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Harmonic Analysis on Finite Groups

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Harmonic Analysis on Finite Groups

Representation Theory, Gelfand Pairs and Markov Chains

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HARMONIC ANALYSIS ON FINITE GROUPS

Line up a deck of 52 cards on a table. Randomly choose two cards and switch them. How many switches are needed in order to mix up the deck?

Starting from a few concrete problems, such as the random walk on the discrete circle and the Ehrenfest diffusion model, this book develops the necessary tools for the asymptotic analysis of these processes. This detailed study culminates with the case-by-case analysis of the cut-off phenomenon discovered by Persi Diaconis.

This self-contained text is ideal for students and researchers working in the areas of representation theory, group theory, harmonic analysis and Markov chains. Its topics range from the basic theory needed for students new to this area, to advanced topics such as Gelfand pairs, harmonics on posets and q-analogs, the complete analysis of the random matchings, and the representation theory of the symmetric group.

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To Katiuscia, Giacomo, and Tommaso

To my parents and Cristina

To my Mom, Rossella, and Stefania

Preface

In September 2003 we started writing a research expository paper on "Finite Gelfand pairs and their applications to probability and statistics" [43] for the proceedings of a conference held in Batumi (Georgia). After a preliminary version of that paper had been circulated, we received several emails of appreciation and encouragement from experts in the field. In particular, Persi Diaconis suggested that we expand that paper to a monograph on Gelfand pairs. In his famous 1988 monograph "Group representations in probability and statistics" [55] there is a short treatement of the theory of Gelfand pairs but, to his and our knowledge, no book entirely dedicated to Gelfand pairs was ever written. We thus started to expand the paper, including some background material to make the book self-contained, and adding some topics closely related to the kernel of the monograph. As the "close relation" is in some sense inductive, we pushed our treatement much further than what Persi was probably expecting. In all cases, we believe that our monograph is in some sense unique as it assembles, for the first time, the various topics that appear in it.

The book that came out is a course in finite harmonic analysis. It is completely self-contained (it only requires very basic rudiments of group theory and of linear algebra). There is also a large number of exercises (with solutions or generous hints) which constitute complements and/or further developments of the topics treated.

For this reason it can be used for a course addressed to both advanced undergraduates and to graduate students in pure mathematics as well as in probability and statistics. On the other hand, due to its completeness, it can also serve as a reference for mature researchers.

It presents a very general treatment of the theory of finite Gelfand pairs and their applications to Markov chains with emphasis on the cutoff phenomenon discovered by Persi Diaconis.

The book by Audrey Terras [220], which bears a similar title, is in some sense orthogonal to our monograph, both in style and contents. For instance, we do not treat applications to number theory, while Terras does not treat the representation theory of the symmetric group.

We present six basic examples of diffusion processes, namely the random walk on the circle, the Ehrenfest and the Bernoulli–Laplace models of diffusion, a Markov chain on the ultrametric space, random transpositions and random matchings. Each of these examples bears its own peculiarity and needs specific tools of an algebraic/harmonic-analytic/probabilistic nature in order to analyze the asymptotic behavior of the corresponding process.

These tools, which we therefore develop in a very self-contained presentation are: spectral graph theory and reversible Markov chains, Fourier analysis on finite abelian groups, representation theory and Fourier analysis of finite groups, finite Gelfand pairs and their spherical functions, and representation theory of the symmetric group. We also present a detailed account of the (distance-regular) graph theoretic approach to spherical functions and on the use of finite posets.

All this said, one can use this monograph as a textbook for at least three different courses on:

(i) **Finite Markov chains** (an elementary introduction oriented to the cutoff phenomenon): Chapters 1 and 2, parts of Chapters 5 and 6, and Appendix 1 (the discrete trigonometric transforms).
(ii) **Finite Gelfand pairs** (and applications to probability): Chapters 1–8 (if applications to probability are not included, then one may omit Chapters 1 and 2 and parts of the other chapters).
(iii) **Representation theory of finite groups** (possibly with applications to probability): Chapters 3, 4 (partially), 9, 10 and 11.

This book would never have been written without the encouragement and suggestions of Persi Diaconis. We thank him with deepest gratitude.

We are also grateful to Alessandro Figà Talamanca who first introduced us to Gelfand pairs and to the work of Diaconis.

We express our gratitude to Philippe Bougerol, Philippe Delsarte, Charles Dunkl, Adriano Garsia, Rostislav I. Grigorchuk, Gerard Letac, Arun Ram, Jan Saxl and Wolfgang Woess for their interest in our work and their encouragement.

We also acknowledge, with warmest thanks, the most precious careful reading of some parts of the book by Reza Bourquin and Pierre de la Harpe who pointed out several inaccuracies and suggested several changes and improvements on our expositions.

