SHELDON M. ROSS

INTRODUCTION TO PROBABILITY AND STATISTICS FOR ENGINEERS AND SCIENTISTS

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INTRODUCTION TO PROBABILITY AND STATISTICS FOR ENGINEERS AND SCIENTISTS

Sheldon M. Ross
University of California, Berkeley

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For
Elise

Preface

This book has been written for an introductory course in probability and statistics for students in engineering, computer science, mathematics, statistics, and the physical sciences. As such, it assumes a knowledge of elementary calculus.

Chapters 1 and 2 introduce the fundamental subject matter of probability theory and Chapter 3 considers certain special types of random variables. A variety of examples indicating the wide applicability of these random variables is presented. Also included in our study of these random variables is a consideration of the question of calculating their probabilities. Indeed, computer programs that compute the probability distribution, and in certain cases their inverse of binomial, Poisson, normal, *t*, *F*, and chi-square random variables, are presented. These programs, which appear in the Appendix of Programs, are also on a diskette—written for an IBM PC—and are included as part of the text.

Chapter 4, Sampling, begins our study of statistics. We consider such topics as the sample mean, sample variance, sample median, as well as histograms, empirical distribution functions, and stem-and-leaf plots. The distributions of certain of the foregoing statistics are presented when the underlying population is normal. Included in this chapter is a program for computing the sample mean and sample variance for a given data set. (The program for computing the sample variance does not utilize the identity $\sum_1^n (x_i - \bar{x})^2 = \sum x_i^2 - n\bar{x}^2$, which though useful when computing by hand, is potentially hazardous—due to computer round-off error—when computing by machine. Instead, a recursive approach, developed in the chapter, is utilized.)

Chapter 5 considers the problem of statistical parameter estimation. Both point and interval estimates are provided, along with programs for computing interval estimates of desired levels of confidence.

Chapter 6 deals with hypothesis testing. The concept of a statistical hypothesis test, along with its significance level and power, is presented, and a variety of such tests concerning the parameters of both 1 and 2 normal populations are considered. Hypothesis tests concerning Bernoulli and Poisson parameters are also presented. For all of these tests we determine the relevant

p-value by directly computing the appropriate tail probability. Programs that perform the computations are provided. For instance, the Fisher-Irwin test of the equality of 2 Bernoulli parameters is presented, and a program for computing its exact p-value (essentially by computing a hypergeometric distribution function) is provided.

Chapter 7 deals with the important topic of regression. Both simple linear regression—including such subtopics as residual analysis and weighted least squares—and multiple linear regression are considered. Chapter 8 deals with problems in the analysis of variance. Both one-way and two-way (with and without interaction) problems are considered. Programs are provided to determine the regression least squares and the analysis of variance estimates as well as providing (in the analysis of variance case) the p-values of the relevant tests.

In Chapter 9, which deals with goodness of fit and nonparametric testing problems, the influence of the modern computer on statistical analysis really becomes apparent. The classical approach to hypothesis testing in these cases is to determine the value of an appropriate test statistic and then approximate the p-value by determining the limiting distribution (usually either a normal or chi-square) of the test statistic when the null hypothesis is true and the sample sizes are large. However, this usually leaves open the question of "how large is large"? The recent advent of fast and inexpensive computational power, however, has given us a way to finesse the above question by opening up two new approaches to obtaining the relevant p-value. The first of these—especially useful in certain nonparametric testing problems—uses computer solved recursive equations to exactly derive the p-value; the second uses a simulation approach to accurately estimate the p-value. Both of these methods, along with the classical approach, are indicated in Chapter 9. In addition, computer programs for implementing the tests are presented.

Chapter 10 deals with problems related to life testing. In this chapter, the exponential distribution, rather than the normal, plays the key role. Inference problems assuming an underlying Weibull life distribution are also considered. Chapter 11 considers the subject matter of quality control. A variety of control charts, including not only the Shewhart control charts but also the more sophisticated moving average and exponentially weighted moving average control charts, are considered. The final chapter on simulation presents techniques for simulating random variables. Variance reduction techniques are also considered in this chapter.

