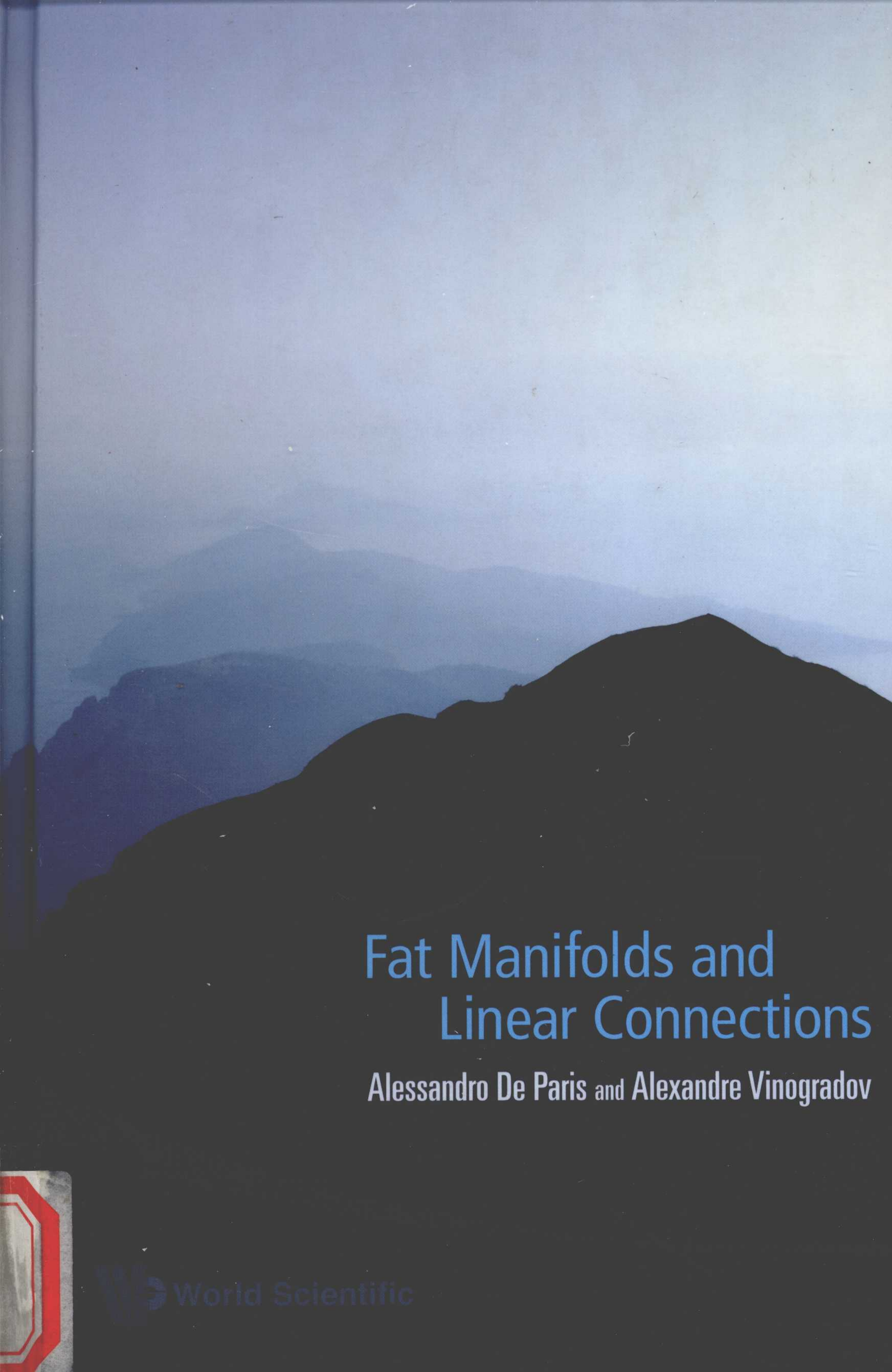

Fat Manifolds and Linear Connections

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Picture of Sorrentine Peninsula taken by Vincenzo Moreno from mount Molare on June 25, 2008

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Preface

In the winter 1999-2000 the second author held a series of lectures on linear connections and gauge transformations. Notes of these lectures taken by G. Manno, F. Pugliese, L. Vitagliano and the first author constituted that raw material on the basis of which he wrote the first version of this text. This version was then substantially revised and new material added. Various parts of these notes were circulated among 'diffiety people' while we were working on them, and the feed back contributed to the final version. Luca Vitagliano and our friend and colleague G. Rotondaro, who tragically left us in October 2004, should be especially mentioned in this connection.

