NONLINEAR PHENOMENA AND COMPLEX SYSTEMS

ALEJANDRO MAASS, SERVET MARTÍNEZ
AND JAIME SAN MARTÍN (EDS.)

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Edited by

Alejandro Maass

Servet Martínez

and

Jaime San Martín

Department of Mathematical Engineering,
Faculty of Physical and Mathematical Sciences,
University of Chile, Santiago, Chile

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The Centre for Nonlinear Physics and Complex Systems (CFNL), Santiago, Chile, and Kluwer Academic Publishers have established this series devoted to nonlinear phenomena and complex systems, which is one of the most fascinating fields of science today, to publish books that cover the essential concepts in this area, as well as the latest developments. As the number of scientists involved in the subject increases continually, so does the number of new questions and results. Nonlinear effects are essential to understand the behaviour of nature, and the methods and ideas introduced to treat them are increasingly used in new applications to a variety of problems ranging from physics to human sciences. Most of the books in this series will be about physical and mathematical aspects of nonlinear science, since these fields report the greatest activity.

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Foreword

This book contains the lectures given at the *Conference on Dynamics and Randomness* held at the Centro de Modelamiento Matemático of the Universidad de Chile from December 11th to 15th, 2000.

This meeting brought together mathematicians, theoretical physicists and theoretical computer scientists, and graduate students interested in fields related to probability theory, ergodic theory, symbolic and topological dynamics.

We would like to express our gratitude to all the participants of the conference and to the people who contributed to its organization. In particular, to Pierre Collet, Bernard Host and Mike Keane for their scientific advise.

We want to thank especially the authors of each chapter for their well-prepared manuscripts and the stimulating conferences they gave at Santiago.

We are also indebted to our sponsors and supporting institutions, whose interest and help was essential to organize this meeting: ECOS-CONICYT, FONDAP Program in Applied Mathematics, French Cooperation, Fundación Andes, Presidential Fellowship and Universidad de Chile.

We are grateful to Ms. Gladys Cavallone for their excellent work during the preparation of the meeting as well as for the considerable task of unifying the typography of the different chapters of this book.

Alejandro Maass Servet Martínez Jaime San Martín

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DIMENSION–LIKE CHARACTERISTICS OF INVARIANT SETS IN DYNAMICAL SYSTEMS

VALENTIN AFRAIMOVICH AND JESUS URIAS
IICO–UASLP, A. Obregón 64
San Luis Potosí, SLP 78000
Mexico
jurias@cactus.iico.uaslp.mx

Abstract. The dimension theory of dynamical systems is certainly not complete. Nevertheless, it has great achievements such as general theory of the Hausdorff dimension of hyperbolic invariant measures (see [22] and references therein).

Recently it was understood that sets of zero measure in the phase space are responsible for such important phenomena as anomalous transport [9, 7]. It is unclear how to apply directly ergodic theory to study asymptotic behavior of orbits in such a situation. One might hope that dimension–like characteristics could help.

In our short lecture notes we did not describe many of those results: we were concentrated mainly on the application of Carathéodory–Pesin theory to dimensions for Poincaré recurrences. We think that this way is useful to introduce some ideas and machinery of the dimension theory, such that Moran geometric constructions, thermodynamic formalism, including Bowen formula, etc.

We believe that a reader who goes through examples and ideas of proofs in the lecture notes will be ready to study more serious literature and we hope that some of the readers will be attracted to this interesting field.

1. Invariant sets as results of inductive procedures

In spite of the fact that dynamical systems are defined by a local rule, say a map $x \mapsto f(x)$ (it could be a system of ODE $\dot{x} = f(x)$, but in these lectures we restrict ourself to the case of discrete time), and this rule is often expressed in a simple form, the global behavior of orbits could be

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amazingly complex. Here, an (semi–)orbit through an initial point x_0 is $\Gamma(x_0) := \bigcup_{i=0}^{\infty} f^i x_i$; a union of orbits Y is an invariant set: $f(Y) \subset Y$. Complexity of such a behavior is reflected in the geometry of invariant sets and can be measured by Hausdorff and box dimensions and other dimension–like characteristics.

