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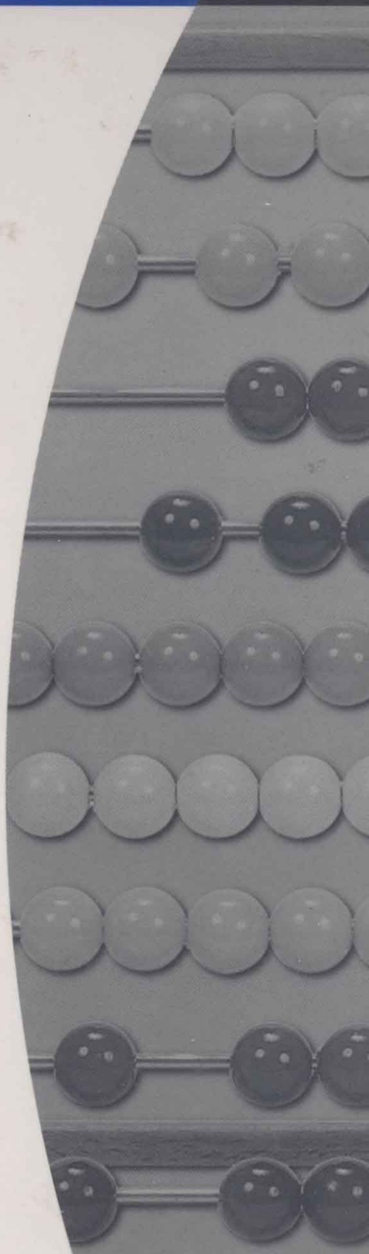

A Discrete Transition to Advanced Mathematics

Bettina Richmond
Thomas Richmond

American Mathematical Society

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Preface

A Discrete Transition to Advanced Mathematics is designed to bridge the gap between more-computational lower level courses and more-theoretical upper level courses in mathematics. While the focus is on building understanding, sharpening critical thinking skills, and developing mathematical maturity, topics from discrete mathematics provide the means.

The text contains more material than can be covered in one semester. There are several reasons for this. First, this makes the book appropriate for Discrete Mathematics courses for second- or third-year mathematics majors, as well as for Introduction to Proofs courses. Second, this will allow each instructor some flexibility in the selection of topics. Perhaps the best reason for the inclusion of so much material, however, is that the book is designed for students who should be learning to read mathematics on their own, and the extra sections should provide enjoyable reading at an appropriate level for these students. Besides more standard topics, the topics mentioned below will distinguish this text from others and, if not presented in class, would provide excellent material for independent projects.

- *Divisibility tests*, long familiar to many students, are explained and proved in Section 3.4.
- The surprising elementary *number patterns* in Section 3.5 emphasize the importance of pattern recognition.
- The *binomial coefficients* are introduced and applied geometrically in Section 4.1 before the formula for them is presented in Section 4.3.
- Modular arithmetic is introduced in Section 5.4 as a quotient construction and *quotient spaces* are used to investigate partial order relations on the blocks of a partition of a set A (i.e., quasiorders on A).
- The study of *sequences* in Chapter 8 provides a discrete version of analysis. *Finite differences* and their relation to sequences generated by polynomials are investigated. *Limits* are treated formally, providing an introduction to epsilon-N proofs for those who may have missed epsilon proofs in the calculus sequence.
- Infinite series, infinite products, and nested radicals in Section 8.5 provide an introduction to some forms of *infinite arithmetic*.
- *Fibonacci numbers* and *Pascal's triangle* in Chapter 9 provide a delightful array of surprising results that provide a unifying synthesis of topics from the previous chapters.
- *Continued fractions* and their applications are discussed in Chapter 10.

The remarkable connections between the Fibonacci numbers, Pascal's triangle, and the golden ratio in Chapter 9 were the original impetus for our writing this text. We considered a course based on these connections and patterns, many of which are very easily grasped. As we debated the appropriate level of presentation, we concluded that these ideas would serve as an excellent capstone to *A Discrete Transition to Advanced Mathematics* course.

