarized the efforts of theirgroup in the past decade that has led to furthering the understanding of spontaneous planar crack propagation in bimaterial systems. The novelty of this chapter is the development of a new numerical technique namely the graded spectral scheme described in Section 2. Comparison of crack propagation characteristics between homogeneous and graded materials are described in Section 3, followed by the comparison between symmetric and unsymmetric graded systems in Section 4. Finally in Section 5 the authors detail the cohesive crack propagation characteristics in bimaterial graded systems, wherein the top and bottom half-space of the system can be any tailored material gradient.

Chapter 2 - Fatigue crack propagation in round bars subjected to mode-I loading has been predicted numerically using the finite element method. Although there are a great number of studies dealing with this topic, the influence of Poisson's ratio on crack shape evolution was not sufficiently analysed. Therefore, in this study, the main objective is to evaluate the influence of Poisson's ratio on crack shape evolution in round bars subjected to remote constant-amplitude axial and bending loads. An automatic linear elastic three-dimensional finite element technique, able to predict the crack shape evolution, was developed and optimised. Dependent parameters, sensitive to crack shape change, were used to evaluate the effect of Poisson's ratio. A significant effect was observed, especially for longer cracks within the interval a/D ε [0.6-0.8]. The effects of Poisson's ratio on fatigue life and on stress intensity factor values were also investigated.

Chapter 3 - This chapter presents boundary element method (BEM) formulations for analysis of crack growth in linear, cohesive and fatigue problems. The proposed formulations are based on the dual BEM, in which singular and hyper-singular integral equations are used. This technique avoids singularities of the resulting algebraic system of equations, despite the fact that the collocation points coincide for the two opposite crack faces. In linear elastic fracture mechanics analyses, the displacement correlation technique is applied to evaluate stress intensity factors. The maximum circumferential stress theory is used to calculate the propagation angle and the effective stress intensity factor. The fatigue model uses Paris' law to predict structural life. For the analysis of crack propagation in linear elastic materials and fatigue, an iterative scheme to predict the crack growth path and the crack length increment at each time step is proposed. This scheme permits to model localisation and coalescence phenomena, which have not been modelled with BEM. The cohesive crack growth analysis is another focus of this chapter. Two non-linear BEM formulations were developed to model this problem. The first formulation uses the concept of constant operator, in which the corrections of the non-linear process are performed only by applying appropriate tractions along the crack surfaces. The second developed BEM formulation is an implicit technique based on the use of a tangent operator. This formulation is accurate, stable and always requires much less iterations to reach the equilibrium within a given load increment in comparison with the classical approach. Examples of simple and multi-fractured structures loaded until rupture are considered. These analyses demonstrate the robustness of the proposed models. In addition, the results indicate that these formulations are accurate and can model localisation and coalescence phenomena.

Chapter 4 - In this chapter, a comprehensive review of the evolution of weight functions for the case of an edge crack in a finite width plate is given. Different techniques for numerical calculation of the weight function are discussed and an accurate weight is derived in a form which is ready for computer modelling.

Chapter 5 - The basic characteristic of the layered materials is existence of an interface, which represents the discontinuity in elastic and thermal characteristics of materials. The concept of the interfacial fracture mechanics is essential for understanding processes like the crack initiation and growth in materials like: fiber reinforced composites, reinforced ceramics, protective coatings, adhesive layers and alike. The attention is devoted to problems of elastic fracture mechanics where the contact zone is small with respect to other relevant geometric dimensions. The basic idea is based on the fracture process in conditions of a stable equilibrium. This Chapter is dealing with theoretical fundamentals of the crack growth at the bimaterial interface: linear elastic fracture mechanics concept for interfacial cracks and the Rice – Thomson model for crack initiation and growth. In considering the LEFM concept for the interfacial cracks the presentation is given of the form of the *stress field* around the interfacial crack tip; cracks are the most simply modeled as surfaces across which no tractions are transmitted. Solutions resulting from that model generally predict material interpenetration very near the tip. It is important to understand that the ductile or brittle response of an interfacial crack depends not only on the structure of the interface, but also on the direction of the crack propagation and local loading conditions. The Rice – Thomson model states that the ductile versus brittle response of an interfacial crack is determined by the competition between dislocation emissions from the crack tip and atomic decohesion of the interface.