1.1. HAUSDORFF DIMENSION

Let X be a metric space with a distance $d(x,y)$, x, $y \in X$. For any subset $Z \subset X$ let $\{U_i\}$ be a finite or countable collection of open sets of diameter less than ϵ such that $\bigcup U_i \supset Z$; here $\mathrm{diam} U_i := \sup\{d(x,y) : \ x,y \in U_i\}$. For any $\alpha > 0$ we introduce

$$m(\alpha,\epsilon,Z) \ = \inf_{\{U_i\}} \sum_i (\mathrm{diam}\, U_i)^\alpha, \tag{1}$$

where infimum is taken over all covers $\{U_i\}$ with diameter less than ϵ, and

$$m(\alpha, Z) = \lim_{\epsilon \to 0} m(\alpha, \epsilon, Z), \tag{2}$$

the α–dimensional Hausdorff measure (the limit exists because of monotonicity of $m(\alpha,\epsilon,Z)$ as a function of ϵ). It is simple to see that $m(\beta,\epsilon,Z) \leq \epsilon^{\beta-\alpha} m(\alpha,\epsilon,Z)$, which implies that there exists a unique critical value α_c of α such that $m(\alpha, Z) = 0$ if $\alpha > \alpha_c$ and $m(\alpha, Z) = \infty$ if $\alpha < \alpha_c$. The quantity $\alpha_c =: \dim_H Z$ is called the Hausdorff dimension.

1.2. GEOMETRIC CONSTRUCTION

Many invariant sets are resulting from so–called geometric constructions [22]. Let (σ, Ω), $\Omega \subset \Omega_p = \{0, \ldots, p-1\}^{\mathbb{N}}$, be a subshift, a closed σ–invariant subset of the full shift with p symbols. The word $(i_0, \ldots, i_{n-1})$ is admissible if the corresponding cylinder $[i_0, \ldots, i_{n-1}]$ has nonempty intersection with Ω. Consider p closed subsets $\Delta_0, \ldots, \Delta_{p-1} \subset \mathbb{R}^m$. Define *basic sets* $\Delta_{i_0,\ldots,i_{n-1}}$ which satisfy the following assumptions:

(A). $\Delta_{i_0,\ldots,i_{n-1}}$ are closed and nonempty if $(i_0, \ldots, i_{n-1})$ is admissible.
(B). $\Delta_{i_0,\ldots,i_{n-1}j} \subset \Delta_{i_0,\ldots,i_{n-1}}$, $j = 0, \ldots, p-1$.
(C). $\mathrm{diam}\Delta_{i_0,\ldots,i_{n-1}} \to 0$ as $n \to \infty$.

We can define now a nonempty set

$$F = \bigcap_{n=1}^{\infty} \ \bigcup_{(i_0,\ldots,i_{n-1})} \ \Delta_{i_0,\ldots,i_{n-1}}. \tag{3}$$

The closed set F becomes a Cantor set, provided that the following "separation condition" hold

(D). $\Delta_{i_0,\ldots,i_{n-1}} \cap \Delta_{j_0,\ldots,j_{n-1}} \cap F = \emptyset$ whenever $(i_0, \ldots, i_{n-1}) \neq (j_0, \ldots, j_{n-1})$.
The *coding map* $\chi : \Omega \to F$ is defined as follows: for any $\omega = (i_0, \ldots, i_{n-1}, \ldots) \in \Omega$, $\chi(\omega) = x$ if $x \in \bigcap \Delta_{i_0,\ldots,i_{n-1}}$.

The simplest constructions are of Moran type. In this case $\Omega = \Omega_p$ and basic sets satisfy additional axioms.

(M1). Every basic set is the closure of its interior.
(M2). For any n, $\mathrm{Int}\Delta_{i_0,\ldots,i_{n-1}} \cap \mathrm{Int}\Delta_{j_0,\ldots,j_{n-1}} = \emptyset$ if $(i_0, \ldots, i_{n-1}) \neq (j_0, \ldots, j_{n-1})$.
(M3). The basic set $\Delta_{i_0,\ldots,i_{n-1},j}$ is homeomorphic to $\Delta_{i_0,\ldots,i_{n-1}}$.
(M4). There are numbers $0 < \lambda_j < 1$, $j = 0, \ldots, p-1$, such that $\mathrm{diam}\Delta_{i_0,\ldots,i_{n-1},j} = \lambda_j \,\mathrm{diam}\Delta_{i_0,\ldots,i_{n-1}}$.