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Foreword

The idea of parallel transport along a path in a Riemannian manifold gave birth to the concept of a linear connection on M at the end of 19th century. Subsequently, it was extended to arbitrary vector bundles and much later, at the time of the Second War, to general bundles. According to the now standard approach, which is mainly due to Ch. Ehresmann, a connection in a fiber bundle is just a distribution of 'horizontal planes' on its total space. Duly specified to various types of fiber bundles this approach leads to connections of a particular interest, such as affine or linear. Geometrical clarity and apparent simplicity is an important advantage of Ehresmann's approach, which, unfortunately, is well balanced by a not negligible disadvantage. Namely, it gives almost no constructive indications on the operative machinery to work with. In particular, it does not allow an immediate natural extension of the theory to some recently emerged situations of a noteworthy importance such as supermanifolds (graded commutative algebras) or secondary calculus (see [Vinogradov (2001)]). Indeed, it would be hardly possible even to imagine what is a secondary ('quantized') connection in terms of a distribution of horizontal planes. Moreover, in field theory one deals directly with fields which may be, or not be interpreted as sections of a vector bundle but not with the bundle as such. So, in this context a connection *must* be defined as a construction which is pertinent to the fields 'in person'. This kind considerations and the fact that differential calculus is, in reality, an aspect of commutative algebra (see [Nestruev]) plainly indicate that a natural framework for the theory of linear connections is differential calculus in the category of modules over a (graded) commutative ground algebra. This point of view combines naturally with the idea to treat a vector bundle as a 'fat' manifold composed of 'fat' points that are its fibers. By using the term 'fat point' we refer to an object possessing an 'inner structure' whose constituents, nevertheless, cannot be directly observed, *i.e.*, something like an elementary particle. In the theory of gauge fields one deals, as a matter of fact, with fat points. In this context unobservability of the constituents is formalized by means of a suitable symmetry group that produce the necessary inseparable mixture.

These and other similar considerations leads to suppose existence of a 'fat' analogue of differential calculus on a fat manifold well adopted to treat various questions concerning a given vector bundle(s) and, in particular, connections in it. Such an analogue positively exists and the gauge freedom is an inherent feature of it. On the other hand, connections in the context of this 'fat' calculus play the role of a mechanism naturally effecting interrelations among fat points.

In these notes we present some basic elements of the fat calculus and then, on its basis, develop the theory of linear connections. In a sense this text may be viewed as a translation of the classical theory of linear connections in smooth vector bundles into its native language. An extension of the domain of the theory of linear connections much beyond its traditional differential geometry frames is one of results of this translation. For instance, this way one discovers that families of vector spaces over a smooth manifold different from vector bundles can also possess connections as well as vector bundles over manifolds with singularities. Another advantage of this new language is that it simplifies noteworthy working techniques and manipulations with connections by offering simple algebraic computations as a substitute for non infrequently ponderous geometrical constructions. In addition, it makes much easier to perceive more delicate aspects of the theory. An instance of that is the notion of compatibility of two connections along a morphism of vector bundles, introduced and studied in these notes for the first time.

These notes are structured along the following lines. The introductive zeroth chapter contains an algebraic interpretation of some basic facts of differential calculus on smooth manifolds that are brought to the form allowing a direct 'fat' generalization. Materials gathered in this chapter make the subsequent exposition self-contained and accessible for graduate students.

Fat manifolds and first elements of 'fat calculus' are introduced and discussed in the 1-st chapter. A fat manifold is simply a pair composed of a smooth manifold and a vector bundle on it. This notion, synonymous by itself to that of vector bundle, acquires, nevertheless, a new meaning in the context of fat calculus. This subtle but important difference is similar to that between 'just a particle' and a charged particle. A general algebraic counterpart of fat manifolds is a pair composed of a commutative algebra and a module over it. A good deal of fat calculus can be developed in this algebraic context and we do that as much as possible. In the 1-st chapter we discuss only simplest elements of fat calculus such as fat tangent vectors, fat vector fields, *etc.*, simultaneously, with their algebraic counterparts. Other fat notions are introduced as required in the course the exposition.