Chapter 6 - The crack most frequently appears at the interface but in some cases the crack that propagates along the interface can kink away from it and continue to propagate in one of the two materials. The attention is focused to the initiation of kinking and the condition that the length of a crack segment that is leaving the interface is smaller than the crack that is at the interface. An important question is what is the role of an interface when the crack is approaching it; whether the crack will be penetrating it or will it be deflecting into the interface and continue to propagate along it. Such questions are of importance in designing the interface between the fiber and the substrate in the fibers composites, where the goal is for the crack, which is propagating in the substrate and is approaching the fiber, to deflect and continue to propagate along the interface leaving the fiber undamaged.

Chapter 7 - The interfacial fracture mechanics concept is essential for understanding processes like the crack initiation and growth in materials like: fiber reinforced composites, reinforced ceramics, protective coatings, adhesive layers and alike. Various problems of cracks between the two thin layers are important for studying behavior of coatings at the substrate. Coatings can be subjected to tensile stresses or in-plane compressive stress which would cause delamination by buckling in form of a blister; while the substrate surface can be flat or cylindrical. For thin layers, the most important problem is that of layers subjected to residual tensile stresses. The thin layers problem, in conditions of residual tensile stresses, gives the appropriate model for solving problems in the area of composite materials manufacturing, electronic devices design, protective coatings problems and others. Ceramic coatings on the metal substrate create significant stresses, when they start to cool down after the coating procedure. These stresses are the consequence of the mismatch in the thermal expansion coefficients of the substrate and coating materials. One of the basic forms of these coatings destruction is the edge delamination. For the ideally brittle interface, the edge delamination of the compressed thin film occurs as the Mode II interfacial fracture. The stresses between the two layers can also be induced by the difference in thermal expansion coefficients of the two materials that are separated by the interface.

Chapter 8 - If the interface is already weakened by existence of flaws, they can act as the initiators for the growth of a crack, which, under certain conditions, can unstably propagate along the interface. Such situations lead to necessity of studying the dynamic crack growth at the interface. In order to formulate the mechanism of initiation and the dynamic crack growth in bimaterial interfaces, it is necessary to know the complete structure of the stress and strain fields that surround the tip of a moving interfacial crack. In this chapter is presented an approach to asymptotic analysis of the strain field around a crack tip that is propagating dynamically along a bimaterial interface. Through asymptotic analysis the problem is being reduced to solving the Riemann-Hilbert's problem, what yields the strain potential that is used for determination of the strain field around a crack tip. The considered field is that of a dynamically propagating crack with a speed that is between zero and shear wave speed of the less stiffer of the two materials, bound along the interface. The analytical results were obtained by application of programming package Mathematica and then compared to experimental results on the dynamic crack growth existing in literature, as well as to some numerical simulations results. This comparison showed that it is necessary to apply the complete expression obtained by asymptotic analysis of optical data and not only its first term as it was done in previous analyses.

Chapter 9 - The stress intensity factor range (SIF) is the inevitable parameter, which must be studied and calculated in fracture mechanics methods. The stress intensity factor, SIF describes the fatigue action at a crack tip in terms of crack propagation. In this chapter, SIFs have been calculated using fracture analysis code 2-dimensional program, FRANC2D. In welded joints, stress concentrations occur at the weld toe and at the weld root which make these regions the points from which fatigue cracks may initiate. These cracks have been considered in the verification processes using FRANC2D software in this chapter. Calculating the fatigue life of welded structures and analyzing the progress of these cracks using fracture mechanics technique requires an accurate calculation of the SIF. The existing SIFs were usually derived for one particular geometry and type of loading. In this chapter, the finite element method FEM (FRANC2D) was used to calculate the SIF during crack propagation steps. It is verified to be highly accurate, with the direction of crack propagation being predicted by using the maximum normal stress criterion. A developed analytical approach for toe cracks in cruciform welded joints has been used. On the other hand, in case of lack of penetration (LOP) a classical equation from Frank and Fisher is used. The calculated SIF results for some notch cases have been verified with available solutions from International Institute of Welding (IIW), British Standards Institution (BSI) and literature. A fairly good correlation was obtained and the results have confirmed the use of FRANC2D to simulate different weld geometries. The results are shown and the agreements are pretty good.

Chapter 10 - This experimental investigation is devoted to the analysis of crack growth in the structural steel. The experimentations were carried out on CVN impact testing with varied load and specimen sizes. The influence of the specimen size and the loading rates on the impact toughness of the material has been determined under room temperature. The formation of the shear lips and the nature of fracture in these shear lips is found to be influenced to a maximum extent by the loading rate and the specimen size. The nature of fracture as a function of load, specimen size and the shear lips is analyzed with the aid of scanning electronic microscopy (SEM). The specimen preparation and the experimentations were carried out according to the ASTM E23 standards.