Moran proved that in this case $\dim_H F = s_0$, where $s = s_0$ is the root of the (Moran) equation

$$\sum_{i=0}^{p-1} \lambda_i^s = 1. \tag{4}$$

Example 1.1 Let J be an invariant set of the map $g : [0,1] \to [0,1]$,

$$g(x) = \begin{cases} x/\lambda_0 & \text{if } x \in [0, \lambda_0] \\ 0 & \text{if } x \in (\lambda_0, 1-\lambda_1) \\ \frac{x}{\lambda_1} - \frac{1-\lambda_1}{\lambda_1} & \text{if } x \in [1-\lambda_1, 1] \end{cases} \tag{5}$$

where $0 < \lambda_0 < \lambda_1 < 1$, $\lambda_0 + \lambda_1 < 1$, consisting of all points of all orbits belonging to $[0,1]$. It is clear that J is a Cantor set.

The set J is constructed with the help of the contractions $u_{0,1} : [0,1] \to [0,1]$,

$$u_0(x) = \lambda_0 x, \qquad u_1(x) = \lambda_1 x + 1 - \lambda_1,$$

such that $g \circ u_i = \mathrm{id}$ on $[0,1]$. For every word $\underline{i} = (w_0, \cdots, w_{i-1}) \in \{0,1\}^i$, define the sets

$$\Delta_{w_0,\cdots,w_{i-1}} := u_{w_{i-1}} \circ \cdots \circ u_{w_0}([0,1]),$$

i.e., the Δ–sets are basic sets of the geometric construction for the set J. Moreover, $\mathrm{diam}\,\Delta_{w_0,\cdots,w_{i-1}} = \lambda_{w_0} \cdots \lambda_{w_{i-1}}$ and

$$dist(\Delta_{\underline{i}0}, \Delta_{\underline{i}1}) = (1 - \lambda_0 - \lambda_1)\lambda_{w_0} \cdots \lambda_{w_{i-1}} > 0 \tag{6}$$

where $dist(x,y) = |x-y|$. Thus, J is resulting from a Moran construction, and $\dim_H J = s_0$, where $s = s_0$ is the root of the equation

$$\lambda_0^s + \lambda_1^s = 1. \tag{7}$$

To make (7) evident, consider the cover of J by basic sets of the n–th generation. Then the sum $\sum_i (\operatorname{diam} U_i)^\alpha$ in (1), up to a constant, becomes

$$\sum_{i_0,\ldots,i_{n-1}} \prod_{k=0}^{n-1} \lambda_{i_k}^\alpha = (\lambda_0^\alpha + \lambda_1^\alpha)^n. \tag{8}$$

If $\alpha > s_0$, then (8) goes to zero as $n \to \infty$, that shows us that $\dim_H J \leq s_0$. To get the opposite inequality, people use the technique of so–called Moran covers [22], see below.

Similar formulas could be obtained in the case when not all words are admissible, i.e., in the case of subshifts. In these cases Hausdorff dimensions of invariant sets can be expressed in terms of topological pressure. It was R. Bowen who introduced this quantity in the theory of dynamical systems [22, 10].

1.3. TOPOLOGICAL PRESSURE

Let us remind the definition for subshifts (the definition for arbitrary dynamical systems can be found in [19]).

Let Ω be a subshift, and ψ a real–valued continuous function on Ω. Let

$$Z_n(\psi, \Omega) = \sum_{|\underline{\omega}|=n} \exp\left(\sup_{\omega \in [\underline{\omega}]} \sum_{j=0}^{|\underline{\omega}|-1} \psi(\sigma^j \omega) \right), \tag{9}$$

where the sum is taken over all cylinders $[\underline{\omega}] \subset \Omega$ of length $|\underline{\omega}| = n$. It is proved in [27] that the limit

$$P_\Omega(\psi) = \lim_{n\to\infty} \frac{1}{n} \log Z_n(\psi, \Omega) \tag{10}$$

exists. The limit is called the topological pressure of the function ψ on Ω with respect to σ. It follows that if $\psi \equiv 0$ then $P_\Omega(0) = h_{\text{top}}(\sigma|\Omega)$, the topological entropy.

Roughly speaking, the system (σ, Ω) has $e^{h_{\text{top}} n}$ different paths of temporal length n (with some accuracy), each of them "costs" $\exp\left(\sum_{j=0}^{|\underline{\omega}|-1} \psi(\sigma^j \omega)\right)$ units, and $e^{nP_\Omega(\psi)}$ is the total price for passing through all of them.

It is known that topological pressure is independent of the metric (preserving a given topology) and is invariant under topological conjugacy [19].