We have taught courses based on Chapters 1–6 with Sections 3.4, 3.5, 4.5, and 6.4 optional, with additional topics and projects selected from the later chapters. Chapters 1–3 are required for all subsequent chapters and should be presented in order. The subsequent chapters need not be covered in order, but Chapter 5 is required for Chapter 6 and Sections 6.1 and 6.2 are needed for Section 7.2 and ' Chapter 8.

The material presented here should be accessible to students with the mathematical maturity provided by two or three semesters of calculus or an introductory linear algebra class. No calculus or linear algebra is used, but on a few occasions, connections to these subjects are noted.

Besides many classic results, we also include many elegant or surprising results which are not as widely known. We have tried to attain an engaging writing style that emphasizes precision through an intuitive understanding of the underlying concepts. However, simply reading the text will not be enough: Every student should work lots of exercises! There are over 650 exercises of varying difficulty designed to reinforce and extend the material presented.

We hope that the selection of topics, examples, and exercises will provide each reader with some of the marvel and amazement we still enjoy.

Ancillaries

The following ancillaries are available:

Student Solutions Manual The Student Solutions Manual provides worked out solutions to selected problems in the text.

Complete Solutions Manual The Complete Solutions Manual provides complete worked out solutions to all of the problems in the text and is available only to instructors.

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Bettina Richmond
Tom Richmond

Contents

1 Sets and Logic

[The universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language...

—Galileo Galilei (1564–1642)

For the things of this world cannot be made known without a knowledge of mathematics.

—Roger Bacon (1214–1294)

1.1 Sets

A *set* is a collection of objects. The objects of the set are called the *elements* of the set. One way to specify a set is to list all the elements inside *set brackets* "{" and "}". For example, {Alabama, Alaska, Arizona, Arkansas} is a set with four elements. We may also specify a set in words. The set given above could be specified by stating "the set of all U.S. states that start with the letter A." It is convenient to give sets names, and conventionally, sets are named by capital letters. Thus, we may write $A =$ {Alabama, Alaska, Arizona, Arkansas}. Alabama is an element of A. Birmingham, Atlanta, and

Wyoming are not elements of A. The symbol for "is an element of" is $\in$. Putting a slash through this symbol gives the symbol for "is not an element of." Thus, we may write

$$\text{Alabama} \in A$$
$$\text{Alaska} \in A$$
$$\text{Birmingham} \notin A$$
$$\text{Atlanta} \notin A$$
$$\text{Wyoming} \notin A.$$

Let us consider the set consisting of the natural numbers less than 6, and let us call this set B. The previous cumbersome sentence may be shortened to this: Let $B = \{1, 2, 3, 4, 5\}$. Here we are listing the elements of B in *roster form* rather than giving a verbal description of the elements. Counting down from 6, you may determine that the set of natural numbers less than 6 should be $\{5, 4, 3, 2, 1\}$. This is also correct. The elements of a set may be listed in any order. Thus, $B = \{1, 2, 3, 4, 5\} = \{5, 4, 3, 2, 1\} = \{3, 5, 2, 1, 4\}$, and there are many more correct representations of the set B.

Any set U must be *well-defined;* that is, for every object x, there must be an unequivocal answer to the question "Is $x \in U$?" We may not always know the answer to this question, but we must know that an unequivocal answer exists. Consider the set F of all living people who have an ancestor with the name Fletcher. Are you a member of this set? Though you may not know the answer, you should recognize that there is an indisputable answer—either yes or no. The set of good books, however, is not a well-defined set. The answer to the question "Is *War and Peace* a good book?" may be subject to dispute. The usage of the word *good* is subjective, and this makes the word an improper choice to use in specifying well-defined sets.

Two sets are *equal* if they contain exactly the same elements. The set $B = \{4, 3, 1, 5, 2\}$ and the set $\{\frac{2}{2}, \sqrt{4}, \sqrt{9}, 2^2, 5\}$ are equal since they contain exactly the same elements, namely $1 = \frac{2}{2}$, $2 = \sqrt{4}$, $3 = \sqrt{9}$, $4 = 2^2$, and 5. The set of kangaroos on the moon is a well-defined set that contains no elements. The set $\{\ \}$ containing no elements is called the *empty set* or *null set* and is denoted $\emptyset$ or $\{\ \}$.