As mentioned previously, a diskette of 35 programs is included as part of the text. Ideally, every student will have access to a personal computer and thus will be able to utilize these programs. For those students without such access we have provided the appropriate tables to enable them to solve all the problems in the book. A solutions manual is available to instructors.

Preface for the Programs

This book comes with a diskette containing 35 programs. Most of these programs give (except for round-off errors) exact answers. However, those programs that either compute or invert continuous distribution functions give, by necessity, approximations. These approximations, with the exception of the one given by the program for inverting the chi-square distribution (Program 3-8-1-b, which should only be used to solve homework problems in the book), are very accurate.

In order to run the program diskette you need to have the file BASICA.COM. It is a DOS command. To load the diskette put it into disk drive A or B and get your computer into this drive. Now type the four letters "read" and press the enter key. There should now appear a message entitled "How to Start," which will give the appropriate instructions. If the instructions do not enable you to load the diskette (because of some kind of incompatibility problem), you can always first get into BASIC in your usual way, put the diskette into drive (say) A, and access the programs by typing LOAD"A:\ PROGRAMS \ name and then pushing the enter key, where name is the name of the program you want to run. For instance, typing LOAD"A:\ PROGRAMS \ 4-3 and pushing the enter key will load program 4-3, which can then be run by typing RUN and pushing the enter key.

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C H A P T E R 1

Elements of Probability

1 INTRODUCTION

The concept of the probability of a particular event of an experiment is subject to various meanings or interpretations. For instance, if a geologist is quoted as saying that "there is a 60 percent chance of oil in a certain region," we all probably have some intuitive idea as to what is being said. Indeed, most of us would probably interpret this statement in one of two possible ways. Either by imagining that

(1) The geologist feels that, over the long run, in 60 percent of the regions whose outward environmental conditions are very similar to the conditions that prevail in the region under consideration, there will be oil; or, by imagining that

(2) the geologist believes that it is more likely that the region will contain oil than it is that it will not; and in fact .6 is a measure of the geologist's belief in the hypothesis that the region will contain oil.

The two foregoing interpretations of the probability of an event are referred to as being the frequency interpretation and the subjective (or personal) interpretation of probability. In the frequency interpretation, the probability of a given outcome of an experiment is considered as being a "property" of that outcome. It is imagined that this property can be operationally determined by continual repetition of the experiment—the probability of the outcome will then be observable as being the proportion of the experiments that result in the outcome. This is the interpretation of probability that is most prevalent among scientists.

In the subjective interpretation, the probability of an outcome is not thought of as being a property of the outcome but rather is considered a statement about the beliefs of the person who is quoting the probability, concerning the chance that the outcome will occur. Thus, in this interpretation, probability becomes a subjective or personal concept and has no meaning outside of expressing one's degree of belief. This interpretation of probability is often favored by philosophers and certain economic decision makers.

No matter which interpretation one gives to probability, however, there is a general consensus that the mathematics of probability are the same in either case. For instance, if you think that the probability that it will rain tomorrow is .3 and you feel that the probability that it will be cloudy but without any rain is .2, then you should feel that the probability that it will either be cloudy or rainy is .5 independently of your individual interpretation of the concept of probability. In this chapter we shall present the accepted rules, or axioms, used in probability theory. As a preliminary to this, however, we need to study the concept of the sample space and the events of an experiment.

2 SAMPLE SPACE AND EVENTS

Consider an experiment whose outcome is not predictable with certainty in advance. Although the outcome of the experiment will not be known in advance, however, let us suppose that the set of all possible outcomes is known. This set of all possible outcomes of an experiment is known as the *sample space* of the experiment and is denoted by S. Some examples are the following:

1. If the outcome of an experiment consists in the determination of the sex of a newborn child, then

$$S = \{g, b\}$$

where the outcome g means that the child is a girl and b that it is a boy.

2. If the experiment consists of the running of a race among the 7 horses having post positions $1, 2, 3, 4, 5, 6, 7$, then

$$S = \{\text{all orderings of } (1, 2, 3, 4, 5, 6, 7)\}$$

The outcomes $(2, 3, 1, 6, 5, 4, 7)$ means, for instance, that the number 2 horse is first, then the number 3 horse, then the number 1 horse, and so on.