CONTENTS

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Chapter 1

Spontaneous Planar Crack Propagation in Inhomogeneous Materials

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Abstract

Nature has evolved inhomogeneous materials in several biological systems such as bones, soft-tissues, cartilages and wood. Material properties such as hardness, Young's modulus or density vary as a function of spatial position. Inhomogeneous variation of the mechanical properties of these materials allows them to effectively respond to external loads enhancing their usability. Mimicking nature humans have adopted the concept of tailoring the mechanical properties in engineering components. A classic example is the case hardening (eg., carburizing or nitriding) of components, in which the surface of the component is hardened and the interior remains tough and ductile. Case hardening treatments are usually performed to increase the wear resistance properties of components such as gears, shafts, bearings where contact is desired and wear is expected. The formal introduction of the nomenclature of *functionally graded materials* (FGMs), however began in the 1980's during the quest for suitable alternative materials to be used as thermal barrier coatings in gas turbine engines. Thermal barrier coatings are systems that are exposed to high temperatures on one hand and on the other require to carry a load without cracking. Ceramics are good at resisting high temperature, however are very brittle and crack easily. Metals on the other hand are tough and ductile but their relatively lower melting point compared to ceramics excludes them to be used in very-high temperature applications. The best of the two is achieved by a functionally graded material, which is a physical mixture of both ceramics and metals and does not contain any physically distinct interface between the constituent phases. From its inception as alternative materials to be used in thermal barrier coatings, functionally graded materials have found numerous engineering applications in wear resistant coatings, MEMs and electronic devices. Combination of two physical constituents other than ceramics and metals are also seen in recent graded materials. Novel processing techniques are also in

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place to obtain bulk graded structures, ranging from plasma sintering, selective UV radiation to centrifugal casting and selective laser sintering. The graded components once manufactured needs to verified whether the desired mechanical properties are attained as designed and classical mechanical testing methods such as indentation, nano-indentation, tensile testing and ultrasonic wave propagation experiments are in place. Solid mechanics analysis of the test methods and experiments are also available to further modify and optimize the graded material systems. Modern machine design methodologies include the concept of fracture mechanics owing to the complex nature of the materials and loading scenarios. Functionally graded materials being relatively newer materials have received some focus in terms of understanding their fracture response and crack propagation characteristics. Experimental, theoretical and numerical fracture mechanics studies are available in the literature dealing with graded and inhomogeneous materials. There is also a lot of ongoing interest in the solid mechanics community to further the understanding of fracture of functionally graded materials. In this chapter, we have summarized the efforts of our group in the past decade that has led to furthering the understanding of spontaneous planar crack propagation in bimaterial systems. The novelty of this chapter is the development of a new numerical technique namely the graded spectral scheme described in Section 2. Comparison of crack propagation characteristics between homogeneous and graded materials are described in Section 3, followed by the comparison between symmetric and unsymmetric graded systems in Section 4. Finally in Section 5 we detail the cohesive crack propagation characteristics in bimaterial graded systems, wherein the top and bottom half-space of the system can be any tailored material gradient.

