Edited by
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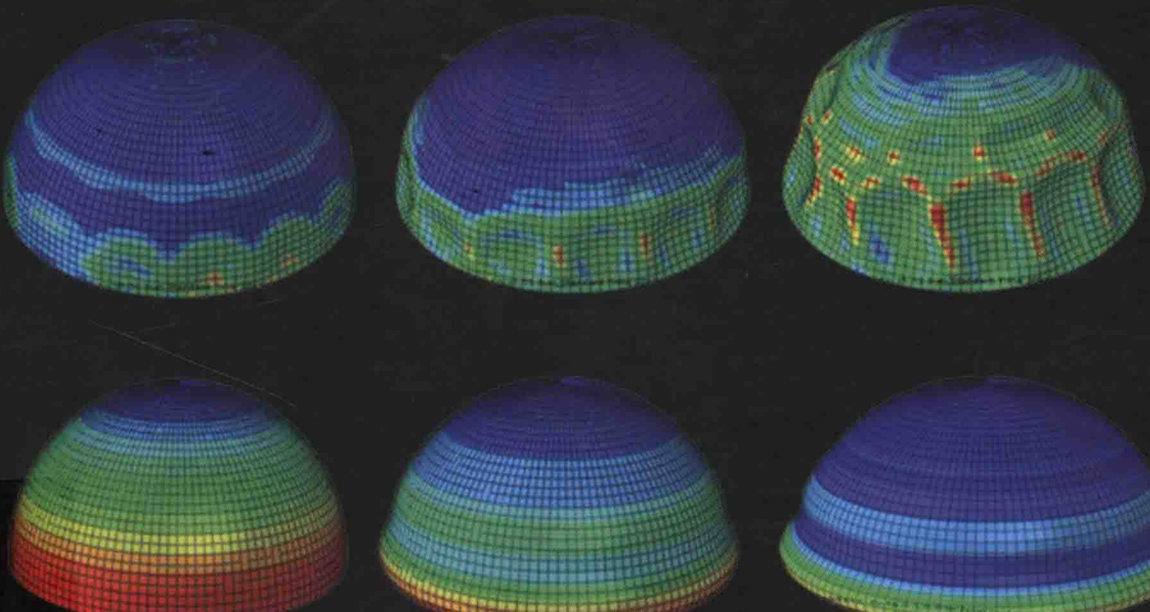
Proceedings of the IJSSD Symposium 201 on Progress in Structural Stability and Dynamics

14–16 April 2012, Nanjing, China



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Preface

International Journal of Structural Stability (IJSSD) has been in existence since 2001. The aim of the journal is to provide a unique forum for the publication and rapid dissemination of original research on stability and dynamics of structures. In support of the journal, conferences and symposia have been organised regularly. The first International Conference on Structural Stability and Dynamics (ICSSD) was held in Taipei in 2000, and it was followed by conferences in Singapore (2002), Orlando, USA (2005) and Jaipur, India (2012). A smaller IJSSD Symposium, that piggy back on the International Conference on Advances in Steel Structures (ICASS), was organised in Hong Kong in 2009. The IJSSD Chief Editors are very grateful to Prof. G.P. Shu (of Southeast University) and Prof. S.L. Chan (of Hong Kong Polytechnic University) for inviting us to continue in this very successful synergetic embedding of the IJSSD symposium in ICASS. This time round, we have 24 papers for presentation at the IJSSD Symposium 2012 to be held in conjunction with the 7th ICASS in Nanjing, China.

This IJSSD Symposium Proceedings consists of 24 papers which cover a broad range of topics such as structural stability and dynamics of thin-walled structural members, elasticas, functionally graded beams and plates, composite structures, spherical shells, bridges, floating structures, carbon nanotubes, graphene sheets and numerical techniques for dynamic analyses.

We hope that the research findings described in this volume of proceedings will inspire researchers, engineers and designers to conceptualize and build even more awesome structures for the betterment of mankind.

C.M. Wang, Y.B. Yang and J.N. Reddy

Editors of IJSSD Symposium Proceedings

Contents

MODIFIED COUPLE STRESS THEORIES OF FUNCTIONALLY GRADED BEAMS AND PLATES

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KEYWORDS

Functionally graded materials, modified couple stress theory, shear deformable beams and plates, the Von Karman nonlinearity.