A fat manifold may be viewed as the result of a 'thickening' of the underlying ordinary manifold, say, M. A natural question is whether this thickening can be extended to other geometrical structures on M. In particular, the problem of a simultaneous thickening of vector fields on M leads to discover the notion of a linear connection in the corresponding vector bundle. In chapter 2 the theory of linear connections is build on the basis of this idea. The main tools in doing that are fat differential calculus on M and its algebraic counterpart. Among other things, here we construct some exotic examples of connections already mentioned above and describe basic operations of linear algebra with connections.

More fine elements of the theory of connections are developed in the 3-rd chapter. Covariant differential, duly interpreted, is the conceptual center of our exposition here. In particular, we show that a connection can be understood as a *cd-module structure* in the graded algebra of *thickened differential forms*. This fact makes possible to introduce the concept of *compatibility* of two connections and the concept of a *connection along a fat map*. From one side, this enriches the standard theory of connections with morphisms and relative objects and, from the other side, allows to develop a more satisfactory theory of the covariant Lie derivation.

The covariant differential of a flat connection transforms the algebra of thickened form into a complex. This kind of cohomology is studied at the beginning of the concluding 4-th chapter. The main result here is the fat homotopy formula, which is surprisingly valid even for cd-modules. As a curiosity we show that the parallel translation along a curve is described naturally by the 'fat Newton-Leibniz formula'.

A cd-module associated with a connection is not, generally, a complex. Nevertheless, there are naturally related with it differential complexes furnishing connections with cohomological invariants. We interpret Maxwell's equations as dynamics of gauge equivalence classes of connections over the fat Minkowski space-time in order to illustrate importance of this aspect in the theory of connections. The theory of characteristic classes of gauge structures is the final accord of these notes. Indeed, many elements of the previously developed theory are here shown in a common action.

Linear connections appear naturally in many areas of mathematics by starting from abstract algebra and up to mathematical physics. A finite separable extension of an algebraic field is supplied canonically with a flat connection. This elementary fact is easily seen from the point of view presented in these notes. On the other hand, the cohomology of the associated with this connection de Rham like complex is an invariant of the extension and a natural question is what are these and, in particular, how to compute them. Some hints about one can extract from differential geometry where flat connection cohomology appears as the de Rham cohomology with 'twisted coefficients'. Moreover, this kind cohomology appears in some situations in (physical) field theory, *etc.* This simple example illustrates why a unified point of view on connections could be of interest and our hopes are that these notes would be useful as a reference point to the subject.

Contents

Chapter 0

Elements of Differential Calculus over Commutative Algebras

In this chapter all necessary notions and facts forming the starting point of the further exposition are collected. First of all this is done in order to make this book self-contained modulo 'undergraduate' mathematics. The suggested reference textbooks are [Singer and Thorpe (1976)] and [Mac Lane and Birkhoff (1967)]. On the other hand, we present some standard elementary topics in a different perspective which better fits our goals. The book [Nestruev (2003)] is highly recommended to the reader who is interested in better understanding the origin and motivation of the algebraic approach to Differential Calculus we follow in this book. Basically, terms that are not explicitly defined here are tacitly assumed to be borrowed from the aforementioned books (in reverse order of priority).

0.1 Algebraic Tools

In this section the needed algebraic terminology is set up. The degree of generality is tuned in view of applications in the subsequent exposition.

0.1.1 *General Conventions*

All *rings* are assumed to posses the identity element 1 (but not all rings will be commutative); all ring homomorphisms are assumed to preserve the identity element. A *k-algebra* is not necessarily commutative, but the base ring k is always assumed to be commutative. Nevertheless, most of the algebras considered here will be commutative. In particular, in most cases there will be no distinctions between left and right modules. When the distinction takes place, 'module' stands for 'left module'. The dual module $\mathrm{Hom}\,(P, A)$ of an A-module P will be denoted by $P^\vee$. We say that a projective and finitely generated A-module P has *constant rank* r if for all maximal ideals $\mathfrak{m}$ of A the dimension of the $A/\mathfrak{m}$–vector space $P/\mathfrak{m}P$ is r.