Let us calculate the topological pressure in the case where $\Omega = \Omega_A$, the topological Markov chain with a $p \times p$ transition matrix A, and the function $\psi(\omega)$ depends only on the first symbol: $\psi(\omega) = \psi(\omega_0)$. In this case

$$Z_n(\psi, \Omega) = \sum_{(i_0,\ldots,i_{n-1})} \exp \sum_{j=0}^{n-1} \psi(i_j) \tag{11}$$

where the sum is taken over all Ω_A–admissible words $(i_0, \dots, i_{n-1})$. Set $\psi(i) = \log \rho_i$, $i = 0, \dots, p-1$, then

$$Z_n(\psi, \Omega_A) = \sum_{(i_0,\dots,i_{n-1})} \prod_{k=0}^{n-1} \rho_k. \tag{12}$$

It is not a difficult algebraic exercise to show that

$$Z_n(\psi, \omega_A) = RB^{n-1}E^T \tag{13}$$

where $R = (\rho_0, \dots, \rho_{p-1})$, $E = (1, \dots, 1)$ and

$$B = A \cdot \mathrm{diag}(\rho_0, \dots, \rho_{p-1}). \tag{14}$$

As a corollary of formula (14) we obtain that $P_\Omega(\psi) = \log \lambda_0$ where λ_0 is the spectral radius of the matrix B.

1.4. TOPOLOGICAL PRESSURE AND HAUSDORFF DIMENSION

Let us show now how the topological pressure is related to the Hausdorff dimension. Assume that a set F is modeled by a Moran construction and the corresponding subshift is a topological Markov chain (σ, Ω_A). Choose a cover of F by basic sets of the n-th generation. Then,

$$\begin{aligned} \sum_{(i_0,\dots,i_{n-1})} (\mathrm{diam}\Delta_{i_0,\dots,i_{n-1}})^\alpha &= \sum_{(i_0,\dots,i_{n-1})} \prod_{k=0}^{n-1} \lambda_{i_k}^\alpha \\ &= \sum_{(i_0,\dots,i_{n-1})} \exp(\alpha \sum_{j=0}^{n-1} \varphi(i_j)) \\ &= Z_n(\alpha\varphi, \Omega_A) \end{aligned} \tag{15}$$

where $\varphi(i_0, \dots,) = \log \lambda_{i_0}$. $Z_n(\alpha\varphi, \Omega_A) \approx \exp(nP_{\Omega_A}(\alpha\varphi))$. Hence, $Z_n(\alpha\varphi, \Omega_A) \gg 1$ if $P_{\Omega_A}(\alpha\varphi) > 0$ and $Z_n(\alpha\varphi, \Omega_A) \ll 1$ if $P_{\Omega_A}(\alpha\varphi) < 0$. It follows that if α_0 is the root of the (Bowen's) equation

$$P_{\Omega_A}(\alpha\varphi) = 0 \tag{16}$$

then $\dim_H F \leq \alpha_0$. The opposite inequality can be proven by using the technique of Moran covers and a dimension–like definition of topological pressure, see below.

1.4.1. *Dimension–Like Definition of Topological Pressure*
For a finite or a countable cover $\mathcal{C}$ of Ω by cylinders of lengths greater than n and $\beta \in \mathbb{R}$ let

$$\mathcal{Z}(\beta, \psi, \mathcal{C}, \Omega) = \sum_{[\underline{\omega}] \in \mathcal{C}} \exp\left(-\beta|\underline{\omega}| + \sup_{\omega \in [\underline{\omega}]} \sum_{j=0}^{|\underline{\omega}|-1} \psi(\sigma^j \omega) \right). \qquad (17)$$

It is proved in [22] that the topological pressure $P_\Omega(\psi)$ coincides with the threshold value

$$P_\Omega(\psi) = \sup \left\{ \beta : \quad \lim_{n\to\infty} (\inf\{\mathcal{Z}(\beta, \psi, \mathcal{C}, \Omega) : |\mathcal{C}| \geq n\}) = \infty \right\}. \qquad (18)$$