ABSTRACT

In this paper an overview of general third-order beam and plate theories that account for (a) geometric nonlinearity, (b) microstructure-dependent size effects, and (c) two-constituent material variation through the thickness (i. e., functionally graded material beams and plates) is presented. A detailed derivation of the equations of motion, using Hamilton's principle, is presented, and it is based on a modified couple stress theory, power-law variation of the material through the thickness, and the von Karman nonlinear strains. The modified couple stress theory includes a material length scale parameter that can capture the size effect in a functionally graded material. The governing equations of motion derived herein for a general third-order theory with geometric nonlinearity, microstructure dependent size effect, and material gradation through the thickness are specialized to classical and shear deformation beam and plate theories available in the literature. The theory presented herein also can be used to develop finite element models and determine the effect of the geometric nonlinearity, microstructure-dependent size effects, and material grading through the thickness on bending and post-buckling response of elastic beams and plates.

INTRODUCTION

The next generation of material systems used in space and other structures as well as in MEMS and NEMS feature *thermo-mechanical coupling*, *functionality*, *intelligence*, and *miniaturization*. These systems may operate under varying conditions. When functionally graded material systems are used in nano- and micro-devices, it is necessary to account for the microstructure-dependent size effect and the geometric nonlinearity. Since beam and plate structural elements are commonly used in these devices and structures, it is useful to develop refined theories of plates that account for size effects, material gradation through thickness, and geometric nonlinearity.

In the context of plate theories, no plate theory exists that accounts for shear deformation while not requiring shear correction factors, material variation through plate thickness, includes microstructure-dependent size effects, and geometric nonlinearity. This very fact motivated the present study. The objective of the current paper is to develop a general third-order plate theory that accounts for through-thickness power-law variation of a two-constituent material with temperature-dependent material properties, modified couple stress theory, and the von Karman nonlinear strains. In particular, we extend the modified couple stress theory of Yang et al. [1] (also see [2-6]) to the case of functionally graded plates using the third-order plate kinematics of Reddy [7-11], and Bose and Reddy [12]. Since most nanoscale devices involve plate-like elements that may be functionally graded and undergo moderately large rotations, the newly developed plate theory can be used to capture the size effects in functionally graded microplates. Moreover, the bending-extensional coupling is captured through the von Karman nonlinear strains.

MODIFIED COUPLE STRESS MODEL

The couple stress theory proposed by Yang et al. [1] is a modification of the classical couple stress theory. They established that the couple stress tensor is symmetric and the symmetric curvature tensor is the only proper conjugate strain measure to have a contribution to the total strain energy of the body. The two main advantages of the modified couple stress theory over the classical couple stress theory are the inclusion of a symmetric couple stress tensor and the involvement of only one length scale parameter, which is a direct consequence of the fact that the strain energy density function depends only on the strain and the symmetric part of the curvature tensor (see Ma, Gao, and Reddy [35] and Reddy [6]). According to the modified couple stress theory, the virtual strain energy δU can be written as

$$\delta U = \int_V \delta \varepsilon : \sigma + \delta \chi : m \mathrm{d}V = \int_V \delta \varepsilon_{ij} : \sigma_{ij} + \delta \chi_{ij} : m_{ij} \,\mathrm{d}V \tag{1}$$

where summation on repeated indices is implied; here σ_{ij} denotes the cartesian components of (the symmetric part of) the stress tensor, ε_{ij} are the strain components, m_{ij} are the components of the deviatoric part of the symmetric couple stress tensor, and χ_{ij} are the components of the symmetric curvature tensor

$$\chi_{ij} = \frac{1}{2}\left(\frac{\partial \omega_i}{\partial x_j} + \frac{\partial \omega_j}{\partial x_i}\right) = -\frac{1}{2} e_{ijk} \frac{\partial u_j}{\partial x_k} \tag{2}$$

FUNCTIONALLY GRADED MATERIALS

Consider a plate of total thickness h. The x and y coordinates are taken in the midplane, denoted with Ω, and the z-axis is taken normal to the plate, as shown in Figure 1. We assume that the material of the plate is isotropic but varies from one kind of material on one side, $z = -h/2$, to another material on the other side, $z = h/2$, as indicated in Figure 2. A typical material property P of the FGM through the plate thickness is assumed to be represented by a power-law (see Praveen and Reddy [13])

$$P(z, T) = [P_c(T) - P_m(T)] f(z) + P_m(T), \quad f(z) = \left(\frac{1}{2} + \frac{z}{h}\right)^n \tag{3}$$

where $P_c(T)$ and $P_m(T)$ are the values of a typical material property P, such as the modulus, density, and conductivity, of the ceramic material and metal, respectively; n denotes the volume fraction exponent, called power-law index. When $n = 0$, we obtain the single-material plate [with the property $P_c(T)$].