There will be generally no a priori choices for universal constructions. For instance, 'direct sum' and 'coproduct' in thus book are synonymous. When a direct sum, tensor product, extension of scalars, *etc.*, is invoked, the reader may fix any object that satisfies the appropriate universal property, unless a particular choice is explicitly indicated.

As usual, *graded algebras* and *graded modules* will be 'internally graded', that is, direct sums of their components (*cf.* [Mac Lane and Birkhoff (1967), Chap. XVI, Appendix to Sect. 3 (p. 546)]). The index set will always be $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, homogeneous components will be denoted by subscripts and a component with a negative subscript will be zero by convention. If $\mathcal{A}$ is a graded k-algebra and $\mathcal{P}$ is a graded k-module equipped with a 'k-compatible' $\mathcal{A}$-module structure, then $\mathcal{P}$ will be called a *graded $\mathcal{A}$-module*, provided that ([1])

$$a_r p_s \in \mathcal{P}_{r+s}, \quad a_r \in \mathcal{A}_r, p_s \in \mathcal{P}_s \ .$$

A graded k-algebra $\mathcal{A}$ will be called *commutative* if it is commutative as a ring; it will be called *graded commutative* ([2]) if

$$a_r a'_s = (-1)^{rs} a'_s a_r, \quad a_r \in \mathcal{A}_r, a'_s \in \mathcal{A}_s$$

A homomorphism

$$\varphi : \mathcal{P} \to \mathcal{Q}$$

of graded k-modules will be called a *graded homomorphism of n-th degree* ($n \in \mathbb{Z}$) if for all s,

$$p_s \in \mathcal{P}_s \Longrightarrow \varphi(p_s) \in \mathcal{Q}_{s+n} \ .$$

When $\mathcal{P}$ and $\mathcal{Q}$ are graded $\mathcal{A}$-modules, with $\mathcal{A}$ being a graded commutative k-algebra, φ is a *graded homomorphism of $\mathcal{A}$-modules* (of n-th degree), if, in addition,

$$\varphi(a_r p_s) = (-1)^{rn} a_r \varphi(p_s), \quad a_r \in \mathcal{A}_r, p_s \in \mathcal{P}_s \ .$$

If $\mathcal{P}$ and $\mathcal{Q}$ are itself graded k-algebras, φ will be a *graded algebra homomorphism* if it is both a ring homomorphism and a zeroth degree homomorphism of graded k-modules.

---

[1]As usual, the components of the direct sums (=coproducts) $\mathcal{A}$ and $\mathcal{P}$ are identified here with their images through the natural monomorphisms $\mathcal{A}_r \hookrightarrow \mathcal{A}$ and $\mathcal{P}_s \hookrightarrow \mathcal{P}$.

[2]The definition of a commutative graded algebra given in [Mac Lane and Birkhoff (1967), Chap. XVI, Sect. 4 (p. 551)] corresponds to the present definition of a *graded commutative* algebra.

In this book, a *cochain complex* (or, for short, *complex*) is a graded module $\mathcal{P}$ together with a first degree homomorphism $\mathrm{d} : \mathcal{P} \to \mathcal{P}$ such that $\mathrm{d} \circ \mathrm{d} = 0$, which is called *differential.* If $(\mathcal{P}, \mathrm{d})$ and $(\mathcal{P}', \mathrm{d}')$ are complexes, a zeroth degree homomorphism $\varphi : \mathcal{P} \to \mathcal{P}'$ will be called a *homomorphism of complexes* or a *cochain homomorphism*, if it commutes with the differentials, *i.e.*, $\varphi \circ \mathrm{d} = \mathrm{d}' \circ \varphi$.

Commutators will always be understood in the sense of ring theory, *i.e.*, $[a, b] = ab - ba$. If $\mathcal{A}$ is a graded k-algebra then the *graded commutator of* $a_r \in \mathcal{A}_r$ *and* $a'_s \in \mathcal{A}_s$ will be

$$[a_r, a'_s]^{(\mathrm{gr})} \stackrel{\mathrm{def}}{=} a_r a'_s - (-1)^{rs} a'_s a_r \,.$$

Similarly, if $\mathcal{P}$ is a graded k-module and φ_r, ψ_s are graded endomorphisms of $\mathcal{P}$ of r-th and s-th degree, respectively, then the graded commutator of φ_r and ψ_s is the graded endomorphism $[\varphi_r, \psi_s]^{(\mathrm{gr})} = \varphi_r \circ \psi_s - (-1)^{rs} \psi_s \circ \varphi_r$.

