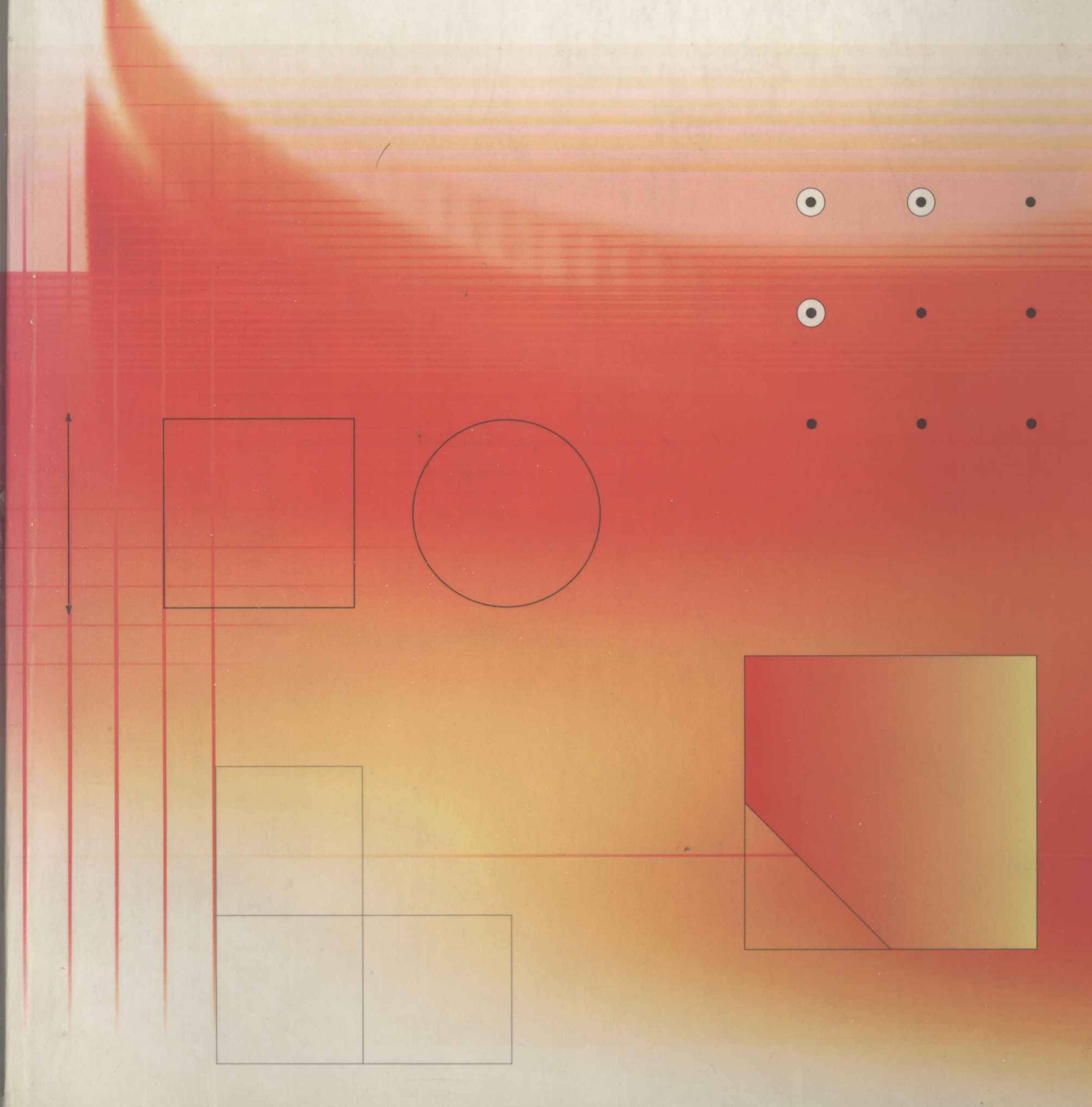
FOUNDATIONS OF

Mathematical Economics

Michael Carter



FOUNDATIONS OF MATHEMATICAL ECONOMICS

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To my parents, Merle and Maurice Carter, who provided a firm foundation for life