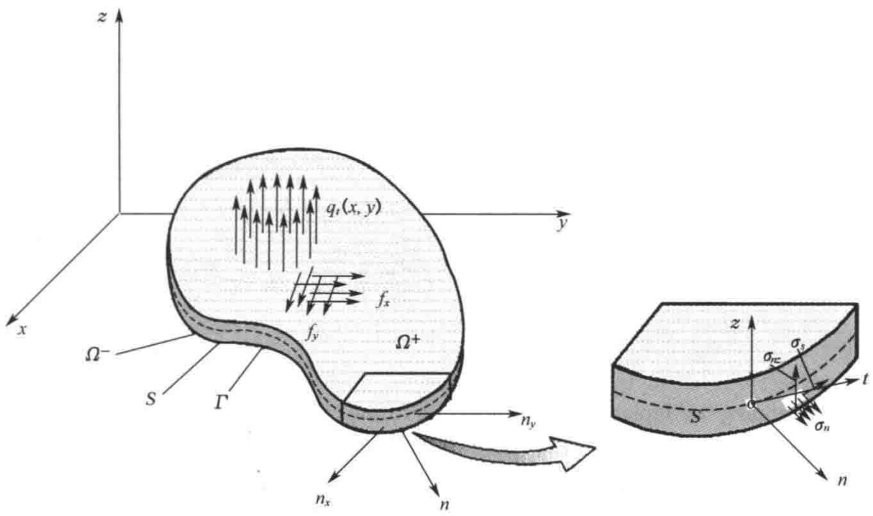

Figure 1 Geometry of a plate loaded with forces.

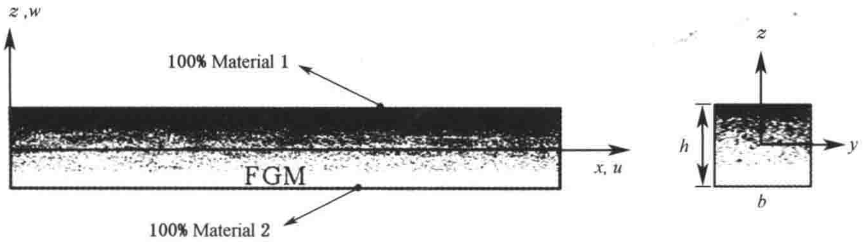

Figure 2 Through-thickness functionally graded plate.

When FGMs are used in high-temperature environment, the material properties are functions of temperature, and they can be expressed as

$$P_\alpha(T) = c_0(c_{-1}T^{-1} + 1 + c_1 T + c_2 T^2 + c_3 T^3),\ \alpha = c \text{ or } m \tag{4}$$

where c_0 is a constant appearing in the cubic fit of the material property with temperature; and c_{-1}, c_1, c_2, and c_3 coefficients obtained after factoring out c_0 from the cubic curve fit of the property. For the analysis with constant properties, the material properties were all evaluated at 25.15℃.

A GENERAL THIRD-ORDER THEORY

Here develop a general third-order theory for the deformation of the plate first and then specialize to the well-known plate theories. We restrict the formulation to linear elastic material behavior, small strains, and moderate rotations and displacements, so that there is no geometric update of the domain, that is, the integrals posed on the deformed configuration are evaluated using the undeformed domain and there is no difference between the Cauchy stress tensor and the second Piola—Kirchhoff stress tensor.