A set is *finite* if there is a whole number that tells the number of elements in the set. The set $B = \{1, 2, 3, 4, 5\}$ is finite, and the number of elements in B is five.

1.1.1 DEFINITION The *cardinality* of a finite set S is the number of elements in the set S and is denoted $|S|$.

Counting the number of elements in a set may not be as easy as it sounds, especially if the set is described instead of listed. How many elements does the set of letters in the word *throughout* have? Stated another way, find the cardinality $|C|$ of the set $C = \{t, h, r, o, u, g, h, o, u, t\}$. To the question "Is $r \in C$?" we should answer "Yes." To the question "Is $t \in C$?" we should answer "Yes, yes." Though it is more emphatic, the affirmative outcome "Yes, yes" is not different from the affirmative outcome "Yes," so the element $t \in C$ only counts as one element, despite the fact that we listed it twice. If we let D be the set of letters in the word *trough*, then $D = \{t, r, o, u, g, h\}$. The sets C and D have exactly the same elements, so $C = D$, and thus $|C| = |D| = 6$. Repeated elements in a set should only be counted once. Recognizing the duplication is frequently more difficult than in this example.

Sometimes we may not be able to count the elements of a finite set. The set F of living people with an ancestor named Fletcher is a finite set, and though we do not know the exact number of elements in F, we know $|F|$ cannot exceed the current world population, and thus must be finite.

If a set is not finite, it is *infinite*. The set of natural numbers, for example, is infinite. We will not be able to list all the elements of an infinite set, but we may indicate an infinite set by a verbal description or by listing several of the elements in a clear pattern followed by an ellipsis ("..."). Some standard notation for some standard sets will illustrate this.

$$\text{The set of natural numbers} = \mathbb{N} = \{1, 2, 3, 4, \ldots\}$$

$$\text{The set of whole numbers} = \mathbb{W} = \{0, 1, 2, 3, \ldots\}$$

$$\begin{aligned}\text{The set of integers} = \mathbb{Z} &= \{0, 1, -1, 2, -2, 3, -3, \ldots\}\\ &= \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}\end{aligned}$$

Another convenient way to specify a set symbolically is by *set-builder notation*, which we illustrate here. The notation $\{x \,|\, x \in \mathbb{N} \text{ and } x < 6\}$ is read "the set of all x such that $x \in \mathbb{N}$ and $x < 6$." In general, $\{x \,|\, * * * * * * *\}$ is read "the set of all x such that x satisfies the properties $* * * * * * *$ stated." Unknown elements of a set are conventionally denoted by lowercase letters, such as the x above. If the elements x are to come from some specified set, we may include this information before the "pipe" symbol "|". The set $\{x \,|\, x \in \mathbb{N} \text{ and } x < 6\}$ could be written as $\{x \in \mathbb{N} \,|\, x < 6\}$ and read "the set of natural numbers x such that $x < 6$." This is the set $B = \{1, 2, 3, 4, 5\}$ we have seen earlier.

We may now introduce the notation for two other frequently used infinite sets.

$$\text{The set of real numbers} = \mathbb{R} = \{x \,|\, x \text{ is a real number}\}$$

$$\text{The set of rational numbers} = \mathbb{Q} = \{\tfrac{a}{b} \,|\, a, b \in \mathbb{Z}, b \neq 0\}$$

The set of rational numbers $\mathbb{Q}$ consists of Quotients of integers, with the usual restriction that division by 0 is not allowed.

If we take an arbitrary set S and remove some, none, or all of its elements, the set T of remaining elements is called a *subset* of S, and we write $T \subseteq S$. Formally, a set T is a subset of S if and only if every element of T is also an element of S. If T is a subset of S, then S is a *superset* of T and we may write $S \supseteq T$. The notation $T \subseteq S$ may also be read "*T is contained in S*." We will illustrate this notation with some examples.