3. Suppose we are interested in determining the amount of dosage that must be given to a patient until that patient reacts positively. One possible sample space for this experiment is to let S consist of all the positive numbers. That is, let

$$S = (0, \infty)$$

where the outcome would be x if the patient reacts to a dosage of value x but not to any smaller dosage.

Any subset E of the sample space is known as an *event*. That is, an event is a set consisting of possible outcomes of the experiment. If the outcome of the experiment is contained in E, then we say that E has occurred. Some examples of events are the following.

In Example 1 if $E = \{g\}$, then E is the event that the child is a girl. Similarly, if $F = \{b\}$, then F is the event that the child is a boy.

In Example 2 if

$$E = \{\text{all outcomes in } S \text{ starting with a } 3\}$$

then E is the event that the number 3 horse wins the race.

For any two events E and F of a sample space S we define the new event $E \cup F$, called the *union* of the events E and F, to consist of all outcomes that are either in E or in F or in both E and F. That is, the event $E \cup F$ will occur if *either* E or F occurs. For instance, in Example 1 if $E = \{g\}$ and $F = \{b\}$, then $E \cup F = \{g, b\}$. That is, $E \cup F$ would be the whole sample space S. In Example 2 if $E = \{\text{all outcomes starting with } 6\}$ is the event that the number 6 horse wins, and $F = \{\text{all outcomes having 6 in the second position}\}$ is the event that the number 6 horse comes in second, then $E \cup F$ is the event that the number 6 horse comes in either first or second.

Similarly, for any two events E and F, we may also define the new event EF, called the *intersection* of E and F, to consist of all outcomes that are in both E and F. That is, the event EF will occur only if both E and F occur. For instance, in Example 3 if $E = (0, 5)$ is the event that the required dosage is less than 5 and $F = (2, 10)$ is the event that it is between 2 and 10 then $EF = (2, 5)$ is the event that the required dosage is between 2 and 5. In Example 2 if $E = \{\text{all outcomes ending in } 5\}$ is the event that horse number 5 comes in last and $F = \{\text{all outcomes starting with } 5\}$ is the event that horse number 5 comes in first, then the event EF does not contain any outcomes, and hence cannot occur. To give such an event a name we shall refer to it as the null event and denote it by $\varnothing$. Thus $\varnothing$ refers to the event consisting of no outcomes. If $EF = \varnothing$, implying that E and F cannot both occur, then E and F are said to be *mutually exclusive*.

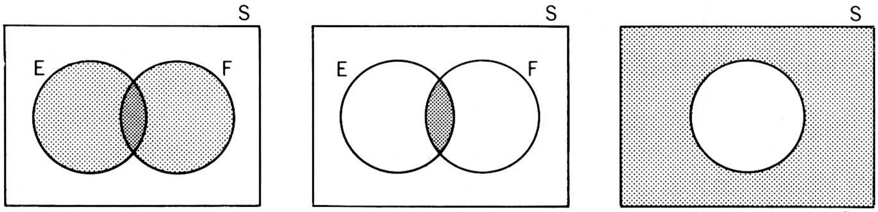
For any event E we define the event E^c, referred to as the *complement* of E, to consist of all outcomes in the sample space S that are not in E. That is, E^c will occur if and only if E does not occur. In Example 1 if $E = \{b\}$ is the event that the child is a boy, then $E^c = \{g\}$ is the event that it is a girl. Also note that since the experiment must result in some outcome, it follows that $S^c = \varnothing$.

For any two events E and F, if all of the outcomes in E are also in F, then we say that E is contained in F and write $E \subset F$ (or equivalently, $F \supset E$). Thus, if $E \subset F$ then the occurrence of E necessarily implies the occurrence of F. If $E \subset F$ and $F \subset E$, then we say that E and F are equal (or identical) and we write $E = F$.