The equations of motion are obtained using Hamilton's principle. The three-dimensional problem is reduced to two-dimensional one by assuming a displacement field that is explicit in the thickness coordinate z. We begin with the following displacement field

$$\begin{aligned} u_1(x, y, z, t) &= u(x, y, t) + z\theta_x + z^2\phi_x + z^3\psi_x \\ u_2(x, y, z, t) &= v(x, y, t) + z\theta_y + z^2\phi_y + z^3\psi_y \\ u_3(x, y, z, t) &= w(x, y, t) + z\theta_z + z^2\phi_z \end{aligned} \tag{5}$$

where (u, v, w) are the displacements along the coordinate lines of a material point on the xy-plane, i. e., $u(x, y, t) = u_1(x, y, 0, t)$, $v(x, y, t) = u_2(x, y, 0, t)$, $w(x, y, t) = u_3(x, y, 0, t)$ and

$$\begin{aligned} &\theta_x = \left(\frac{\partial u_1}{\partial z}\right)_{z=0}, \ \theta_y = \left(\frac{\partial u_2}{\partial z}\right)_{z=0}, \ \theta_z = \left(\frac{\partial u_3}{\partial z}\right)_{z=0} \\ &2\phi_x = \left(\frac{\partial^2 u_1}{\partial z^2}\right)_{z=0}, \quad 2\phi_y = \left(\frac{\partial^2 u_2}{\partial z^2}\right)_{z=0}, \quad 2\phi_z = \frac{\partial^2 u_3}{\partial z^2}, \\ &6\psi_x = \frac{\partial^3 u_1}{\partial z^3}, \ 6\psi_y = \frac{\partial^3 u_2}{\partial z^3} \end{aligned} \tag{6}$$

The reason for expanding the inplane displacements up to the cubic term and the transverse displacement up to the quadratic term in z is to obtain a quadratic variation of the transverse shear strains $\gamma_{xz} = 2\varepsilon_{xz}$ and $\gamma_{yz} = 2\varepsilon_{yz}$ through the plate thickness. Note that all three displacements contribute to the quadratic variation. In the most general case represented by the displacement field in Eqn. (5), there are 11 generalized displacements $(u, v, w, \theta_x, \theta_y, \theta_z, \phi_x, \phi_y, \phi_z, \psi_x, \psi_y)$ and, therefore, 11 differential equations will be required to determine them.

The von Karman nonlinear strain-displacement relations associated with the displacement field in Eqn. (17) can be obtained by assuming that the strains are small and rotations are moderately large; that is, we assume

$$\begin{aligned} &\left(\frac{\partial u_\alpha}{\partial x}\right)^2 \approx 0, \ \left(\frac{\partial u_\alpha}{\partial y}\right)^2 \approx 0, \ \left(\frac{\partial u_3}{\partial x}\right)^2 \approx \left(\frac{\partial w}{\partial x}\right)^2 \\ &\left(\frac{\partial u_3}{\partial y}\right)^2 \approx \left(\frac{\partial w}{\partial y}\right)^2, \ \left(\frac{\partial u_3}{\partial x}\right)\left(\frac{\partial u_3}{\partial y}\right) \approx \frac{\partial w}{\partial x}\frac{\partial w}{\partial y} \end{aligned} \tag{7}$$

for $\alpha = 1, 2$. Thus the nonzero strains of the general third-order theory with the von Karman nonlinearity are

$$\begin{Bmatrix} \varepsilon_{xx} \\ \varepsilon_{yy} \\ \varepsilon_{xy} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \varepsilon_{xy}^{(0)} \end{Bmatrix} + z\begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \varepsilon_{xy}^{(1)} \end{Bmatrix} + z^2\begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{yy}^{(2)} \\ \varepsilon_{xy}^{(2)} \end{Bmatrix} + z^3\begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \varepsilon_{xy}^{(3)} \end{Bmatrix} \tag{8}$$

$$\begin{Bmatrix} \varepsilon_{zz} \\ \gamma_{xz} \\ \gamma_{yz} \end{Bmatrix} = \begin{Bmatrix} \varepsilon_{zz}^{(0)} \\ \gamma_{xz}^{(0)} \\ \gamma_{yz}^{(0)} \end{Bmatrix} + z\begin{Bmatrix} \varepsilon_{zz}^{(1)} \\ \gamma_{xz}^{(1)} \\ \gamma_{yz}^{(1)} \end{Bmatrix} + z^2\begin{Bmatrix} \varepsilon_{zz}^{(2)} \\ \gamma_{xz}^{(2)} \\ \gamma_{yz}^{(2)} \end{Bmatrix} \tag{9}$$