$$\{\text{red, white, blue}\} \subseteq \{\text{red, white, blue, green}\}$$

$$\{1, 3, 5\} \subseteq \{1, 2, 3, 4, 5\}$$

$$\{2\} \subseteq \{1, 2, 3, 4, 5\}$$

$$\emptyset = \{\ \} \subseteq \{1, 2, 3, 4, 5\}$$

$$\{1, 2, 3, 4, 5\} \subseteq \{1, 2, 3, 4, 5\}$$

$$\{5, 6, 7\} \not\subseteq \{1, 2, 3, 4, 5\}$$

The symbol $\not\subseteq$ used in the last example above means "is not a subset of." It is critical to use the correct terminology and symbols for subsets and elements of a set. Observe that $3 \in \{1, 3, 5\}$ but $3 \not\subseteq \{1, 3, 5\}$. Since 3 is not a set, it cannot be a subset of anything. Similarly, $\{3\} \subseteq \{1, 3, 5\}$ but $\{3\} \notin \{1, 3, 5\}$.

If $T \subseteq S$ but $T \neq S$, we say T is a *proper subset* of S and write $T \subset S$. (Compare this notation to $<$ and $\leqslant$.) While $\{1, 3\}$ is a subset of $\{1, 2, 3, 4, 5\}$, denoted $\{1, 3\} \subseteq \{1, 2, 3, 4, 5\}$, we could be more explicit and say that it is a proper subset, denoted $\{1, 3\} \subset \{1, 2, 3, 4, 5\}$. Since $\{1, 2, 3, 4, 5\}$ is a subset of itself but not a proper subset of itself, we could write $\{1, 2, 3, 4, 5\} \subseteq \{1, 2, 3, 4, 5\}$ but $\{1, 2, 3, 4, 5\} \not\subset \{1, 2, 3, 4, 5\}$.

Suppose $A \subseteq B$ and $B \subseteq C$. Then every element of A is an element of B and every element of B is an element of C. It follows that every element of A is an element of C; that is, $A \subseteq C$. Thus, $A \subseteq B$ and $B \subseteq C$ implies $A \subseteq C$. We may depict this situation using a *Venn diagram* as shown in Figure 1.1. Venn diagrams provide informal graphical illustrations of shared elements of several sets.

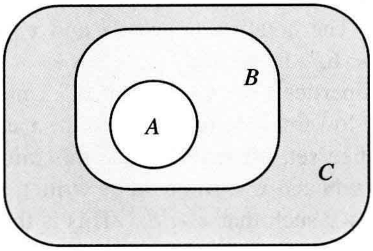

Figure 1.1
If $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

If $A \subseteq B$ and $B \subseteq A$, then every element of A is an element of B and every element of B is an element of A. Thus, A and B have exactly the same elements, so $A = B$. This is a standard way to show that two sets are equal: show A is contained in B and show B is contained in A. (For example, see Example 1.2.2 in the next section.)

Every set is a subset of itself, and the empty set $\emptyset$ is a subset of any set. Thus, for any set S, we have $\emptyset \subseteq S \subseteq S$. This seems to show that every set S has at least two subsets, namely the empty set and itself. This is true unless $S = \emptyset$, in which case these "two" subsets are really one and the same. We can properly state that every *nonempty* set S has at least two subsets, namely $\emptyset$ and S. Counting the number of subsets of a given set is an important problem we will consider from several approaches in the chapters that follow. Let us count all the subsets of a few small sets. To count them, we need a systematic way to find all the subsets.

1.1.2 EXAMPLE How many subsets does the set $\{1, 2\}$ have?

Solution There is one subset of $\{1, 2\}$ with zero elements: $\emptyset$.
There are two subsets of $\{1, 2\}$ with one element: $\{1\}$ and $\{2\}$.
There is one subset of $\{1, 2\}$ with two elements: $\{1, 2\}$.
This gives a total of $1 + 2 + 1 = 4$ subsets of the two-element set $\{1, 2\}$. ∎

1.1.3 EXAMPLE How many subsets does the set $\{1, 2, 3\}$ have?

Solution There is one subset of $\{1, 2, 3\}$ with zero elements: $\emptyset$.
There are three subsets of $\{1, 2, 3\}$ with one element: $\{1\}$, $\{2\}$, and $\{3\}$.
There are three subsets of $\{1, 2, 3\}$ with two elements: $\{1, 2\}$, $\{1, 3\}$, and $\{2, 3\}$.
There is one subset of $\{1, 2, 3\}$ with three elements: $\{1, 2, 3\}$.
This gives a total of $1 + 3 + 3 + 1 = 8$ subsets of the three-element set $\{1, 2, 3\}$. ■