Introduction

Economics made progress without mathematics, but has made faster progress with it. Mathematics has brought transparency to many hundreds of economic arguments.
—Deirdre N. McCloskey (1994)

Economists rely on models to obtain insight into a complex world. Economic analysis is primarily an exercise in building and analyzing models. An economic model strips away unnecessary detail and focuses attention on the essential details of an economic problem. Economic models come in various forms. Adam Smith used a verbal description of a pin factory to portray the principles of division of labor and specialization. Irving Fisher built a hydraulic model (comprising floating cisterns, tubes, and levers) to illustrate general equilibrium. Bill Phillips used a different hydraulic model (comprising pipes and colored water) to portray the circular flow of income in the national economy. Sir John Hicks developed a simple mathematical model (IS-LM) to reveal the essential differences between Keynes's *General Theory* and the "classics." In modern economic analysis, verbal and physical models are seen to be inadequate. Today's economic models are almost exclusively mathematical.

Formal mathematical modeling in economics has two key advantages. First, formal modeling makes the assumptions explicit. It clarifies intuition and makes arguments transparent. Most important, it uncovers the limitations of our intuition, delineating the boundaries and uncovering the occasional counterintuitive special case. Second, the formal modeling aids communication. Once the assumptions are explicit, participants spend less time arguing about what they really meant, leaving more time to explore conclusions, applications, and extensions.

Compare the aftermath of the publication of Keynes's *General Theory* with that of von Neumann and Morgenstern's *Theory of Games and Economic Behavior*. The absence of formal mathematical modeling in the *General Theory* meant that subsequent scholars spent considerable energy debating "what Keynes really meant." In contrast, the rapid development of game theory in recent years owes much to the advantages of formal modeling. Game theory has attracted a predominance of practitioners who are skilled formal modelers. As their assumptions are very explicit, practitioners have had to spend little time debating the meaning of others' writings. Their efforts have been devoted to exploring ramifications and applications. Undoubtedly, formal modeling has enhanced the pace of innovation in game-theoretic analysis in economics.

Economic models are not like replica cars, scaled down versions of the real thing admired for their verisimilitude. A good economic model strips away all the unnecessary and distracting detail and focuses attention on the essentials of a problem or issue. This process of stripping away unnecessary detail is called abstraction. Abstraction serves the same role in mathematics. The aim of abstraction is not greater generality but greater simplicity. Abstraction reveals the logical structure of the mathematical framework in the same way as it reveals the logical structure of an economic model.

Chapter 1 establishes the framework by surveying the three basic sources of structure in mathematics. First, the order, geometric and algebraic structures of sets are considered independently. Then their interaction is studied in subsequent sections dealing with normed linear spaces and preference relations.

Building on this foundation, we study mappings between sets or functions in chapters 2 and 3. In particular, we study functions that preserve the structure of the sets which they relate, treating in turn monotone, continuous, and linear functions. In these chapters we meet the three fundamental theorems of mathematical economics—the (continuous) maximum theorem, the Brouwer fixed point theorem, and the separating hyperplane theorem, and outline many of their important applications in economics, finance, and game theory.

A key tool in the analysis of economic models is the approximation of smooth functions by linear and quadratic functions. This tool is developed in chapter 4, which presents a modern treatment of what is traditionally called multivariate calculus.

Since economics is the study of rational choice, most economic models involve optimization by one or more economic agents. Building and analyzing an economic model involves a typical sequence of steps. First, the model builder identifies the key decision makers involved in the economic phenomenon to be studied. For each decision maker, the model builder must postulate an objective or criterion, and identify the tools or instruments that she can use in pursuit of that objective. Next, the model builder must formulate the constraints on the decision maker's choice. These constraints normally take the form of a system of equations and inequalities linking the decision variables and defining the feasible set. The model therefore portrays the decision maker's problem as an exercise in constrained optimization, selecting the best alternative from a feasible set.