with

$$\begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \gamma_{xy}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\ \dfrac{\partial v}{\partial y} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2 \\ \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} \end{Bmatrix}, \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \gamma_{xy}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \theta_x}{\partial x} \\ \dfrac{\partial \theta_y}{\partial y} \\ \dfrac{\partial \theta_x}{\partial y} + \dfrac{\partial \theta_y}{\partial x} \end{Bmatrix} \tag{10}$$

$$\begin{Bmatrix} \varepsilon_{xx}^{(2)} \\ \varepsilon_{yy}^{(2)} \\ \gamma_{xy}^{(2)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \phi_x}{\partial x} \\ \dfrac{\partial \phi_y}{\partial y} \\ \dfrac{\partial \phi_x}{\partial y} + \dfrac{\partial \phi_y}{\partial x} \end{Bmatrix}, \begin{Bmatrix} \varepsilon_{xx}^{(3)} \\ \varepsilon_{yy}^{(3)} \\ \gamma_{xy}^{(3)} \end{Bmatrix} = \begin{Bmatrix} \dfrac{\partial \psi_x}{\partial x} \\ \dfrac{\partial \psi_y}{\partial y} \\ \dfrac{\partial \psi_x}{\partial y} + \dfrac{\partial \psi_y}{\partial x} \end{Bmatrix} \tag{11}$$

$$\begin{Bmatrix} \varepsilon_{zz}^{(0)} \\ \gamma_{xz}^{(0)} \\ \gamma_{yz}^{(0)} \end{Bmatrix} = \begin{Bmatrix} \theta_z \\ \theta_x + \dfrac{\partial w}{\partial x} \\ \theta_y + \dfrac{\partial w}{\partial y} \end{Bmatrix}, \begin{Bmatrix} \varepsilon_{zz}^{(1)} \\ \gamma_{xz}^{(1)} \\ \gamma_{yz}^{(1)} \end{Bmatrix} = \begin{Bmatrix} \phi_z \\ 2\phi_x + \dfrac{\partial \theta_z}{\partial x} \\ 2\phi_y + \dfrac{\partial \theta_z}{\partial y} \end{Bmatrix}, \begin{Bmatrix} \varepsilon_{zz}^{(2)} \\ \gamma_{xz}^{(2)} \\ \gamma_{yz}^{(2)} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 3\psi_x + \dfrac{\partial \phi_z}{\partial x} \\ 3\psi_y + \dfrac{\partial \phi_z}{\partial y} \end{Bmatrix} \tag{12}$$

In view of the displacement field in Eq. (5), components of the rotation vector and curvature tensor take the form (with $\omega_1 = \omega_x$, $\omega_2 = \omega_y$, $\omega_3 = \omega_z$, $\chi_{11} = \chi_{xx}$, $\chi_{22} = \chi_{yy}$, and so on)

$$\begin{aligned} \omega_x &= \frac{1}{2}\left[\left(\frac{\partial w}{\partial y} + z\frac{\partial \theta_z}{\partial y} + z^2\frac{\partial \phi_z}{\partial y}\right) - (\theta_y + 2z\phi_y + 3z^2\psi_y)\right] = \omega_x^{(0)} + z\omega_x^{(1)} + z^2\omega_x^{(2)} \\ \omega_y &= \frac{1}{2}\left[(\theta_x + 2z\phi_x + 3z^2\psi_x) - \left(\frac{\partial w}{\partial x} + z\frac{\partial \theta_z}{\partial x} + z^2\frac{\partial \phi_z}{\partial x}\right)\right] = \omega_y^{(0)} + z\omega_y^{(1)} + z^2\omega_y^{(2)} \\ \omega_z &= \frac{1}{2}\left[\left(\frac{\partial v}{\partial x} + z\frac{\partial \theta_y}{\partial x} + z^2\frac{\partial \phi_y}{\partial x} + z^3\frac{\partial \psi_y}{\partial x}\right) - \left(\frac{\partial u}{\partial y} + z\frac{\partial \theta_x}{\partial y} + z^2\frac{\partial \phi_x}{\partial y} + z^3\frac{\partial \psi_x}{\partial y}\right)\right] \\ &= \omega_z^{(0)} + z\omega_z^{(1)} + z^2\omega_z^{(2)} + z^3\omega_z^{(3)} \end{aligned} \tag{13}$$

