Algebraic Topology: An Introduction

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Foreword

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Preface

This textbook is designed to introduce advanced undergraduate or beginning graduate students to algebraic topology as painlessly as possible. The principal topics treated are 2-dimensional manifolds, the fundamental group, and covering spaces, plus the group theory needed in these topics. The only prerequisites are some group theory, such as that normally contained in an undergraduate algebra course on the junior-senior level, and a one-semester undergraduate course in general topology.

The topics discussed in this book are "standard" in the sense that several well-known textbooks and treatises devote a few sections or a chapter to them. This, I believe, is the first textbook giving a straightforward treatment of these topics, stripped of all unnecessary definitions, terminology, etc., and with numerous examples and exercises, thus making them intelligible to advanced undergraduate students.

The subject matter is used in several branches of mathematics other than algebraic topology, such as differential geometry, the theory of Lie groups, the theory of Riemann surfaces, or knot theory. In the development of the theory, there is a nice interplay between algebra and topology which causes each to reinforce interpretations of the other. Such an interplay between different topics of mathematics breaks down the often artificial subdivision of mathematics into different "branches" and emphasizes the essential unity of all mathematics.

Undoubtedly some experts will be shocked that a textbook purporting to be an introduction to algebraic topology does not even mention homology theory. It is certainly true that homology and cohomology theory form the core of algebraic topology. However, it is difficult to motivate the student who is learning these subjects for the first time, and their systematic treatment requires the patient development of a great deal of machinery. Only after several months of classroom lectures and study can interesting applications be given which show that the development of all the machinery was worthwhile. For these reasons, I believe that it is easier for the student to understand and appreciate homology theory after he has studied the fundamental group and allied topics presented in this book.

To those with a strictly logical mind, Chapter I, which discusses 2-dimensional manifolds, will perhaps seem the least rigorous part of the book. There certainly would be no real problem in giving a strictly rigorous treatment of this subject matter. However, such a treatment would be rather dull and tedious, with long-winded proofs of facts that are visually obvious. Moreover, the results of Chapter I are not basic to the main theorems in the rest of the book; rather, they furnish examples, illustrations, and applications of the results of the later chapters.

Chapter II gives the definition and basic properties of the fundamental group and the homomorphism induced by a continuous map. General methods for determining the structure of the fundamental group of a space are developed later, in Chapter IV, after certain essential group-theoretic notions have been introduced in Chapter III.

In Chapters III and IV the characterization of certain mathematical structures as the solutions of "universal mapping problems" is emphasized for two different reasons. First, it seems that the most efficient method of determining the structure of the fundamental group of a wide variety of spaces is by use of the Seifert-Van Kampen theorem (Chapter IV); the best formulation of this essential theorem involves the notion of a universal mapping problem. Second, this method of characterizing various mathematical structures as solutions to universal mapping problems seems to be one of the truly unifying mathematical principles to have emerged since 1945, and it should be brought into the mathematics curriculum as early as possible.

Chapter V contains a rather thorough discussion of covering spaces. The relationship between covering spaces and the fundamental group is emphasized throughout.

In Chapters VI and VII are given topological proofs of several well-known theorems of group theory, especially the Nielsen-Schreier theorem on subgroups of a free group, the Kurosh theorem on subgroups of a free product, and the Grushko theorem on the decomposition of a finitely generated group as a free product. These theorems belong to a section of group theory whose original development was largely motivated by combinatorial topology. I believe that the proofs of these theorems using the fundamental groups and covering spaces of certain low-dimensional complexes are more easily comprehended than the purely algebraic proofs. I hope the unified treatment of these theorems by these essentially geometric methods will make this section of group theory less formidable and more readily accessible.

Chapter VIII is rather brief and of a strictly descriptive nature; no theorems are proved. Its purpose is to help the student make the transition to the study of more advanced topics in algebraic topology.