INTRODUCTION

Nature has evolved inhomogeneous material properties in natural biological systems to respond efficiently to the external loading. Different biological systems have various types of inhomogeneities. Human and animal teeth for example have a hard-enamel on the outside that transits into the soft interior with a continuous density and modulus variation (Yang and Xiang, 2007; Nagata et al., 2009). The next extensively studied biological system is the bone, which has a complex anisotropic (directional dependent) inhomogeneity as explained by several studies by (Bourne et al., 2004; Tawara et al., 2005; Yoshibash et al., 2007; Baca et al., 2008; Trabelsi et al., 2009). Experiments and analysis on bones have obtained 3-D maps of bone density as a function of spatial- position using computed-tomography (CT) scans and converted them into a Hounsfield-Unit (HU) that relates to the spatially-varying bone density. Those experiments obtained an empirical relation between the bone density and the modulus as a function of density, which in-turn is treated as a known spatial inhomogeneous variation of modulus. Cartilages are another class of biological systems studied extensively due to its importance in the understanding and treatment of arthritis. Cartilages are soft-tissues that exhibit inhomogeneous density and strength variations in order to effectively reduce the friction between load bearing joints (Chin-Purcell and Lewis, 1996; Krishnan et al., 2003; Simha et al., 2004; Beatty et al., 2008; Federico and Herzog, 2008; Mathews et al., 2008; Bian et al., 2009; Natoli and Athanasiou, 2009; Smith et al., 2009; Genin et al., 2009; Arakaki et al., 2010). One of the main motivations of analyzing naturally occurring inhomogeneous biological systems (such as tooth, bones, cartilages, tendons etc.) is to replace them with man-made implants that exactly mimic their properties. Most recent experiments and analysis of biological systems have also been devoted to understanding the variations of the inhomogeneous properties. One of the necessary requirements of a biological implant material is bio-compatibility. Apart from bio-compatibility, design or manufacture of biological implants that solely mimic the observed inhomogeneous strength and modulus distribution might not be adequate. Naturally occurring biological systems such as bones and cartilages also experience extreme external mechanical loading leading to fracture, cuts or tears that require a thorough understanding of their fracture behavior in designing and manufacturing appropriate implants. Hence, forms as one of the motivations of the present study of fracture mechanics analysis of inhomogeneous materials. There has been wide interest in understanding the fracture behavior of naturally occurring biological systems taking into effect the inhomogeneous variation of density, modulus and fracture properties of materials (Chin-Purcell et al., 1996; Simha et al., 2004; Beatty et al., 2008; Myung et al., 2008; Nagata et al., 2009). There has also been recent interest in manufacturing and designing of inhomogeneous soft-tissue simulants based on hydrogels (Pompe et al., 2003; Mathews et al., 2008; Myung et al., 2008; Singh et al., 2008; Smith et al., 2009; Arakaki et al., 2010). Inhomogeneously varying porosity is one technique developed to manufacture novel bone-implants using either polymeric or ceramic based material systems (Muthutantri et al., 2008; Yao et al., 2009). Bones, cartilages and many other biological systems of interest to our current study not only exhibit inhomogeneous material property variations but also exhibit nonlinear stress-strain responses, whose effect is not understood very well.

Metal based inhomogeneous systems commonly used in thermal barrier coatings and wear-resistant coatings have been extensively studied also exhibit nonlinearity and inhomogeneity. The present chapter aims at developing a basic understanding of the fracture mechanics of engineering, biological and bio-material simulant materials that exhibit a combination of inhomogeneity and nonlinear stress-strain response. Recently, Titanium based graded engineering materials have been developed and analyzed for their superior corrosion resistance properties, while retaining desired properties of metals such as high fracture toughness and machinability (Quast et al., 2008; Kidane and Shukla, 2008 & 2010). Inhomogeneous systems comprising of Nickel and ZrO_2 (ceramic based) graded materials have been developed and analyzed recently by Yam et al., (2009). Metal, polymer or ceramic based inhomogeneous engineering materials have been extensively studied to understand their fracture resistance behavior as a function of the inhomogeneous property variation assuming the stress-strain constitutive response to be linear-elastic.

Similar to the multitude of manufacturing processes and applications of inhomogeneous materials, solid mechanics analyses and in particular to the present study fracture mechanics analyses of inhomogeneous materials exist and has received wide attention since its beginning in the 1980s. The work of Delale and Erdogan (1983) was first of its kind to analyze a stationary crack in functionally graded materials, which concluded that the stress-intensity-factor (SIF) based design methodology to be applicable for systems made of graded materials too. Asymptotic analysis around a stationary crack in an elastic functionally graded material by Eischen (1987) confirmed the conclusions of Delale and Erdogan (1983). A fracture mechanics analysis of inhomogeneous materials is of current prevailing interest (as of October 2011) as illustrated by the recent articles available in the literature. Fracture analyses have been performed at various spatial and temporal resolution levels, starting from specimen level testing to asymptotic (near crack-tip) solutions. Analyses assuming linear stress-strain behavior to inertial time-dependent and rapid-crack propagation analyses are available in the literature. Full field cohesive zone model based finite element fracture analysis of cement concrete was performed by Roesler et al., (2007) and Park et al., (2010). Recent analysis of Wadgaonkar and Parameswaran (2009) around stationary cracks in transversely graded material systems has obtained the structure of the near-tip stress field, which they have confirmed by their experiments (Komanna and Parameswaran, 2009). Near-tip fields were obtained recently by Chalivendara (2008, 2009) for stationary cracks embedded in inhomogeneous orthotropic medium under mode-1 loading and mixed-mode loading, respectively. Fracture analysis and experiments in inhomogeneous materials considering the dynamic effects (inertial effects of the medium) assuming linear elastic response of the medium have also received recent attention (Chalivendra, 2007; Jain and Shukla, 2007; Shukla et al., 2007; Lee et al., 2008; Kirugulige and Tippur, 2008; Kubair and Lakshmana, 2008). As mentioned earlier, fracture analysis of inhomogeneous biological materials has also drawn attention in order to develop, design and manufacture efficient bio-implants.