We can also define unions and intersections of more than two events. In particular, the union of the events $E_1, E_2, \ldots, E_n$, denoted either by $E_1 \cup E_2 \cup \cdots \cup E_n$ or by $\bigcup_1^n E_i$, is defined to be the event consisting of all outcomes that are in E_i for at least one $i = 1, 2, \ldots, n$. Similarly the intersection of the events E_i, $i = 1, 2, \ldots, n$, denoted by $E_1 E_2 \cdots E_n$, is defined to be the event consisting of those outcomes that are in all of the events E_i, $i = 1, 2, \ldots, n$. In other words, the union of the E_i occurs when *at least* one of the events E_i occurs; while the intersection occurs when *all* of the events E_i occur.

3 VENN DIAGRAMS AND THE ALGEBRA OF EVENTS

A graphical representation of events that is very useful for illustrating logical relations among them is the Venn diagram. The sample space S is represented as consisting of all the points in a large rectangle, and the events $E, F, G, \ldots,$ are represented as consisting of all the points in given circles within the rectangle. Events of interest can then be indicated by shading appropriate regions of the diagram. For instance, in the following three Venn diagrams, the shaded areas represent respectively the events $E \cup F$, EF, and E^c,



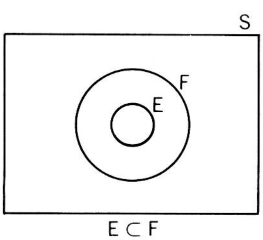
(a) Shaded region: $E \cup F$. (b) Shaded region: EF. (c) Shaded region: E^C

The Venn diagram in Figure 1.3.1 indicates that $E \subset F$.

The operation of forming unions, intersections, and complements of events obey certain rules not dissimilar to the rules of algebra. We list a few of these

Commutative law	$E \cup F = F \cup E$	$EF = FE$
Associative law	$(E \cup F) \cup G = E \cup (F \cup G)$	$(EF)G = E(FG)$
Distributive law	$(E \cup F)G = EG \cup FG$	$EF \cup G = (E \cup G)(F \cup G)$

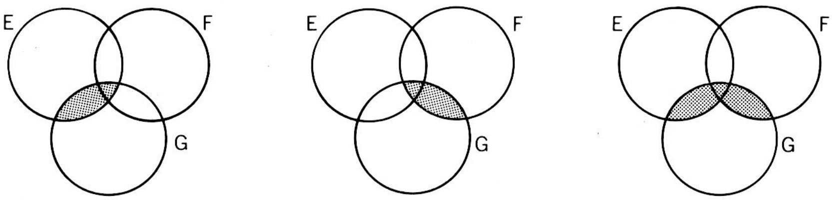
These relations are verified by showing that any outcome that is contained in the event on the left side of the equality is also contained in the event on the right side and vice versa. One way of showing this is by means of Venn



$E \subset F$

FIGURE 1.3.1 Venn diagrams

diagrams. For instance, the distributive law may be verified by the following sequence of diagrams.



(*a*) Shaded region: EG. (*b*) Shaded region: FG. (*c*) Shaded region: (E ∪ F)G.
(E ∪ F)G = EG ∪ FG

The following useful relationship between the three basic operations of forming unions, intersections, and complements of events is known as *DeMorgan's laws*.

$$(E \cup F)^c = E^cF^c$$
$$(EF)^c = E^c \cup F^c$$

4 AXIOMS OF PROBABILITY

It appears to be an empirical fact that if an experiment is continually repeated under the exact same conditions, then for any event E, the proportion of time that the outcome is contained in E approaches some constant value as the number of repetitions increases. For instance, if a coin is continually flipped, then the proportion of flips resulting in heads will approach some value as the number of flips increases. It is this constant limiting frequency that we often have in mind when we speak of the probability of an event.

From a purely mathematical viewpoint we shall suppose that for each event E of an experiment having a sample space S there is a number, denoted by $P(E)$, which is in accord with the following three axioms.

AXIOM 1

$$0 \leq P(E) \leq 1$$

AXIOM 2

$$P(S) = 1$$

AXIOM 3

For any sequence of mutually exclusive events $E_1, E_2, \ldots$ (that is, events for which $E_iE_j = \varnothing$ when $i \neq j$)

$$P\left(\bigcup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i), \qquad n = 1, 2, \ldots, \infty$$

We call $P(E)$ the probability of the event E.