Although triangulations of 2-manifolds are used in Chapter I, and the CW-complexes of J. H. C. Whitehead are introduced in the last chapter, there is no systematic treatment of simplicial complexes in this book. This may surprise some readers in view of the fact that many treatises on algebraic topology start off with just such a discussion. However, it is difficult to see how it could have materially simplified the exposition. Moreover, it is my personal opinion that any such discussion must of necessity be rather dull. One of the tendencies of algebraic topology during the last fifteen years or so has been the replacement of simplicial complexes by CW-complexes as the main object of study.

The sections listed below are not absolutely necessary to the further developments of the theory, and they can be omitted completely or given less emphasis in a briefer course or on a first reading of the book:

Chapter I, Sections 9–13.
Chapter II, Sections 7 and 8.
Chapter III, Section 7.
Chapter IV, Section 6.
Chapter V, Sections 10–12.
Chapter VI, Section 8.
Chapter VII, Sections 5 and 6.

Also, a briefer course could be built around the material in the first five chapters, omitting the same sections.

This book has developed from lectures given at Yale University to both graduate and undergraduate students over a period of several years. It is a pleasure to acknowledge my indebtedness to these students. Their questions, criticisms, and suggestions have given me many insights. I am also deeply indebted to my colleagues for many discussions of the ideas presented in this book. Most of the theorems and definitions in this book may be found in well-known textbooks or articles in mathematical journals. In this regard, special mention must be made of the following German textbooks: B. Kerekjarto, *Topologie* (Springer, 1923); K. Reidemeister, *Einführung in die Kombinatorische Topologie* (Teubner, 1932), H. Seifert and W. Threlfall, *Lehrbuch der Topologie* (Vieweg, 1934). In many cases I have tried to indicate the person or persons to whom I thought an idea or theorem should be credited. However, in a subject such as this, whose development spans most of the past century and which has been the joint work of many mathematicians in many countries, it is inevitable that I have committed some errors in assigning credit. To those whose names have been inadvertently omitted, I apologize; I trust that they will be understanding.

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Note to the Student

Prerequisites This book assumes that the student knows enough group theory to understand such standard terms as group, subgroup, normal subgroup, homomorphism, quotient group, coset, abelian group, and cyclic group. Moreover, it is hoped that he has seen enough examples and has worked enough exercises to have some feeling for the true significance of these concepts. An appendix on permutation and transformation groups is supplied for the benefit of those who are unfamiliar with this topic. Most of the additional topics needed in group theory are developed in the text, especially in Chapter III.

The necessary background in point set topology can be obtained from a one-semester undergraduate course in the subject. Because most textbooks for such a course either treat the subject very briefly or omit it entirely, a short discussion of quotient spaces is appended. No knowledge of any branch of algebra other than group theory is needed; in particular, nothing is used from the theory of rings, fields, modules, or vector spaces.

Terminology and notation Since most terminology and notation is standard in contemporary mathematics books on this level, little explanation is needed. In group theory, all groups (with a few standard exceptions, such as the additive group of integers) are written multiplicatively, not additively. A homomorphism from one group to another is called an *epimorphism* if it is onto, a *monomorphism* if it is one-to-one (i.e., the kernel contains only the identity), and an *isomorphism* if it is both one-to-one and onto. A diagram of groups and homomorphisms, such as

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g'}\downarrow & & \downarrow{\scriptstyle g} \\ C & \xrightarrow[f']{} & D \end{array}$$

is said to be *commutative* if all possible homomorphisms from one group to another in the diagram are equal. In the above diagram, there are two homomorphisms from group A to group D, namely, gf (i.e., f followed by g) and $f'g'$. Thus, requiring that this diagram be commutative is equivalent to requiring that $gf = f'g'$. Note that the requirement that a diagram be commutative has nothing to do with whether or not any of the groups involved is commutative or abelian. For example, the above diagram could be commutative even if A, B, C, and D were non-abelian groups.