0.1.2 *Differential Operators*

Let A be a commutative k-algebra, with k being a field, and P, Q modules over A. If $a \in A$ and $\Delta : P \to Q$ is a k-homomorphism, the commutator

$$[\Delta, a] : P \to Q$$

makes sense provided that a is identified with the multiplication by a operators in P and Q, respectively. Define inductively

$$\mathrm{Diff}_0(P, Q) \stackrel{\mathrm{def}}{=} \mathrm{Hom}_A(P, Q) = \{\Delta : [\Delta, a] = 0 \ \forall a \in A\},$$

$$\mathrm{Diff}_n(P, Q) \stackrel{\mathrm{def}}{=} \{\Delta : [\Delta, a] \in \mathrm{Diff}_{n-1}(P, Q) \ \forall a \in A\},$$

$$\mathrm{Diff}(P, Q) \stackrel{\mathrm{def}}{=} \bigcup_n \mathrm{Diff}_n(P, Q) \,.$$

Equivalently, $\Delta \in \mathrm{Diff}_n(P, Q)$ if and only if

$$[\ldots [[\Delta, a_0], a_1], \ldots, a_n] = 0, \qquad \forall a_0, a_1, \ldots, a_n \in A \,.$$

These sets admit two natural A-module structures

$$a\Delta \stackrel{\mathrm{def}}{=} a \circ \Delta, \quad a^+\Delta \stackrel{\mathrm{def}}{=} \Delta \circ a \,.$$

The notation $\mathrm{Diff}(P, Q)$ usually refers to the first one, while $\mathrm{Diff}^+(P, Q)$ is used for the second, and $\mathrm{Diff}^{(+)}(P, Q)$ is used to denote the bimodule. Elements of these modules are called *linear differential operators from* P *to* Q. The interested reader is referred to [Nestruev (2003), 9.66, 9.67] for more details.

0.1.3 *Derivations*

Let A be a commutative k-algebra and P an A-module. A *derivation of A into P* is a linear over k function

$$\Delta : A \to P$$

that fulfills the *Leibniz rule*

$$\Delta(ab) = a\Delta(b) + b\Delta(a), \quad a, b \in A \,.$$

Such a function is sometimes also called *k-derivation* (this may be useful when more than one algebra structure on the same ring are under consideration). The set of all derivations of A into P, equipped with the natural A-module structure

$$(a\Delta)(a') \overset{\text{def}}{=} a\left(\Delta(a')\right), \quad a, a' \in A \,,$$

will be denoted by $\mathrm{D}(P)$, or sometimes by $\mathrm{D}_k(P)$. In particular, $\mathrm{D}(A)$ is the A-module of all derivations of A into itself (often shortly called 'derivations of A'). Take notice that $\mathrm{D}(A)$ is not, generally, a subring of $\mathrm{End}_k(A)$ (with the operation of function composition). However, it is easily checked that the commutator of elements of $\mathrm{D}(A)$ lies again in $\mathrm{D}(A)$ (see, *e.g.*, [Nestruev (2003), 9.53]).

If $\varphi : A \to B$ is a homomorphism of commutative k-algebras, a *derivation along φ* will be a derivation $A \to B$ with B considered as an A-module via φ. The set of all derivations along φ, equipped with the natural B-module structure

$$(b\Delta)(a) \overset{\text{def}}{=} b\left(\Delta(a)\right), \quad a \in A, b \in B \,,$$

will be denoted by $\mathrm{D}(A)_\varphi$.

Let $\mathcal{A}$ be a graded commutative algebra. An n-th degree graded module endomorphism $\Delta : \mathcal{A} \to \mathcal{A}$ is called a *graded derivation of $\mathcal{A}$* (into itself) if it fulfills the following *graded Leibniz rule*:

$$\Delta(a_s a') = \Delta(a_s)a' + (-1)^{ns} a_s \Delta(a'), \quad a_s \in \mathcal{A}_s, a' \in \mathcal{A} \,,$$

for all s.