We finally express our deep gratitude to David Tranah, Peter Thompson, Bethan Jones and Mike Nugent from Cambridge University Press for their constant and kindest help at all the stages of the editing process.

Roma, 14 February 2007 TCS, FS and FT

Contents

Part I

Preliminaries, Examples and Motivations

1
Finite Markov chains

1.1 Preliminaries and notation

Let X be a finite set and denote by $L(X) = \{f : X \to \mathbb{C}\}$ the vector space of all complex-valued functions defined on X. Clearly $\dim L(X) = |X|$, where $|\cdot|$ denotes cardinality.

For $x \in X$ we denote by δ_x the *Dirac function* centered at x, that is

$$\delta_x(y) = \begin{cases} 1 & \text{if } y = x \\ 0 & \text{if } y \neq x. \end{cases}$$

The set $\{\delta_x : x \in X\}$ is a natural basis for $L(X)$ and if $f \in L(X)$ then $f = \sum_{x \in X} f(x)\delta_x$.

The space $L(X)$ is endowed with the scalar product defined by setting

$$\langle f_1, f_2 \rangle = \sum_{x \in X} f_1(x)\overline{f_2(x)}$$

for $f_1, f_2 \in L(X)$, and we set $\|f\|^2 = \langle f, f \rangle$. Note that the basis $\{\delta_x : x \in X\}$ is orthonormal with respect to $\langle \cdot, \cdot \rangle$. Sometimes we shall write $\langle \cdot, \cdot \rangle_{L(X)}$ to emphasize the space where the scalar product is defined if other spaces are also considered.

If $Y \subseteq X$, the symbol $\mathbf{1}_Y$ denotes the *characteristic function* of Y:

$$\mathbf{1}_Y(x) = \begin{cases} 1 & \text{if } x \in Y \\ 0 & \text{if } x \notin Y; \end{cases}$$

in particular, if $Y = X$ we write $\mathbf{1}$ instead of $\mathbf{1}_X$.

For $Y_1, Y_2, \ldots, Y_m \subseteq X$ we write $X = Y_1 \coprod Y_2 \coprod \cdots \coprod Y_m$ to indicate that the Y_j's constitute a *partition* of X, that is $X = Y_1 \cup Y_2 \cup \cdots \cup Y_m$ and $Y_i \cap Y_j = \emptyset$ whenever $i \neq j$. In other words the symbol $\coprod$ denotes a *disjoint union.* In particular, if we write $Y \coprod Y'$ we implicitly assume that $Y \cap Y' = \emptyset$.

If $A : L(X) \to L(X)$ is a linear operator, setting $a(x,y) = [A\delta_y](x)$ for all $x, y \in X$, we have that

$$[Af](x) = \sum_{y \in X} a(x,y) f(y) \tag{1.1}$$

for all $f \in L(X)$ and we say that the *matrix* $a = (a(x,y))_{x,y\in X}$, *indexed* by X, *represents* the operator A.

If the linear operators $A_1, A_2 : L(X) \to L(X)$ are represented by the matrices a_1 and a_2, respectively, then the composition $A_1 \circ A_2$ is represented by the corresponding product of matrices $a = a_1 \cdot a_2$ that is

$$a(x,y) = \sum_{z \in X} a_1(x,z) a_2(z,y).$$

For $k \in \mathbb{N}$ we denote by $a^k = \left(a^{(k)}(x,y)\right)_{x,y\in X}$ the product of k copies of a, namely

$$a^{(k)}(x,y) = \sum_{z \in X} a^{(k-1)}(x,z) a(z,y).$$

We remark that (1.1) can be also interpreted as the product of the matrix a with the column vector $f = (f(x))_{x \in X}$.

Given a matrix a and a column or, respectively, a row vector f, we denote by a^T and by f^T the transposed matrix (i.e. $a^T(x,y) = a(y,x)$ for all $x, y \in X$) and the row, respectively column transposed vector. This way we also denote by $f^T A$ the function given by

$$[f^T A](y) = \sum_{x \in X} f(x) a(x,y). \tag{1.2}$$

With our notation, the identity operator is represented by the identity matrix which may be expressed as $I = (\delta_x(y))_{x,y\in X}$. If X is a set of cardinality $|X| = n$ and $k \le n$, then a k-subset of X is a subset $A \subseteq X$ such that $|A| = k$.

If $v_1, v_2, \ldots, v_m$ are vectors in a vector space V, then $\langle v_1, v_2, \ldots, v_m \rangle$ will denote their linear span.

1.2 Four basic examples

This section is an informal description of four examples of finite diffusion processes. Their common feature is that their structure is rich in symmetries so that one can treat them by methods and techniques from finite harmonic analysis.