1.4.2. *Moran Covers*
We describe them in slightly different form than in [22]. Given an open ball $B \subset \mathbb{R}^m$, a basic set $\Delta_{i_0,\dots,i_{n-1}}$ is called B–related if $\Delta_{i_0,\dots,i_{n-1}} \cap B \neq \emptyset$, $\mathrm{diam}\Delta_{i_0,\dots,i_{n-2}} \geq \mathrm{diam}\, B$, but $\mathrm{diam}\Delta_{i_0,\dots,i_{n-1}} < \mathrm{diam}\, B$. Let $R(B)$ be the collection of all B–related basic sets. It is known that if $\mathrm{diam}|B| \ll 1$ then $\#R(B) \leq M$ where M is constant depending only on m. Therefore,

$$(\mathrm{diam}\, B)^\alpha \geq \frac{1}{M} \sum_{\Delta^j \in R(B)} (\mathrm{diam}\, \Delta^j)^\alpha \qquad (19)$$

for any nonnegative α. We consider now an arbitrary finite cover of F by balls B_i of diameters $\epsilon_i < \epsilon$, $i = 0, \dots, N-1$. Then, collection $R(B_i)$, $i = 0, \dots, N-1$, form a cover, say $\mathcal{C}$, of F which is called the Moran cover. Because of the inequality (19), we have

$$\sum_{i=0}^{N-1} \epsilon_i^\alpha \geq \frac{1}{M} \sum_{i=0}^{N-1} \sum_{\Delta_{i_0,\dots,i_k} \in R(B_i)} (\mathrm{diam}\, \Delta_{i_0,\dots,i_k})^\alpha, \qquad (20)$$

where the second sum is taken over all B_i–related basic sets. Given $\epsilon > 0$, there is $n = n(\epsilon)$ such that for any B_i–related basic set $\Delta_{i_0,\dots,i_k}$ we have $k > n(\epsilon)$. Moreover, $n(\epsilon) \to \infty$ as $\epsilon \to 0$. By using (15), we obtain that the right hand side of (20) is bounded from below by

$$\frac{1}{M} \sum_{(i_0,\dots,i_{n-1})} \prod_{k=0}^{n-1} \lambda_{i_k}^\alpha = \frac{1}{M} \sum_{(i_0,\dots,i_{n-1})} \exp(\alpha \sum_{k=0}^{n-1} \log \lambda_{i_k}) \qquad (21)$$

where the sum is taken over all words $(i_0, \dots, i_{n-1})$ corresponding to B_i–related basic sets, for all i.

Let us rewrite the statistic sum (17) for this particular case:

$$\mathcal{Z}(\beta, \alpha \log \lambda_{i_k}, \mathcal{C}, \Omega) = \sum_{\Delta_{\underline{\omega}} \in \mathcal{C}} \exp\left(-\beta|\underline{\omega}| + \alpha \sum_{k=0}^{|\underline{\omega}|-1} \log \lambda_{i_k} \right). \tag{22}$$

Assume that $\alpha < \beta_c = P_\Omega(\alpha \log \lambda_0)$, then, for any $K > 0$, there is $n_0 = n_0(K)$ such that $\mathcal{Z}(\beta_c, \alpha \log \lambda_{i_k}, \mathcal{C}, \Omega) > K$ provided that $n(\epsilon) > n_0$. It follows that

$$\sum_{\Delta_{\underline{\omega}} \in \mathcal{C}} \exp\,(\alpha \sum_{k=0}^{|\underline{\omega}|-1} \log \lambda_{i_k}) \geq K e^{\beta_c n(\epsilon)} \tag{23}$$

i.e,

$$\sum_{i=0}^{N-1} \epsilon_i^\alpha \geq \frac{K}{M} e^{\beta_c n(\epsilon)}. \tag{24}$$

This implies that $\dim_H F \geq P_\Omega(\alpha \log \lambda_0)$. The opposite inequality has been already obtained. Thus, we proved that $\dim_H F = \alpha_0$, the root of the Bowen equation (16).

1.4.3. *Examples*

Example 1.2 Let us come back to Example 1.1. In this case $A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$, $0 < \lambda_{0,1} < 1$ are rates of contraction, $\psi(i_0, i_1, \ldots) = \alpha \log \lambda_{i_0} = \alpha\varphi(i_0, i_1, \ldots)$. Thus, $\rho_i = \lambda_i^\alpha$, $i = 0,\, 1$, $B = \begin{pmatrix} \lambda_0^\alpha & \lambda_1^\alpha \\ \lambda_0^\alpha & \lambda_1^\alpha \end{pmatrix}$ and $P_{\Omega_2}(\alpha\varphi) = \log\,(\lambda_0^\alpha + \lambda_1^\alpha)$. The Bowen's equation (16) becomes the Moran's equations (7).