0.1.4 *Additive Functions on Tensor Products*

Let A be a commutative ring. As usual, an A-module homomorphism

$$P \otimes Q \to R$$

will often be determined by means of an assignment such as

$$p \otimes q \mapsto b(p, q) ,$$

provided that the expression $b(p, q)$ is A-bilinear. Occasionally in this book, there will be needed functions $P \otimes Q \to R$ that are not A-module homomorphisms. To recognize if an assignment

$$p \otimes q \mapsto f(p, q) ,$$

gives a well-defined additive function, it suffices to check that $f : P \times Q \to R$ is biadditive and satisfies

$$f(ap, q) = f(p, aq), \quad a \in A, p \in P, q \in Q$$

(see, *e.g.*, [Hilton and Stammbach (1971), Chap. III, Theorem 7.2]).

0.1.5 *Some Basic Facts*

Let $\varphi : A \to B$ be a homomorphism of commutative rings, P, $P_1, \ldots, P_n$ modules over A, Q a module over B, Q_A the A-module obtained from Q by restriction of scalars, P_B, $P_{1B}, \ldots, P_{nB}$ the B-modules obtained from P, $P_1, \ldots, P_n$ by extension of scalars, and $\nu : P \to P_B$, $\nu_1 : P_1 \to P_{1B}$, ..., $\nu_n : P_n \to P_{nB}$ the universal homomorphisms. In the sequel the following simple facts are supposed to be known.

(1) If P is projective, then P_B is projective (see, *e.g.*, [Nestruev (2003), 11.52]).
(2) For every multilinear function of A-modules

$$b : P_1 \times \cdots \times P_n \to Q_A ,$$

there is exactly one multilinear function of B-modules

$$\bar{b} : P_{1B} \times \cdots \times P_{nB} \to Q$$

such that

$$b = \bar{b} \circ (\nu_1 \times \cdots \times \nu_n) .$$

(3) In the above situation, if $P_1 = \cdots = P_n$ and b is alternating or symmetric, then $\bar{b}$ is, respectively, alternating or symmetric.
(4) There exists exactly one graded A-homomorphism between (fixed) exterior algebras

$$\bigwedge^{\bullet} Q_A \to \bigwedge^{\bullet} Q$$

such that the first degree component is the identity map of Q ([3]).

[3]As usual, the first degree components of tensor, symmetric and exterior algebras of a module are supposed to be identified with the module itself. We use the symbol $\bigwedge^{\bullet}$ for exterior algebras.

(5) The graded algebra obtained from $\bigwedge^\bullet P$ by extension of scalars is an exterior algebra of P_B:

$$B \otimes_A \left(\bigwedge^\bullet P\right) = \bigwedge^\bullet (B \otimes_A P)$$

(it follows form (3)).

(6) If P is projective and finitely generated then the natural homomorphism $P \to P^{\vee\vee}$ is an isomorphism ([4]).

(7) If either P or P_1 is projective and finitely generated then the natural homomorphism

$$P^\vee \otimes P_1 \to \mathrm{Hom}\,(P, P_1)$$

is an isomorphism.

(8) If P is projective and finitely generated, then ${P_B}^\vee$ is a module obtained from $P^\vee$ by extension of scalars via φ, where the universal homomorphism

$$\mu : P^\vee \to {P_B}^\vee$$

is determined by

$$\mu\,(\alpha)\,(\nu(p)) = \varphi\,(\alpha(p))\,, \qquad p \in P, \alpha \in P^\vee$$

(it follows from (7)).

(9) More generally, if P is projective and finitely generated, then $\mathrm{Hom}_B\,(P_B, {P_1}_B)$ is a module obtained from $\mathrm{Hom}\,(P, P_1)$ by extension of scalars via φ.

(10) If P is projective, finitely generated and of constant rank 1 then all its endomorphisms are multiplication by scalars operators ([5]).

(11) There exists a natural decomposition

$$\bigwedge^\bullet (P \oplus P_1) = \bigwedge^\bullet P \otimes \bigwedge^\bullet P_1$$

(see [Bourbaki (1989), Chap. III, Sect. 7.7]).

(12) If P is projective and finitely generated, then $\bigwedge^n P$ is projective and finitely generated for all $n \in \mathbb{N}_0$ (it follows from (11)).

---

[4]This result and the following (7) easily follow from the fact that the natural homomorphisms involved are compatible with finite direct sums.

[5]It follows from (9) and Nakayama's Lemma (see, *e.g.*, [Atiyah and Macdonald (1969), Proposition 2.6]; take also into account [Atiyah and Macdonald (1969), Chap. 2, Exercises, n. 10 (p. 32) and Proposition 3.9]).