where

$$\begin{aligned} &\omega_x^{(0)} = \frac{1}{2}\left(\frac{\partial w}{\partial y} - \theta_y\right), \omega_x^{(1)} = \frac{1}{2}\left(\frac{\partial \theta_z}{\partial y} - 2\phi_y\right), \omega_x^{(2)} = \frac{1}{2}\left(\frac{\partial \phi_z}{\partial y} - 3\psi_y\right), \omega_y^{(0)} = \frac{1}{2}\left(\theta_x - \frac{\partial w}{\partial x}\right) \\ &\omega_y^{(1)} = \frac{1}{2}\left(2\phi_x - \frac{\partial \theta_z}{\partial x}\right), \omega_y^{(2)} = \frac{1}{2}\left(3\psi_x - \frac{\partial \phi_z}{\partial x}\right), \omega_z^{(0)} = \frac{1}{2}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right), \omega_z^{(1)} = \frac{1}{2}\left(\frac{\partial \theta_y}{\partial x} - \frac{\partial \theta_x}{\partial y}\right) \\ &\omega_z^{(2)} = \frac{1}{2}\left(\frac{\partial \phi_y}{\partial x} - \frac{\partial \phi_x}{\partial y}\right), \omega_z^{(3)} = \frac{1}{2}\left(\frac{\partial \psi_y}{\partial x} - \frac{\partial \psi_x}{\partial y}\right) \end{aligned} \tag{14}$$

and

$$\begin{aligned} \chi_{xx} &= \chi_{xx}^{(0)} + z\chi_{xx}^{(1)} + z^2\chi_{xx}^{(2)} \\ \chi_{yy} &= \chi_{yy}^{(0)} + z\chi_{yy}^{(1)} + z^2\chi_{yy}^{(2)} \\ \chi_{zz} &= \chi_{zz}^{(0)} + z\chi_{zz}^{(1)} + z^2\chi_{zz}^{(2)} \\ \chi_{xy} &= \chi_{xy}^{(0)} + z\chi_{xy}^{(1)} + z^2\chi_{xy}^{(2)} \\ \chi_{xz} &= \chi_{xz}^{(0)} + z\chi_{xz}^{(1)} + z^2\chi_{xz}^{(2)} + z^3\chi_{xz}^{(3)} \\ \chi_{yz} &= \chi_{yz}^{(0)} + z\chi_{yz}^{(1)} + z^2\chi_{yz}^{(2)} + z^3\chi_{yz}^{(3)} \end{aligned} \tag{15}$$

with

$$\begin{aligned} &\chi_{xx}^{(0)} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial w}{\partial y} - \theta_y\right), \chi_{xx}^{(1)} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \theta_z}{\partial y} - 2\phi_y\right), \chi_{xx}^{(2)} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \phi_z}{\partial y} - 3\psi_y\right) \\ &\chi_{yy}^{(0)} = \frac{1}{2}\frac{\partial}{\partial y}\left(\theta_x - \frac{\partial w}{\partial x}\right), \chi_{yy}^{(1)} = \frac{1}{2}\frac{\partial}{\partial y}\left(2\phi_x - \frac{\partial \theta_z}{\partial x}\right), \chi_{yy}^{(2)} = \frac{1}{2}\frac{\partial}{\partial y}\left(3\psi_x - \frac{\partial \phi_z}{\partial x}\right) \\ &\chi_{zz}^{(0)} = \frac{1}{2}\left(\frac{\partial \theta_y}{\partial x} - \frac{\partial \theta_x}{\partial y}\right), \chi_{zz}^{(1)} = \left(\frac{\partial \phi_y}{\partial x} - \frac{\partial \phi_x}{\partial y}\right), \chi_{zz}^{(2)} = \frac{3}{2}\left(\frac{\partial \psi_y}{\partial x} - \frac{\partial \psi_x}{\partial y}\right) \\ &\chi_{xy}^{(0)} = \frac{1}{2}\left[\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial y} - \theta_y\right) + \frac{\partial}{\partial x}\left(\theta_x - \frac{\partial w}{\partial x}\right)\right], \chi_{xy}^{(1)} = \frac{1}{2}\left[\frac{\partial}{\partial y}\left(\frac{\partial \theta_z}{\partial y} - 2\phi_y\right) + \frac{\partial}{\partial x}\left(2\phi_x - \frac{\partial \theta_z}{\partial x}\right)\right] \\ &\chi_{xy}^{(2)} = \frac{1}{2}\left[\frac{\partial}{\partial y}\left(\frac{\partial \phi_z}{\partial y} - 3\psi_y\right) + \frac{\partial}{\partial x}\left(3\psi_x - \frac{\partial \phi_z}{\partial x}\right)\right], \chi_{xz}^{(0)} = \frac{1}{2}\left[\left(\frac{\partial \theta_z}{\partial y} - 2\phi_y\right) + \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right] \\ &\chi_{xz}^{(1)} = \left(\frac{\partial \phi_z}{\partial y} - 3\psi_y\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \theta_y}{\partial x} - \frac{\partial \theta_x}{\partial y}\right), \chi_{xz}^{(2)} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \phi_y}{\partial x} - \frac{\partial \phi_x}{\partial y}\right), \chi_{xz}^{(3)} = \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial \psi_y}{\partial x} - \frac{\partial \psi_x}{\partial y}\right) \\ &\chi_{yz}^{(0)} = \frac{1}{2}\left[\left(2\phi_x - \frac{\partial \theta_z}{\partial x}\right) + \frac{\partial}{\partial y}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\right], \chi_{yz}^{(1)} = \left(3\psi_x - \frac{\partial \phi_z}{\partial x}\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \theta_y}{\partial x} - \frac{\partial \theta_x}{\partial y}\right) \\ &\chi_{yz}^{(2)} = \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \phi_y}{\partial x} - \frac{\partial \phi_x}{\partial y}\right), \chi_{yz}^{(3)} = \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial \psi_y}{\partial x} - \frac{\partial \psi_x}{\partial y}\right) \end{aligned} \tag{16}$$