Example 1.3 Consider now the "golden mean" topological Markov chain with the transition matrix $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$. Assume that $0 < \lambda_{0,1} < 1$ are rates of contraction. Here again $\rho_i = \lambda_i^\alpha$ but the matrix B has the form $B = \begin{pmatrix} \lambda_0^\alpha & \lambda_1^\alpha \\ \lambda_0^\alpha & 0 \end{pmatrix}$. The characteristic equation of matrix B is $\mu^2 - \mu\lambda_0^\alpha - (\lambda_0\lambda_1)^\alpha = 0$ and spectral radius is $r = (1/2)(\lambda_0^\alpha + \sqrt{\lambda_0^{2\alpha} + 4(\lambda_0\lambda_1)^\alpha})$. Thus, the Hausdorff dimension of the corresponding set F is the root of the Bowen's equation $\log\,((1/2)(\lambda_0^\alpha + \sqrt{\lambda_0^{2\alpha} + 4(\lambda_0\lambda_1)^\alpha})) = 0$. If $\lambda_0 = \lambda_1 = \lambda$ then the equation becomes $\alpha \log \lambda + \log\,((1+\sqrt{5})/2)$, i.e., $\dim_H F = \alpha_0 = \log\,((1+\sqrt{5})/2)/-\log\lambda$. If you take into account that $\log\,((1+\sqrt{5})/2) = h_{\text{top}}$, the topological entropy of the topological Markov chain (σ, Ω_A), then we obtain the relation ([17])

$$\dim_H F = \frac{h_{\text{top}}}{-\log \lambda}.$$

1.4.4. *General Subshifts*
It was shown in [22, 23] that the Bowen's equation (16) holds not only for topological Markov chains and not only for finitely many values of rates of contraction. Consider $\Omega \subset \Omega_p$ an arbitrary subshift with positive topological entropy and let $\lambda : \Omega \to \mathbb{R}^+$ be an arbitrary Hölder continuous positive function, such that $\lambda(\omega) < 1$ for any $\omega \in \Omega$. We may replace the Moran axiom (M4) by the following assumptions: there are positive constants $\underline{c}$ and $\overline{c}$ such that

$$\operatorname{diam} \Delta_{i_0,\dots,i_{n-1}} \geq \underline{c} \inf_{\omega \in [\underline{\omega}]} \prod_{j=0}^{|\underline{\omega}|-1} \lambda(\sigma^j \omega) \tag{25}$$

$$\operatorname{diam} \Delta_{i_0,\dots,i_{n-1}} \leq \overline{c} \sup_{\omega \in [\underline{\omega}]} \prod_{j=0}^{|\underline{\omega}|-1} \lambda(\sigma^j \omega) \tag{26}$$

It was shown in [22, 23] that the $\dim_H F = \alpha_c$, where α_c is the root of the Bowen's equation $P_\Omega(\alpha \log \lambda) = 0$. Similar formula was obtained for conformal repellers [24] and in many other situations [22].

1.5. STICKY SETS RESULTING FROM GEOMETRIC CONSTRUCTIONS

An area preserving map f of the plane, possessing an infinite hierarchy of islands–around–islands structure, has invariant sets of zero Lebesgue measure on which it behaves similarly to multipermutative systems [7, 1].

Let $\Omega_p = \{0, 1, \dots, p-1\}^{\mathbb{N}_0}$ with the usual metric. The elements in Ω_p will be denoted here by ω.

1.5.1. *Multipermutative Systems*
A map $T : \Omega_p \to \Omega_p$ is said to be *multipermutative* if for every $\omega \in \Omega_p$ the sequence $T\omega$ is given by

$$T\omega = (\omega_0 + p_0,\ \omega_1 + p_1(\omega_0),\ \dots,\ \omega_i + p_i(\omega_0, \dots, \omega_{i-1}),\ \dots)$$

with $p_i : A^i \to A$ for $i > 0$ and $p_0 \in A = \{0,\ \dots,\ p-1\}$. At every coordinate the addition is understood to be modulo p.

Example 1.4 The *p–adic adding machine* is a multipermutative system (Ω_p, S) such that

$$S\omega = (\omega_0 + 1,\ \omega_1 + s_1(\omega_0),\ \dots,\ \omega_i + s_i(\omega_0, \dots, \omega_{i-1}),\ \dots),$$

with $s_i(\omega_0, \ldots, \omega_{i-1}) = 1$ if $(\omega_0, \ldots, \omega_{i-1})$ is maximal and $s_i(\omega_0, \ldots, \omega_{i-1}) = 0$ otherwise. The word $(\omega_0, \ldots, \omega_{i-1})$ is maximal when $\omega_j = p-1$ for $j = 0, \ldots, i-1$.