0.1.6 *Equivalence of Categories*

A functor $\mathcal{E} : \mathfrak{A} \to \mathfrak{C}$ is called an *equivalence of categories* if there exists a functor $\mathcal{F} : \mathfrak{C} \to \mathfrak{A}$ and natural isomorphisms $\eta : \mathrm{I}_{\mathfrak{C}} \xrightarrow{\sim} \mathcal{E} \circ \mathcal{F}$ and $\varepsilon : \mathcal{F} \circ \mathcal{E} \xrightarrow{\sim} \mathrm{I}_{\mathfrak{A}}$, where $\mathrm{I}_{\mathfrak{A}}$, $\mathrm{I}_{\mathfrak{C}}$ denote the identity functors (see [Mac Lane (1971), Chap. 4, Sect. 4 (p. 91)]).

Suppose that, in addition, the following *triangular identities* are fulfilled for all objects C of $\mathfrak{C}$ and A of $\mathfrak{A}$:

$$\mathcal{E}\left(\varepsilon_A\right) \circ \eta_{\mathcal{E}(A)} = \mathrm{id}_{\mathcal{E}(A)}, \quad \varepsilon_{\mathcal{F}(C)} \circ \mathcal{F}\left(\eta_C\right) = \mathrm{id}_{\mathcal{F}(C)} \,. \tag{0.1}$$

Then η and ε determine an adjunction φ (6): see [Mac Lane (1971), Chap. IV, Sect. 1, Theorem 2, (v) (p. 81)]; *cf.* also [Mac Lane and Birkhoff (1967), Chap. XV, Sect. 8, Exercise 12 (p. 535)]. The transformation η is called the *unit* and ε the *counit* of the adjunction. In this case the triple $(\mathcal{F}, \mathcal{E}, \varphi)$ is called an *adjoint equivalence*: see [Mac Lane (1971), Chap. IV, Sect. 4 (p. 91)] (7).

A functor is said to be *full* if, for all pairs of objects, the map on morphisms are surjective. The notion of a *faithful* functor is obtained by replacing 'surjective' with 'injective'. Every equivalence $\mathcal{E} : \mathfrak{A} \to \mathfrak{C}$ is a full and faithful functor with the property that every object of $\mathfrak{C}$ is isomorphic to $\mathcal{E}(A)$ for some object A of $\mathfrak{A}$: see [Mac Lane (1971), Chap. IV, Sect. 4, Theorem 1 (p. 91)]. By the same theorem, if a full and faithful functor $\mathfrak{A} \to \mathfrak{C}$ is such that every object of $\mathfrak{C}$ is isomorphic to the correspondent of some object of $\mathfrak{A}$, then it is part of an *adjoint* equivalence. In particular, if $\eta : \mathrm{I}_{\mathfrak{C}} \xrightarrow{\sim} \mathcal{E} \circ \mathcal{F}$ and $\varepsilon : \mathcal{F} \circ \mathcal{E} \xrightarrow{\sim} \mathrm{I}_{\mathfrak{A}}$ are natural isomorphisms, then $\mathcal{E}$ is part of an adjoint equivalence. However, this does *not* imply, generally, that η and ε satisfy the triangular identities (0.1), because the unit and counit of the so-obtained adjoint equivalence do not necessarily coincide with η and ε.

0.2 Smooth Manifolds

In this section, we recall some basic facts concerning the algebraic interpretation of the theory of smooth manifolds. For additional information, see [Nestruev (2003)].

[6] Be aware that, when φ is an adjunction of $\mathcal{F} : \mathfrak{C} \to \mathfrak{A}$ to $\mathcal{E} : \mathfrak{A} \to \mathfrak{C}$ in the sense of our reference book [Mac Lane and Birkhoff (1967)], then the triple $(\mathcal{F}, \mathcal{E}, \varphi)$ is called an adjunction from $\mathfrak{C}$ to $\mathfrak{A}$ in [Mac Lane (1971)].

[7] In [Mac Lane (1971)], when an adjunction of $\mathcal{F}$ to $\mathcal{E}$ is determined by η and ε, it is also denoted by $\langle \mathcal{F}, \mathcal{E}, \eta, \varepsilon \rangle$: see [Mac Lane (1971), Chap. IV, Sect. 1 (p. 81)].