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positively ordered semigroups

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POSITIVELY ORDERED SEMIGROUPS

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TO

VENKATESWARA SWAMY

PREFACE

The study of positively totally ordered semigroups drew the attention of Hölder in connection with the problem of determining when a totally ordered semigroup is embeddable in the additive semigroup of positive real numbers. In recent years, interest in this study has been evinced by several people (A. H. Clifford and L. Füchs, for example). The theory of positively totally ordered semigroups has also found applications in measurement theory. The recent survey article by Jabovich (1976) on the fully ordered semigroups has stimulated the author to present a systematic account of positively totally ordered semigroups in greater detail. Most of the material in these notes was presented to faculty and graduate students at Bowling Green State University in a seminar for nearly two quarters during 1976-1977. The latest findings of the author are also included in these notes. This is the first attempt of the author to bring the results in this area in a consolidated form. Suggestions of the readers will be highly appreciated. It is hoped that this work will benefit many working in this field and stimulate them to handle the unsolved problems, some of which are mentioned at the end of this monograph. Finally, it is a great pleasure to acknowledge the help of C. S. Nagore for the useful changes in the original manuscript.

M. Satyanarayana

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CHAPTER 1

INTRODUCTION

The purpose of this chapter is to introduce some basic concepts and the machinery needed to explain the new concepts which occur in the later chapters. We mention only some important definitions and results, which are used in the sequel. For the detailed proofs of some results mentioned here, we refer the reader to [1] for semigroup theory and to [2] for ordered structures. We begin with concepts in the theory of semigroups. Throughout this chapter let S denote an arbitrary semigroup; 0 is called a zero of S if $0 \cdot s = s \cdot 0 = 0$ for every $s \in S$. S is said to contain an identity 1 if $1 \cdot s = s \cdot 1 = s$ for every $s \in S$. If S contains 1, call $S = S^1$. Otherwise S^1 denotes $S \cup \{1\}$. If e is an element in S and if $e = e^2$, then e is called an idempotent. An ordering can be introduced among idempotents. If e and f are idempotents, $e < f$ if $e = ef = fe$. An element x in S is nilpotent if $x^n = 0$ for some natural number n, provided S contains 0. An element x in S is a left (right) zero-divisor if $xy = 0$ $(yx = 0)$ for some nonzero y in S. x is a right (left) cancellable element in S if $yx = zx$ $(xy = xz)$ for some y,z in S, then $y = z$. x is called cancellable if x is both right and left cancellative. x is a periodic element if $x^n = x^m$ for some natural numbers n and m.

A monoid is a semigroup with identity. If every element of S is nilpotent, then S is called a nilsemigroup. A nilsemigroup S is nilpotent if there exists a natural number n such that $x^n = 0$ for every $x \in S$. S is right (left) cancellative if every element of S is right (left) cancellative. S is cancellative if it is both right and left cancellative. S is periodic if every element of S is periodic. S is semisimple if $x \in SxSxS$ for every $x \in S$ or equivalently $A = A^2$ for every ideal A in S. S is called regular if, for every $a \in S$, there exists an $x \in S$ such that $a = axa$. S is a band if every element of S is an idempotent. A commutative band is called a semilattice. A chain is a semilattice in which the idempotents are linearly ordered under the ordering introduced above. If $S = S^2$, S is called globally idempotent. S is said to be periodic if every one of its elements is periodic.

NOTATION. '$\subset$' denotes inclusion and   denotes proper inclusion. If A and B are subsets, $A \setminus B$ is the set of all elements in A which are not in B.

A nonempty subset T of S is a right (left) ideal if $s \in S$, $t \in T$ imply $ts \in T$ ($st \in T$); T is a two-sided ideal or simply an ideal if it is both a right and left ideal; a right or left or two-sided ideal is called proper if it is different from S. One-sided or two-sided ideal T is trivial if $S \setminus T$ is a singleton.

An ideal T is prime if $AB \subset T$ for some ideals A and B (or for some one-sided ideals of the same type), then $A \subset T$ or $B \subset T$; T is called completely prime if $ab \in T$ for some a and b in S, then $a \in T$ or $b \in T$; T is a maximal right (left) ideal if it is not contained in any proper right (left) ideal. R*, L*, M*, P* and Q* denote the intersection of all maximal right ideals, maximal left ideals, maximal ideals, prime ideals and completely prime ideals respectively. Kernel of S is the intersection of all ideals. If S and T are semigroups, then a map $f : S \to T$ is called a homomorphism provided $f(ab) = f(a)f(b)$ for every a and b in S. With every ideal

A in S, we can define a canonical onto homomorphism from S onto S/A, where the elements of S/A are 0 (the image of A) and the other elements in S\A (S/A is called Rees-quotient semigroup); i.e., if $\bar{a},\bar{b} \in S/A$, then either a = b or $a,b \in A$.

Let $\{T_\alpha\}$ be a family of ideals in S such that $\bigcap T_\alpha = \emptyset$. Then S is said to be a subdirect product of S/T_α. This definition is a particular case of the general definition expressed in terms of congruences. R is a relation in S defined by: aRb iff $aS^1 = bS^1$; similarly L is a relation in S defined by: aLb iff $S^1a = S^1b$ and D is a relation defined by: aDb iff there exists an c such that aRc and cLb or vice versa. These relations partition S into disjoint subsets, which are called R-classes, L-classes and D-classes. A congruence relation ρ in S is an equivalence relation such that $xs\rho ys$ and $sx\rho sy$ for every $s \in S$, whenever $x\rho y$. The classes form a semigroup under the usual multiplication and is denoted by S/ρ. The association of the elements of S with its σ-equivalence classes is a homomorphism from S onto S/ρ. A congruence ρ is called a semilattice congruence if S/ρ is a semilattice.