The p–adic adding machine is a minimal system, and, as it was shown in [1], every minimal multipermutative system defined above is topologically conjugate to the p–adic adding machine. The map T is not chaotic and its topological entropy is zero. A set F on which f is topologically conjugate to T, nevertheless, may appear as a result of a Moran type geometric construction.

1.5.2. *Sticky Sets* [1, 7]

Sticky sets are the sets of all limiting points of an infinite hierarchy of islands.

A closed topological disk P is said to be an island of stability if $f^n(P) = P$ for some integer n. We now give a definition of an infinite hierarchy of islands–around–islands structure (sticky riddle) for the general case when not all words $\underline{i} = (i_0, \ldots, i_{n-1})$ might be admissible.

A collection $\mathcal{P}$ of islands $\{P_{\underline{i}} : \underline{i}$ is Ω–admissible$\}$ is said to be a sticky riddle if the sets $P_{\underline{i}}$ are pairwise disjoint, are contained in a compact set, and

(i) for any island $P_{\underline{i}} \in \mathcal{P}$ there is an island $P_{\underline{j}} \in \mathcal{P}$, $|\underline{i}| = |\underline{j}|$, such that $f(P_{\underline{i}}) = P_{\underline{j}}$;

(ii) if $f(P_{\underline{i}}) = P_{\underline{j}}$ then for any admissible $\underline{i}k$ there is $s \in \{0, 1, \ldots, q-1\}$ such that $f(P_{\underline{i}k}) = P_{\underline{j}s}$;

(iii) $\mathrm{diam}(P_{\underline{i}}) \to 0$ as $|\underline{i}| \to \infty$;

(iv) for any $\omega = (i_0, i_1, \ldots) \in \Omega$, if $x_n \in P_{i_0,\ldots,i_{n-1}}$, $n > 0$, then $\lim_{n\to\infty} x_n$ exists;

(v) if $x_n \in P_{\underline{i}}$, $y_n \in P_{\underline{j}}$, $|\underline{i}| = |\underline{j}| = n$, $n > 0$, and $\underline{i} \neq \underline{j}$ at least for one value of n then $\lim_{n\to\infty} x_n \neq \lim_{n\to\infty} y_n$.

These axioms reflect our understanding of an infinite islands–around–islands hierarchy:

(i) an island of the n–th generation is mapped into an island of the same generation;

(ii) if an island $P_{\underline{i}k}$ lies in the vicinity of the island $P_{\underline{i}}$ then its image $P_{\underline{j}s}$ lies in a vicinity of $P_{\underline{j}}$;

(iii) to be packed into a compact set, the islands of the n–th generation should be small if $n >> 1$;

(iv) there should be only one point of accumulation of islands $P_{i_0,\ldots,i_{n-1}}$ for any fixed $\omega = (i_0, \ldots, i_{n-1}, \ldots)$;

(v) for different points $\omega = (\omega_0, \omega_1, \ldots)$, $\omega' = (\omega'_0, \omega'_1, \ldots)$ in Ω the corresponding points of accumulation of islands should be different.

Let $\mathcal{P}$ be a sticky riddle. For any $\omega = (i_0, i_1, \ldots) \in \Omega$ and any sequence $x_n \in P_{i_0,\ldots,i_{n-1}}$, define $x = x(\omega) := \lim_{n\to\infty} x_n$. The set $\Lambda = \{x(\omega) : \omega \in \Omega\}$ is said to be a sticky set. It is well defined thanks to Axioms (iii)–(v).

It was shown in [1] that $f|\Lambda$ is topologically conjugate to a multipermutative system, i.e., $f|\Lambda$ has zero topological entropy.

1.5.3. *Geometric Constructions for Sticky Sets*

Some numerical observations [9] show that sometimes every island of stability $P_{\underline{i}}$, together with all its satellites $P_{\underline{ij}}$, belongs to a basic set $\Delta_{\underline{i}}$ of a geometric construction. So, the set Λ can be resulted from this construction. Axiomatically, the conditions for that can be expressed as follows.

(P1) There exists a collection of sets $\{\Delta_{\underline{i}} : \underline{i} \text{ is admissible}\}$ that are closed, and for each admissible word $\underline{i}$, $P_{\underline{ij}} \subset \Delta_{\underline{i}}$ for every admissible word $\underline{ij}$.

(P2) $P_{\underline{i}} \cap \Delta_{\underline{ij}} = \emptyset$ for every admissible $\underline{i}$ and $\underline{ij}$.