Typically analysis of an optimization model has two stages. In the first stage, the constrained optimization problem is solved. That is, the optimal choice is characterized in terms of the key parameters of the model. After a general introduction, chapter 5 first discusses necessary and sufficient conditions for unconstrained optimization. Then four different perspectives on the Lagrangean multiplier technique for equality constrained problems are presented. Each perspective adds a different insight contributing to a complete understanding. In the second part of the chapter, the analysis is extended to inequality constraints, including coverage of constraint qualification, sufficient conditions, and the practically important cases of linear and concave programming.

In the second stage of analysis, the sensitivity of the optimal solution to changes in the parameters of the problem is explored. This second stage is traditionally (in economics) called comparative statics. Chapter 6 outlines four different approaches to the comparative static analysis of optimization models, including the traditional approaches based on the implicit function theorem or the envelope theorem. It also introduces a promising new approach based on order properties and monotonicity, which often gives strong conclusions with minimal assumptions. Chapter 6 concludes with a brief outline of the comparative static analysis of equilibrium (rather than optimization) models.

The book includes a thorough treatment of some material often omitted from introductory texts, such as correspondences, fixed point theorems, and constraint qualification conditions. It also includes some recent developments such as supermodularity and monotone comparative statics. We have made a conscious effort to illustrate the discussion throughout with economic examples and where possible to introduce mathematical concepts with economic ideas. Many illustrative examples are drawn from game theory.

The completeness of the real numbers is assumed, every other result is derived within the book. The most important results are stated as theorems or propositions, which are proved explicitly in the text. However, to enhance readability and promote learning, lesser results are stated as exercises, answers for which will be available on the internet (see the note to the reader). In this sense the book is comprehensive and entirely self-contained, suitable to be used as a reference, a text, or a resource for self-study.

The sequence of the book, preceding from sets to functions to smooth functions, has been deliberately chosen to emphasize the structure of the underlying mathematical ideas. However, for instructional purposes or for self-study, an alternative sequence might be preferable and easier to motivate. For example, the first two sections of chapter 1 (sets and ordered sets) could be immediately followed by the first two sections of chapter 2 (functions and monotone functions). This would enable the student to achieve some powerful results with a minimum of fuss. A second theme could then follow the treatment of metric spaces (and the topological part of section 1.6) with continuous functions culminating in the continuous maximum theorem and perhaps the Banach fixed point theorem. Finally the course could turn to linear spaces, linear functions, convexity, and linear functionals, culminating in the separating hyperplane theorem and its applications. A review of fixed point theorems would then highlight the interplay of linear and topological structure in the Brouwer fixed point theorem and its generalizations. Perhaps it would then be advantageous to proceed through chapters 4, 5, and 6 in the given sequence. Even if chapter 4 is not explicitly studied, it should be reviewed to understand the notation used for the derivative in the following chapters.

The book can also be used for a course emphasizing microeconomic theory rather than mathematical methods. In this case the course would follow a sequence of topics, such as monotonicity, continuity, convexity, and homogeneity, interspersed with analytical tools such as constrained optimization, the maximum, fixed point, and separating hyperplane theorems, and comparative statics. Each topic would be introduced and illustrated via its role in the theory of the consumer and the producer.

Achieving consistency in notation is a taxing task for any author of a mathematical text. Wherever I could discern a standard notation in the economics literature, I followed that trend. Where diversity ruled, I have tended to follow the notation in Hal Varian's *Microeconomic Analysis*, since it has been widely used for many years. A few significant exceptions to these rules are explicitly noted.

Many people have left their mark on this book, and I take great pleasure in acknowledging their contribution. Foremost among my creditors is Graeme Guthrie whose support, encouragement, and patient exposition of mathematical subtleties has been invaluable. Richard Edlin and Mark Pilbrow drafted most of the diagrams. Martin Osborne and Carolyn Pitchik made detailed comments on an early draft of the manuscript and Martin patiently helped me understand intricacies of TEX and LATEX.