The elements of a set may take any form. That is, we may take sets of any kind of objects. We may form sets of words, such as {black, white}, or sets of letters, such as $\{b, l, a, c, k, w, h, i, t, e\}$. We may form sets of numbers, such as $\{2, 4, 6, 8, 10\}$, or even sets of *sets*, such as $\{\ \{2, 4, 6, 8, 10\}, \{3, 6, 9\}, \{4, 8\}, \{5, 10\}\ \}$. To avoid confusion about the context of the word *set*, a set whose elements are sets will be called a *collection* of sets or a *family* of sets. Collections of sets are typically denoted with script capital letters. Thus, $\mathscr{C} = \{\ \{2, 4, 6, 8, 10\}, \{3, 6, 9\}, \{4, 8\}, \{5, 10\}, \{6\}\ \}$ is a collection of five sets. We have $\{3, 6, 9\} \in \mathscr{C}$, $\{4, 8\} \in \mathscr{C}$, and $\{6\} \in \mathscr{C}$, but $6 \notin \mathscr{C}$, $3 \notin \mathscr{C}$, $\{3\} \notin \mathscr{C}$, and $\{2, 4\} \notin \mathscr{C}$.

A *subcollection* of a collection $\mathscr{C}$ is a collection $\mathscr{S}$ such that every set in the collection $\mathscr{S}$ is also a set in the collection $\mathscr{C}$. Thus, $\{\ \{4, 8\}, \{5, 10\}\ \} \subseteq \mathscr{C}$ says that $\{\ \{4, 8\}, \{5, 10\}\ \}$ is a subcollection of $\mathscr{C}$. The definition of subcollection is precisely the definition of subset but with a shift of terminology to compensate for the fact that our "set" of "elements" is, in this case, called a "collection" of "sets." Some other examples may clarify these definitions.

Consider the collection of all subsets of $\{1, 2\}$. In Example 1.1.2 we found all the elements of this collection. The collection of all subsets of $\{1, 2\}$ is $\{\ \emptyset, \{1\}, \{2\}, \{1, 2\}\ \}$. This is an important construction and has a special name.

1.1.4 DEFINITION The collection of all subsets of a given set S is called the *power set of* S, denoted $\mathscr{P}(S)$. Thus, $\mathscr{P}(S) = \{A \mid A \subseteq S\}$.

Example 1.1.3 shows that the power set of $\{1, 2, 3\}$ is

$$\mathscr{P}(\{1, 2, 3\}) = \{\ \emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\ \}.$$

We may consider subcollections of $\mathscr{P}(\{1, 2, 3\})$ such as the collection $\mathscr{D}$ of all subsets of $\{1, 2, 3\}$ that contain the element 2:

$$\mathscr{D} = \{A \in \mathscr{P}(\{1, 2, 3\}) \mid 2 \in A\} = \{\ \{2\}, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\ \}.$$

The collection $\mathscr{F}$ of subsets of $\{1, 2, 3\}$ that have cardinality 2 is

$$\mathscr{F} = \{A \in \mathscr{P}(\{1, 2, 3\}) \mid |A| = 2\} = \{\ \{1, 2\}, \{1, 3\}, \{2, 3\}\ \}.$$

We have $\mathscr{D} \subseteq \mathscr{P}(\{1, 2, 3\})$ and $\mathscr{F} \subseteq \mathscr{P}(\{1, 2, 3\})$. Observe the distinction between "containing two elements" and "containing the element 2." As with any set, we may consider the cardinality of a set of sets—that is, of a collection. Here we have $|\mathscr{D}| = 4$, $|\mathscr{F}| = 3$ and $|\mathscr{P}(\{1, 2, 3\})| = 8$.

The following example will reinforce the importance of distinguishing between an element of a set and a subset of a set.