In this chapter we summarize the efforts of our computational dynamic fracture mechanics group towards understanding the spontaneous propagation of planar cracks subjected to an anti-plane shear mode-3 loading. The crack is assumed to develop a finite sized cohesive zone ahead of it tip while propagating. The derivation of the spectral formulation and its implementation are described next. The effects of the inhomogeneous variation of the rigidity modulus and density on the spontaneous crack propagation characteristics are understood by our numerical results obtained from the systematic parametric simulations of various graded systems.

2. FORMULATION

The spectral based elastodynamic relations pertaining to an inhomogenous bimaterial system are derived here. The model elastodynamic problem considered is depicted in Figure 1. The bimaterial interface is assumed to be coincident with the x_2 plane and is also the weak-plane. An interfacial crack of initial length a_o is allowed to propagate spontaneously on this weak-plane due to the action of an external loading τ_o. In our spectral formulation, the external load can be any arbitrary function of time and spatial position. The continuum above and below the interface/weak-plane are designated by a plus (+) and minus (-), respectively. The rigidity modulus (μ) and density (ρ) are allowed to vary exponentially in the direction perpendicular to the weak-plane as

$$\mu(x_2^{\pm}) = \mu_o^{\pm} \exp\left(\frac{x_2^{\pm}}{L_g^{\pm}}\right), \qquad \rho(x_2^{\pm}) = \rho_o^{\pm} \exp\left(\frac{x_2^{\pm}}{L_g^{\pm}}\right), \tag{1}$$

where $\mu_o^{\pm}$ and $\rho_o^{\pm}$ are the rigidity modulus and density on the weak-plane ($x_2 = 0$) for the top (+) and bottom (-) materials, respectively.

In the present bimaterial formulation, the material properties are allowed to suffer a discontinuous jump across the weak-plane. In Equation (1) $L_g^{\pm}$ corresponds to the natural inhomogeneity length scale of the FGM, which controls how fast or slow the material properties vary along the chosen spatial direction. The inhomogeneity length scale L_g can be in the order of a few microns (as in the case of thermal barrier coatings) or in the order of several meters as in tectonic plates.

The trivial case of a homogeneous material is obtained by assuming the inhomogeneity length scale to be infinite. The inhomogeneity length scale can be either positive or negative in the spectral formulation. In the top half a positive value of the inhomogeneity length scale leads to an inhomogeneous material that progressively becomes more rigid and denser away from the weak-plane and is termed as a "hardening" or "strengthening" type functionally graded material. When the inhomogeneity length scale is set to be less than zero (negative) the material properties degenerate away from the weak-plane and such a material is termed as a "softening" or "weakening" type functionally graded material. A combination of both a softening material in one half and a hardening material in the other half with different inhomogeneities is possible in our bimaterial formulation.

The particular cases of the unsymmetric functionally graded material can be obtained by assuming $\mu_0^+ = \mu_o^-$, $\rho_0^+ = \rho_o^-$ and $L_g^+ = L_g^-$. Similarly a symmetric functionally graded material can be obtained by assuming $\pm L_g^+ = \mp L_g^-$, with identical rigidity modulii and densities for the top and bottom materials. As mentioned earlier, in the present bimaterial formulation we allow the material parameters, namely the rigidity modulii, density and the inhomogeneity length scale to be different in the two halves. As seen from Equation (1), the inhomogeneous variation of the rigidity modulus and densities in the two halves are assumed to be identical mathematical functions. This assumption leads to a homogenous shear wave speed $c_s^{\pm} = \sqrt{\mu_o^{\pm} / \rho_o^{\pm}}$ in each of the half-spaces. The formulation described here is for the anti-plane shear case, where the only non-vanishing displacement is $u_3^{\pm}(x_1, x_2, t)$. The kinematic relation between the displacement and strains are given by

$$\varepsilon_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right), \tag{2}$$

where $i, j \to 1, 2 \,\&\, 3$ and repeated indices imply summation convention as in tensor algebra. For the anti-plane shear case the strain tensor will be