(17)

The equations of motion are obtained by using the principle of virtual displacements or Hamilton's principle (see Reddy [14])

$$\int_0^T (\delta K - \delta U - \delta V)\,\mathrm{d}t = 0 \tag{18}$$

where δK is the virtual kinetic energy, δU is the virtual strain energy, and δV is the virtual work done by external forces.

The details of deriving the expressions for the virtual energies are not given here due to the space restrictions (see Reddy and Kim [15]). The equations of motion are

$$\delta u:\ \frac{\partial M_{xx}^{(0)}}{\partial x} + \frac{\partial M_{xy}^{(0)}}{\partial y} + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial M_{zx}^{(0)}}{\partial x} + \frac{\partial M_{yz}^{(0)}}{\partial y}\right) + F_x^{(0)} + \frac{1}{2}\frac{\partial c_z^{(0)}}{\partial y} = m_0\ddot{u} + m_1\ddot{\theta}_x + m_2\ddot{\phi}_x + m_3\ddot{\psi}_x$$

$$\delta v:\ \frac{\partial M_{xy}^{(0)}}{\partial x} + \frac{\partial M_{yy}^{(0)}}{\partial y} - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial M_{zx}^{(0)}}{\partial x} + \frac{\partial M_{yz}^{(0)}}{\partial y}\right) + F_y^{(0)} - \frac{1}{2}\frac{\partial c_z^{(0)}}{\partial x} = m_0\ddot{v} + m_1\ddot{\theta}_y + m_2\ddot{\phi}_y + m_3\ddot{\psi}_y$$

$$\begin{aligned}\delta w:\ & \frac{\partial}{\partial x}\left(\frac{\partial w}{\partial x}M_{xx}^{(0)} + \frac{\partial w}{\partial y}M_{xy}^{(0)}\right) + \frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}M_{xy}^{(0)} + \frac{\partial w}{\partial y}M_{yy}^{(0)}\right) + \frac{\partial M_{xz}^{(0)}}{\partial x} + \frac{\partial M_{yz}^{(0)}}{\partial y}\\ & - \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial M_{xx}^{(0)}}{\partial x} + \frac{\partial M_{xy}^{(0)}}{\partial y}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial M_{yy}^{(0)}}{\partial y} + \frac{\partial M_{xy}^{(0)}}{\partial x}\right) + F_z^{(0)} + \frac{1}{2}\left(\frac{\partial c_y^{(0)}}{\partial x} - \frac{\partial c_x^{(0)}}{\partial y}\right)\\ & = m_0\ddot{w} + m_1\ddot{\theta}_z + m_2\ddot{\phi}_z\end{aligned}$$