1.1.5 EXAMPLE

Let $A = \{\text{Alabama, Alaska, Arizona, Arkansas}\}$

$$E = \emptyset$$

$$I = \{\text{Illinois, Indiana, Iowa}\}$$

$$O = \{\text{Ohio, Oklahoma, Oregon}\}, \text{ and}$$

$$U = \{\text{Utah}\}.$$

Now, if we let $\mathscr{V} = \{A, E, I, O, U\}$, then $\mathscr{V}$ is a collection of five sets and we have

$$\text{Alaska} \in A$$

$$\{\text{Alaska, Arizona}\} \subseteq A$$

$$\text{Alaska} \notin \mathscr{V}$$

$$\{\text{Alaska, Arizona}\} \notin \mathscr{V}$$

$$\{\text{Alaska, Arizona}\} \not\subseteq \mathscr{V}$$

$$I = \{\text{Illinois, Indiana, Iowa}\} \in \mathscr{V}$$

$$I \not\subseteq \mathscr{V}$$

$$\{O, E, U\} \subseteq \mathscr{V}$$

$$\{E, U\} = \{\,\{\,\}, \{\text{Utah}\}\,\} \subseteq \mathscr{V}$$

$$\{U\} \subseteq \mathscr{V}.$$

Note that Utah $\in U$ and $\{\text{Utah}\} \subseteq U$ (in fact, $\{\text{Utah}\} = U$), but Utah $\not\subseteq U$. Furthermore, $U \in \mathscr{V} = \{A, E, I, O, U\}$ and $\{\,\{\text{Utah}\}\,\} = \{U\} \subseteq \mathscr{V}$, but $\{\text{Utah}\} = U \not\subseteq \mathscr{V}$, Utah $\not\subseteq \mathscr{V}$, and Utah $\notin \mathscr{V}$. There are also some subtleties involving the empty set in this example. From the definition of $\mathscr{V}$, we see that $E = \emptyset \in \mathscr{V}$. The empty set, however, is a subset of any set, and in particular, the empty collection is a subcollection of any collection. For our collection $\mathscr{V}$, we have $\emptyset \subseteq \mathscr{V}$. Now we have shown that $E = \emptyset \in \mathscr{V}$ and $E = \emptyset \subseteq \mathscr{V}$. This is a rare occurrence. Only in extraordinary circumstances will an element of a set also be a subset of that set. Note that $E = \emptyset \subseteq \mathscr{V}$ and also $\{E\} = \{\emptyset\} \subseteq \mathscr{V}$, but $\emptyset \neq \{\emptyset\}$. Generally, $x \neq \{x\}$, and there is no exception for $x = \emptyset$. While $\emptyset$ has no elements, $\{\emptyset\}$ has one element, namely $\emptyset$. Before leaving this example, we should note that the collection $\mathscr{V}$ of the five sets A, E, I, O, and U is not the same as the set {Alabama, Alaska, Arizona, Arkansas, Illinois, Indiana, Iowa, Ohio, Oklahoma, Oregon, Utah} of the 11 states that start with a vowel. In the next section we will see that this latter set is the union of the collection $\mathscr{V}$.

Large collections of sets are often expressed using an "index" for each set. For example, suppose a certain class meets for 36 days. Let S_1 be the set of students present on the first day, S_2 be the set of students present on the second day, and in general, let S_k be the set of students present on the k-th day ($k \in \{1, 2, 3, \ldots, 36\}$). The subscript k is called the *index* (plural: *indices*). The collection $\mathscr{S}$ of all the sets $\{S_1, S_2, \ldots, S_{36}\}$

may be represented as an *indexed collection* using various notations:

$$\begin{aligned}\mathscr{S} &= \{S_k \,|\, k = 1, 2, 3, \ldots, 36\} \\ &= \{S_k \,|\, k \in \{1, 2, 3, \ldots, 36\}\ \} \\ &= \{S_k \,|\, k \in I\} \text{ where } I = \{1, 2, 3, \ldots, 36\} \\ &= \{S_k\}_{k=1}^{36} \\ &= \{S_k\}_{k \in I} \text{ where } I = \{1, 2, 3, \ldots, 36\}.\end{aligned}$$

The index k is a "dummy variable"—we could just as well use i, j, λ, or any other symbol. The set of values that the index may assume is called the *index set*. In the example at hand, the index set is $I = \{1, 2, 3, \ldots, 36\}$.