In set theory, the notation

$$\prod_{i \in I} S_i$$

denotes the product (or cartesian product) of the family of sets S_i, $i \in I$. An element x of the cartesian product is a function that assigns to each index $i \in I$ an element $x_i \in S_i$. The element $x_i \in S_i$ is also called the *coordinate* of the element x corresponding to the index $i \in I$.

If A is a subset of B, then there is a uniquely defined *inclusion map* of A into B: It assigns to each element $x \in A$ the element x itself. In symbols, if $i : A \to B$ denotes the inclusion map, then $i(x) = x$ for any $x \in A$. If C is another set and $f : B \to C$ is any function from B to C, then $f \mid A$ denotes the *restriction* of f to the subset A; i.e., for any $a \in A$, $(f \mid A)(a) = f(a) \in C$.

The following notation is fixed throughout the book:

$\mathbf{Z}$ = set of all integers, positive and negative.

$\mathbf{Q}$ = set of all rational numbers.

$\mathbf{R}$ = set of all real numbers.

$\mathbf{C}$ = set of all complex numbers.

The notation $\mathbf{R}^n$ (respectively, $\mathbf{C}^n$) for any integer $n > 0$ denotes the set of all n-tuples $(x_1, \ldots, x_n)$ of real (respectively, complex) numbers; $\mathbf{R}^n$ is the *Euclidean n-space* and has its usual topology. If $x = (x_1, \ldots, x_n)$ is a point of $\mathbf{R}^n$, then the *norm* or *absolute value* of x, denoted by $|x|$, is defined as usual:

$$|x| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}.$$

With this notation, we define the following standard subsets of Euclidean n-space for any $n > 0$:

$$E^n = \{x \in \mathbf{R}^n : |x| \leqq 1\},$$

$$U^n = \{x \in \mathbf{R}^n : |x| < 1\},$$

$$S^{n-1} = \{x \in \mathbf{R}^n : |x| = 1\}.$$

These spaces are called the *closed n-dimensional disc* or *ball*, the *open n-dimensional disc* or *ball*, and the $(n - 1)$-*dimensional sphere*, respectively. Each is topologized as a subset of $\mathbf{R}^n$. The same names are sometimes applied to any topological space homeomorphic to one of the spaces just mentioned.

If a and b are real numbers such that $a < b$, then the following standard notation is used for the open and closed intervals with a and b as end points:

$$(a, b) = \{x \in \mathbf{R} : a < x < b\},$$

$$[a, b] = \{x \in \mathbf{R} : a \leqq x \leqq b\},$$

$$(a, b] = \{x \in \mathbf{R} : a < x \leqq b\}.$$

We say two spaces are of the same *topological type* if they are homeomorphic.

References A reference to Theorem or Lemma III. 8.4 indicates Theorem or Lemma 4 in Section 8 of Chapter III; if the reference is simply to Theorem 8.4, then the theorem is in Section 8 of the same chapter in which the reference occurs.

At the end of each chapter is a brief bibliography. Numbers in square brackets in the text refer to items in the bibliography.

On studying this book The exercises and examples are an integral part of the text; without them it would be much more difficult to gain an understanding of the subject. Many assertions are made without proof, and the details of certain proofs are omitted. Regard the filling in of the missing details as an exercise that tests whether you really understand the ideas involved.

Remember that the path from ignorance to knowledge in any subject is not straight and true, but is almost always rather zigzagged. One seems to learn things by a method of successive approximations to the truth. Thus, the first attempt to master some of the more difficult theorems in this book is not likely to be completely successful. However, do not give up. Rather, proceed with the study of the exercises and examples and some of the later material, confident that your perseverance will be rewarded with a deeper understanding of the ideas involved.

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CHAPTER ONE

Two-Dimensional Manifolds

1 Introduction

The topological concept of a surface or 2-dimensional manifold is a mathematical abstraction of the familiar concept of a surface made of paper, sheet metal, plastic, or some other thin material. A surface or 2-dimensional manifold is a topological space with the same local properties as the familiar plane of Euclidean geometry. An intelligent bug crawling on a surface could not distinguish it from a plane if he had a limited range of visibility.