S is called simple if S has no proper ideals. If S has no proper completely prime ideals, then S is called N-simple [1] and is also called ζ-indecomposable. The following is proved in [1].

FACT 1.1. *Every semigroup is a semilattice of N-simple semigroups; i.e., there exists a semilattice congruence, all of whose classes are N-simple semigroups.*

A right (left) ideal A in S is said to be finitely generated if $A = \bigcup_{i=1}^{n} x_i S^1$ $(A = \bigcup_{i=1}^{n} S^1 x_i)$. S is right (left) Noetherian if every right (left) ideal is finitely generated or equivalently every chain of right (left) ideals terminates at a finite stage. S is Archimedean if for every x and y in S there exists a natural number n such that $x^n \in SyS$. Clearly Archimedean semigroups are ζ-indecomposable but the converse need not be true.

A semigroup S is called a totally ordered semigroup, for short t.o. semigroup, if S is a totally ordered set under $\leq$ such that $a \leq b$ implies $ac \leq bc$ and $ca \leq cb$ for all $c \in S$. If '$a < b$' is not true, we write $a \geq b$. Assume that S is a t.o. semigroup here afterwords. For every element s in S such that $s \leq a$ ($a \leq s$) then a is called a maximal (minimal) element of S. An element a in S is a positive element if $x \leq ax$ and $x \leq xa$ for every s in S. If the inequality is strict, then a is called a strictly positive element. Similarly negative and strictly negative elements can be defined. S is called order-Archimedean (o-Archimedean) if a and b are positive elements such that $a^n < b$ for every positive integer n then a is the identity of S, if it exists and also if a and b are negative elements such that $a^n > b$ for every positive integer n then a is the identity of S, if it exists.

A semigroup S is said to satisfy right (left) quotient condition if, for every pair of elements a and b in S, there exist x and y in S such that $ax = by$ ($xa = yb$). Now, as mentioned in [2], we have

FACT 1.2. *Let S be a cancellative t.o. semigroup satisfying right quotient condition. Then S can be embedded in a t.o. group G of quotients* $g = ab^{-1}$, *where* $a,b \in S$. *If e is the identity of G, set* $g = ab^{-1} > e$ *iff* $a > b$. *G is then a t.o. group.*

A convex subset A of a t.o. semigroup S is a set in which $a < b < c$ with $a,c \in A$ imply $b \in A$. A congruence σ on a t.o. semigroup S is called convex if every σ-congruence class is a convex subset of S. If σ is a convex congruence on a t.o. semigroup S, then S/σ is a t.o. semigroup by prescribing:

$$\bar{a} \geq \bar{b},\ \bar{a},\bar{b} \in S/\sigma \text{ iff } a \geq b, \text{ where } a \in \bar{a},\ b \in \bar{b}$$

If S and T are t.o. semigroups, then f is called an order homomorphism (o-homomorphism) if f is a mapping such that

$f(ab) = f(a)f(b)$ for every a and b in S and $f(a) \leq f(b)$ whenever $a \leq b$. Order-isomorphism (o-isomorphism) is one to one onto o-homomorphism. If σ is a convex congruence on a t.o. semigroup S, then the canonical mapping $S \twoheadrightarrow S/\sigma$ is an o-homomorphism. In particular if A is a convex ideal in a t.o. semigroup S, then the natural map $S \twoheadrightarrow S/A$ is an o-homomorphism.

FACT 1.3. *If S is a t.o. semigroup, then the set E of all idempotents is S, if nonempty, is a subsemigroup of S.*

PROOF. Let $e,f \in E$. For definiteness let $e \leq f$. Then $e \leq ef$ and $fe \leq f$. This implies $ef = e^3f = e^2(ef) \leq efef$ and $efef = e(fe)f \leq ef^2 = ef$. Thus $ef = (ef)^2$. Hence $ef \in E$ and so E is a subsemigroup.

FACT 1.4. *If a t.o. semigroup contains positive and negative elements, then S contains idempotents.*

PROOF. Let x and y be positive and negative elements respectively. Then $xy \geq x$ and $xy \leq y$, which implies $x \leq y$. Since $x \leq x^3$, $xy \leq x^3y$ and $x \leq y$ implies $x^2 \leq xy$ and $yx \leq y^2$. Also $y^3 \leq y$. Hence $xy \leq x^3y = x^2(xy) \leq xyxy = x(yx)y \leq xy^3 \leq xy$. Thus xy is an idempotent.

CHAPTER 2

POSITIVELY TOTALLY ORDERED SEMIGROUPS

In this chapter we study the properties and structure of positively t.o. semigroups. A t.o. semigroup S is positively ordered if, for all a,b in S, ab $\geq$ a and ab $\geq$ b. The first decomposition theorem of positively t.o. semigroups is due to Clifford and Klein-Barmen. They proved that every positively t.o. semigroup is an ordinal sum of ordinally irreducible semigroups. But it is a hopeless task to study the structure of ordinally irreducible semigroups since positively t.o. semigroups which are cancellative but do not contain identity are always ordinally irreducible and so, much cannot be said. Saito provided another decomposition for arbitrary t.o. semigroups [3]. According to Saito, two elements a and b in a t.o. semigroup are strongly o-Archimedean equivalent or said to belong to the same A-class, if there exist natural numbers p,q,r,s such that $a^p < b^q$ and $b^r < a^s$. The membership of the same A-class is an equivalence relation. If a t.o. semigroup contains only one A-class, then we say that S is strongly o-Archimedean. Saito proved that every t.o. semigroup is a pair-wise disjoint union of convex strongly o-Archimedean subsemigroups. In this representation, we do not know how to multiply the elements of two different strongly o-Archimedean classes. There exist only partial results in this direction [4].