EXERCISES

1. **(a)** True or false? {Red, White, Blue} = {White, Blue, Red}.
(b) What is wrong with this statement: Red is the first element of the set {Red, White, Blue}?

2. Which has the larger cardinality? The set of letters in the word *MISSISSIPPI* or the set of letters in the word *FLORIDA*?

3. Fill in the blank with the appropriate symbol, $\in$ or $\subseteq$.
(a) $\{1, 2, 3\}$____$\{1, 2, 3, 4\}$
(b) 3____$\{1, 2, 3, 4\}$
(c) $\{3\}$____$\{1, 2, 3, 4\}$
(d) $\{a\}$____$\{\{a\}, \{b\}, \{a, b\}\}$
(e) $\emptyset$____$\{\{a\}, \{b\}, \{a, b\}\}$
(f) $\{\{a\}, \{b\}\}$____$\{\{a\}, \{b\}, \{a, b\}\}$

4. Draw a Venn diagram showing the proper relationship between these sets: $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{W}$, and $\mathbb{Z}$.

5. **(a)** How many subsets does the empty set have?
(b) How many subsets does the set $\{1\}$ have?
(c) Noting the number of subsets of a two-element set and of a three-element set from Examples 1.1.2 and 1.1.3, how many subsets do you think a four-element set $\{1, 2, 3, 4\}$ would have?
(d) List all the subsets of the four-element set $\{1, 2, 3, 4\}$.
(e) How many subsets do you think a five-element set would have? A six-element set? An n-element set?

6. Determine whether the sets below are well-defined or not. For each well-defined set, state whether it is finite or infinite.
(a) The set of women pregnant with twins at some time during this year.
(b) The set of kangaroos in Australia.
(c) The set of tall buildings.
(d) The set of grains of sand on the earth.
(e) The set of even integers.
(f) The set of hairs on your head.
(g) The collection of all subsets of the set of hairs on your head.
(h) The set of people who shook hands with George Washington.

7. **(a)** Are there well-defined sets in Exercise 6 for which we may not know the answer to the question "Is x an element of this set?" for every object x?
(b) Are there finite sets in Exercise 6 for which we do not know the cardinality?

8. Let $S_1 = \{o, n, e\}$, $S_2 = \{t, w, o\}$, $S_3 = \{t, h, r, e, e\}$, and so on.
(a) Find all $k \in \{1, 2, \ldots, 10\}$ with $|S_k| = 4$.
(b) Find distinct indices $j, k \in \mathbb{N}$ with $S_j = S_k$.
(c) Find the smallest value of $k \in \mathbb{N}$ with $a \in S_k$.
(d) Let $\mathscr{S} = \{S_k\}_{k=1}^{40}$. Determine whether the following statements are true or false.

i. $S_{13} = \{n, e, i, t, h, e, r\}$
ii. $\{n, e, t\} \subseteq S_{20}$
iii. $S_1 \in \mathscr{S}$
iv. $S_3 \subseteq \mathscr{S}$
v. $\emptyset \in \mathscr{S}$
vi. $\emptyset \subset \mathscr{S}$
vii. $\emptyset \subseteq \mathscr{S}$
viii. $S_1 \subseteq S_{11}$
ix. $S_1 \subseteq S_{21}$
x. $S_1 \subset S_{21}$
xi. $\{n, i, e\} \in \mathscr{S}$
xii. $\{\{f, o, u, r\}\} \subseteq \mathscr{S}$
xiii. $u \in S_{40}$
xiv. $\mathscr{P}(S_9) \subseteq \mathscr{P}(S_{19})$
xv. $\{s, i\} \in \mathscr{P}(S_6)$
xvi. $w \in \mathscr{P}(S_2)$