The natural, higher dimensional analog of a surface is an n-dimensional manifold, which is a topological space with the same local properties as Euclidean n-space. Because they occur frequently and have application in many other branches of mathematics, manifolds are certainly one of the most important classes of topological spaces. Although we define and give some examples of n-dimensional manifolds for any positive integer n, we devote most of this chapter to the case $n = 2$. Because there is a classification theorem for compact 2-manifolds, our knowledge of 2-dimensional manifolds is incomparably more complete than our knowledge of the higher dimensional cases. This classification theorem gives a simple procedure for obtaining all possible compact 2-manifolds. Moreover, there are simple computable invariants which enable us to decide whether or not any two compact 2-manifolds are homeomorphic. This may be considered an ideal theorem. Much research in topology has been directed toward the development of analogous classification theorems for other situations. Unfortunately, no such theorem is known for compact 3-manifolds, and logicians have shown that we cannot even hope for such a complete result for n-manifolds, $n \geqq 4$. Nevertheless, the theory of higher dimensional manifolds is currently a very active field of mathematical research, and will probably continue to be so for a long time to come.

We shall use the material developed in this chapter, especially in Sections 1–8, later in the book.

2 Definition and examples of *n*-manifolds

Assume n is a positive integer. An *n-dimensional manifold* is a Hausdorff space (i.e., a space that satisfies the T_2 separation axiom) such that each point has an open neighborhood homeomorphic to the open n-dimensional disc U^n ($= \{x \in \mathbf{R}^n : |x| < 1\}$). Usually we shall say "n-manifold" for short.

Examples

2.1 Euclidean n-space $\mathbf{R}^n$ is obviously an n-dimensional manifold. We can easily prove that the unit n-dimensional sphere

$$S^n = \{x \in \mathbf{R}^{n+1} : |x| = 1\}$$

is an n-manifold. For the point $x = (1, 0, \ldots, 0)$, the set $\{(x_1, \ldots, x_{n+1}) \in S^n : x_1 > 0\}$ is a neighborhood with the required properties, as we see by orthogonal projection on the hyperplane in $\mathbf{R}^{n+1}$ defined by $x_1 = 0$. For any other point $x \in S^n$, there is a rotation carrying x into the point $(1, 0, \ldots, 0)$. Such a rotation is a homeomorphism of S^n onto itself; hence, x also has the required kind of neighborhood.

2.2 If M^n is any n-dimensional manifold, then any open subset of M^n is also an n-dimensional manifold. The proof is immediate.

2.3 If M is an m-dimensional manifold and N is an n-dimensional manifold, then the product space $M \times N$ is an $(m + n)$-dimensional manifold. This follows from the fact that $U^m \times U^n$ is homeomorphic to U^{m+n}. To prove this, note that, for any positive integer k, U^k is homeomorphic to $\mathbf{R}^k$, and $\mathbf{R}^m \times \mathbf{R}^n$ is homeomorphic to $\mathbf{R}^{m+n}$.

In addition to the 2-sphere S^2, the reader can easily give examples of many other subsets of Euclidean 3-space $\mathbf{R}^3$, which are 2-manifolds, e.g., surfaces of revolution, etc.

As these examples show, an n-manifold may be either connected or disconnected, compact or noncompact. In any case, an n-manifold is always locally compact.

What is not so obvious is that a connected manifold need not satisfy the second axiom of countability (i.e., it need not have a countable base). The simplest example is the "long line."[1] Such manifolds are usually regarded as pathological, and we shall restrict our attention to manifolds with a countable base.

Note that in our definition we required that a manifold satisfy the Hausdorff separation axiom. We must make this requirement explicit

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[1] See *General Topology* by J. L. Kelley. Princeton, N.J.: Van Nostrand, 1955. Exercise L, p. 164.