(P3) $\Delta_{\underline{ij}} \subset \Delta_{\underline{i}}$, for every admissible words $\underline{i}$ and $\underline{ij}$.

(P4) $\mathrm{diam}\Delta_{i_0\ldots i_{n-1}} \to 0$ as $n \to \infty$.

(P5) Separation axiom. $\Delta_{\underline{i}} \cap \Delta_{\underline{j}} \cap F = \emptyset$ if $\underline{i} \neq \underline{j}$, $|\underline{i}| = |\underline{j}|$, where

$$F = \bigcap_{n=1}^{\infty} \bigcup_{\substack{i_0,\ldots,i_{n-1} \\ \text{is admissible}}} \Delta_{i_0\ldots i_{n-1}}$$

Thus, if these axioms are satisfied, then $\Lambda = F$. Let us emphasize that an invariant set with nonchaotic dynamics is resulted from a geometric construction, modeled by a full subshift (σ, Ω_p) or a subshift with positive topological entropy. In other words, we have a "contradiction" between temporal and spatial behavior of a system. To describe such a situation, we need characteristics which could take into account both temporal and spatial behavior. We introduce them in the next Lecture.

1.6. PROBLEMS

It is well–known that nonuniformity of hyperbolicity of invariant sets causes a lot of troubles in the study of behavior of orbits. We believe that the problems below reflect some of these difficulties by projecting them to field of dimension theory.

1.6.1. *Geometric Constructions Modeled by Topological Markov Chains with some Contractions Rates* $\lambda_i = 1$

The problem is to find necessary and sufficient conditions for the validity of the following statement: the root of Bowen's equation

$$P_{\Omega_A}(\alpha \log \lambda_{i_0}) = 0 \tag{27}$$

is equal to $\dim_H F$, where F is resulted from a Moran–type geometric construction, modeled by the topological Markov chain (σ, Ω_A), with contraction rates λ_0, λ_1, $\ldots \lambda_{p-1}$, in the case when $\lambda_i = 1$ for some i's.

1.6.2. *Moran Geometric Constructions with Nonuniform Contraction.* In the situation of subsection 1.4.4, provided that $\lambda(\omega) = 1$ for some $\omega \in \Omega$, find conditions under which the root of the Bowen's equation (16) is equal to the Hausdorff dimension of the set F (the article [18] could be helpful).

2. Generalized Carathéodory construction and spectra of dimensions for Poincaré recurrences

The examples of sticky sets in the previous lecture and the construction for the Feigenbaum attractor below show us that, in general, we should apply a wider notion than the Hausdorff dimension to describe simultaneously behavior of orbits on invariant sets and their geometric origination. The generalized Carathéodory construction allows us to do it.

2.1. CARATHÉODORY–PESIN CONSTRUCTION [22]

We describe here a general approach developed by Ya. Pesin on the basis of classical Carathéodory results. We describe it not in full generality but in sufficient details for our purpose.

Assume that X is a metric space with a distance ρ, and $\mathcal{F}$ is a collection of subsets of X such that for any $Z \subset X$ and every $\epsilon > 0$ there is a finite or countable subcollection $\{U_i\}$ of $\mathcal{F}$ with $\psi(U_i) \le \epsilon$ covering Z. Here $\psi : \mathcal{F} \to \mathbb{R}^+$ is a nonnegative function such that $\psi(U) = 0$ iff $U = \emptyset$. In a standard example $\mathcal{F}$ is the collection all open sets (or open balls) and $\psi(U) = \operatorname{diam} U$.

Consider functions ξ, $\eta : \mathcal{F} \to \mathbb{R}^+$ such that $\eta(U) = 0$ iff $U = \emptyset$. We assume also that for any $\delta > 0$ one can find $\epsilon > 0$ such that $\eta(U) \le \delta$ for any $U \in \mathcal{F}$ with $\psi(U) \le \epsilon$.

The quadruple $(\mathcal{F}, \psi, \xi, \eta)$ is said to be a Carathéodory structure.

Given $Z \subset X$, let us consider a finite or countable cover $G = \{U_i\}$ of Z by elements of $\mathcal{F}$, with $\psi(U_i) \le \epsilon$. Then, introduce the sum

$$M_\xi(\alpha, \epsilon, G, Z) = \sum_i \xi(U_i)\eta(U_i)^\alpha$$

and consider its minimum

$$M_\xi(\alpha, \epsilon, Z) = \inf_G \sum_i \xi(U_i)\eta(U_i)^\alpha, \tag{28}$$