9. For $k \in \{1, 2, \ldots, 20\}$, let $D_k = \{x \mid x \text{ is a prime number that divides } k\}$ and let $\mathscr{D} = \{D_k \mid k \in \{1, 2, \ldots, 20\}\}$.
(a) Find D_1, D_2, D_{10}, and D_{20}.
(b) True or false:

i. $D_2 \subset D_{10}$
ii. $D_7 \subseteq D_{10}$
iii. $D_{10} \subset D_{20}$
iv. $\emptyset \in \mathscr{D}$
v. $\emptyset \subset \mathscr{D}$
vi. $5 \in \mathscr{D}$
vii. $\{5\} \in \mathscr{D}$
viii. $\{4, 5\} \in \mathscr{D}$
ix. $\{\{3\}\} \subseteq \mathscr{D}$
x. $\mathscr{P}(D_9) \subseteq \mathscr{P}(D_6)$
xi. $\mathscr{P}(\{3, 4\}) \subseteq \mathscr{D}$
xii. $\{2, 3\} \in \mathscr{P}(D_{12})$

(c) Find $|D_{10}|$ and $|D_{19}|$.
(d) Find $|\mathscr{D}|$.

10. Give an example of an indexed collection $\mathscr{S} = \{S_k\}_{k=1}^{5}$ with $|\mathscr{S}| = 3$.

1.2 Set Operations

There are some standard set operations used to derive new sets from given sets.

Intersection and Union

Given sets S and T, the *intersection* of S and T, denoted $S \cap T$, is the set of elements which are in both S and T. The *union* of sets S and T, denoted $S \cup T$, is the set of all elements which are in either S or T or both.

$$S \cap T = \{x \mid x \in S \text{ and } x \in T\}$$

$$S \cup T = \{x \mid x \in S \text{ or } x \in T\}$$

It is clear from the definitions that for any sets S and T, $S \cap T = T \cap S$ and $S \cup T = T \cup S$. The Venn diagrams in Figure 1.2 depict the sets S and T, the intersection $S \cap T$, and the union $S \cup T$.

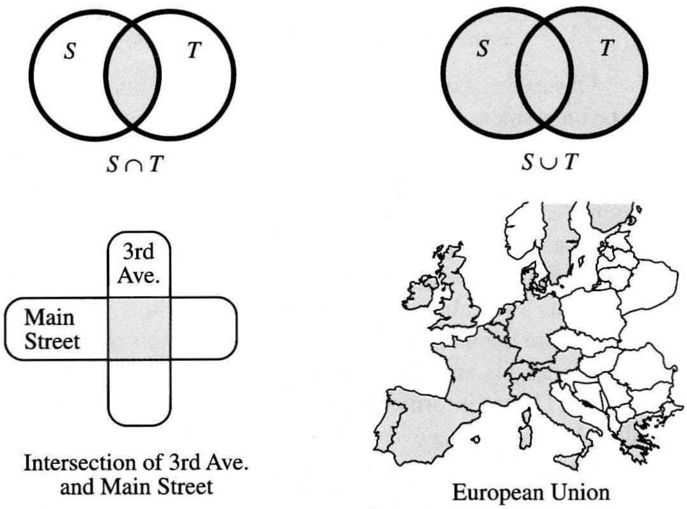

Figure 1.2
Intersections and unions.

For example, suppose $A = \{1, 3, 5, 7\}$, $B = \{3, 4, 5, 6\}$, and $C = \{2, 4\}$. Then we have

$$A \cap B = \{3, 5\}$$

$$A \cup B = \{1, 3, 4, 5, 6, 7\}$$

$$A \cap C = \emptyset$$

$$A \cup C = \{1, 2, 3, 4, 5, 7\}$$

$$B \cap C = \{4\}$$

$$B \cup C = \{2, 3, 4, 5, 6\}$$

Two sets with no "overlap," such as A and C above, are said to be *disjoint*. Formally, sets S and T are *disjoint* if $S \cap T = \emptyset$.

Let us consider another example. Let U be the set of students enrolled at Ottawa University. In this example, we will only consider subsets of this set U. Such a set U containing all the objects to be considered is called the *universal set* for the problem in question.

Let $H = \{x \in U \mid x \text{ has black hair}\}$.

Let $E = \{x \in U \mid x \text{ has green eyes}\}$.