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A Note to the Reader

Few people rely solely on any social science for their pleasures, and attaining a suitable level of ecstasy involves work. . . . It is a nuisance, but God has chosen to give the easy problems to the physicists.
—Lave and March (1975)

Some people read mathematics books for pleasure. I assume that you are not one of this breed, but are studying this book to enhance your understanding of economics. While I hope this process will be enjoyable, to make the most of it will require some effort on your part. Your reward will be a comprehension of the foundations of mathematical economics, you will appreciate the elegant interplay between economic and mathematical ideas, you will know why as well as how to use particular tools and techniques.

One of the most important requirements for understanding mathematics is to build up an appropriate mental framework or structure to relate and integrate the various components and pieces of information. I have endeavored to portray a suitable framework in the structure of this book, in the way it is divided into chapters, sections, and so on. This is especially true of the early mathematical chapters, whose structure is illustrated in the following table:

Sets	Functions
Ordered sets	Monotone functions
Metric spaces	Continuous functions
Linear spaces	Linear functions
Convex sets	Convex functions
Cones	Homogeneous functions

This is the framework to keep in mind as you proceed through the book.

You will also observe that there is a hierarchy of results. The most important results are stated as theorems. You need to be become familiar with these, their assumptions and their applications. Important but more specialized results are stated as propositions. Most of the results, however, are given as exercises. Consequently exercise has a slightly different meaning here than in many texts. Most of the 820 exercises in the book are not "finger exercises," but substantive propositions forming an integral part of the text. Similarly examples contain many of the key ideas and warrant careful attention.

There are two reasons for this structure. First, the exercises and examples break up the text, highlighting important ideas. Second, the exercises provide the potential for deeper learning. It is an unfortunate fact of life that for most of us, mathematical skills (like physical skills) cannot be obtained by osmosis through reading and listening. They have to be acquired through practice. You will learn a great deal by attempting to do these exercises. In many cases elaborate hints or outlines are given, leaving you to fill in the detail. Then you can check your understanding by consulting the comprehensive answers, which are available on the Internet at *http://mitpress.mit.edu/carter-foundations.*

FOUNDATIONS OF MATHEMATICAL ECONOMICS

Contents

1 Sets and Spaces

All is number.
—Pythagoras

God created the integers; everything else is the work of man
—L. Kronecker

One of the most important steps in understanding mathematics is to build a framework to relate and integrate the various components and pieces of information. The principal function of this introductory chapter is to start building this framework, reviewing some basic concepts and introducing our notation. The first section reviews the necessary elements of set theory. These basics are developed in the next three sections, in which we study sets that have a specific structure. First, we consider ordered sets (section 1.2), whose elements can be ranked by some criterion. A set that has a certain form or structure is often called *a space*. In the following two sections, we tour in turn the two most important examples: metric spaces and linear spaces. Metric spaces (section 1.3) generalize the familiar properties of Euclidean geometry, while linear spaces (section 1.4) obey many of the usual rules of arithmetic while. Almost all the sets that populate this book will inhabit a linear, metric space (section 1.5), so a thorough understanding of these sections is fundamental to the remainder of the book. The chapter ends with an extended example (section 1.6) in which we integrate the order, algebraic, and geometric perspectives to study preference relations that are central to the theory of the consumer and other areas of economics.

1.1 Sets

A *set* is a collection of objects (called *elements*) such as the set of people in the world, books in the library, students in the class, weekdays, or commodities available for trade. Sometimes we denote a set by listing all its members between braces { }, for example,

Weekdays = {Monday, Tuesday, Wednesday, Thursday, Friday}

Some of the elements may be omitted from the list when the meaning is clear, as in the following example:

alphabet = $\{A, B, C, \ldots, Z\}$