$$\begin{aligned}\delta\theta_x:\ & \frac{\partial M_{xx}^{(1)}}{\partial x} + \frac{\partial M_{xy}^{(1)}}{\partial y} - M_{xz}^{(0)} + \frac{1}{2}\left(\frac{\partial M_{xy}^{(0)}}{\partial x} + \frac{\partial M_{yy}^{(0)}}{\partial y} - \frac{\partial M_{zz}^{(0)}}{\partial y}\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial M_{xz}^{(1)}}{\partial x} + \frac{\partial M_{yz}^{(1)}}{\partial y}\right)\\ & + F_x^{(1)} + \frac{1}{2}c_y^{(0)} + \frac{1}{2}\frac{\partial c_z^{(1)}}{\partial y} = m_1\ddot{u} + m_2\ddot{\theta}_x + m_3\ddot{\phi}_x + m_4\ddot{\psi}_x\end{aligned}$$

$$\begin{aligned}\delta\theta_y:\ & \frac{\partial M_{xy}^{(1)}}{\partial x} + \frac{\partial M_{yy}^{(1)}}{\partial y} - M_{yz}^{(0)} - \frac{1}{2}\left(\frac{\partial M_{xx}^{(0)}}{\partial x} + \frac{\partial M_{xy}^{(0)}}{\partial y} - \frac{\partial M_{zz}^{(0)}}{\partial x}\right) - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial M_{xz}^{(1)}}{\partial x} + \frac{\partial M_{yz}^{(1)}}{\partial y}\right)\\ & + F_y^{(1)} - \frac{1}{2}c_x^{(0)} - \frac{1}{2}\frac{\partial c_z^{(1)}}{\partial x} = m_1\ddot{v} + m_2\ddot{\theta}_y + m_3\ddot{\phi}_y + m_4\ddot{\psi}_y\end{aligned}$$

$$\begin{aligned}\delta\phi_x:\ & \frac{\partial M_{xx}^{(2)}}{\partial x} + \frac{\partial M_{xy}^{(2)}}{\partial y} - 2M_{xz}^{(1)} + \left(\frac{\partial M_{xy}^{(1)}}{\partial x} + \frac{\partial M_{yy}^{(1)}}{\partial y} - \frac{\partial M_{zz}^{(1)}}{\partial y} - M_{yz}^{(0)}\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial M_{xz}^{(2)}}{\partial x} + \frac{\partial M_{yz}^{(2)}}{\partial y}\right)\\ & + F_x^{(2)} + c_y^{(1)} + \frac{1}{2}\frac{\partial c_z^{(2)}}{\partial y} = m_2\ddot{u} + m_3\ddot{\theta}_x + m_4\ddot{\phi}_x + m_5\ddot{\psi}_x\end{aligned}$$

$$\begin{aligned}\delta\phi_y:\ & \frac{\partial M_{xy}^{(2)}}{\partial x} + \frac{\partial M_{yy}^{(2)}}{\partial y} - 2M_{yz}^{(1)} - \left(\frac{\partial M_{xx}^{(1)}}{\partial x} + \frac{\partial M_{xy}^{(1)}}{\partial y} - \frac{\partial M_{zz}^{(1)}}{\partial x} - M_{xz}^{(0)}\right) - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial M_{xz}^{(2)}}{\partial x} + \frac{\partial M_{yz}^{(2)}}{\partial y}\right)\\ & + F_y^{(2)} - c_x^{(1)} - \frac{1}{2}\frac{\partial c_z^{(2)}}{\partial x} = m_2\ddot{v} + m_3\ddot{\theta}_y + m_4\ddot{\phi}_y + m_5\ddot{\psi}_y\end{aligned}$$

$$\begin{aligned}\delta\psi_x:\ & \frac{\partial M_{xx}^{(3)}}{\partial x} + \frac{\partial M_{xy}^{(3)}}{\partial y} - 3M_{xz}^{(2)} + \frac{3}{2}\left(\frac{\partial M_{xy}^{(2)}}{\partial x} + \frac{\partial M_{yy}^{(2)}}{\partial y} - \frac{\partial M_{zz}^{(2)}}{\partial y} - 2M_{yz}^{(1)}\right) + \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial M_{xz}^{(3)}}{\partial x} + \frac{\partial M_{yz}^{(3)}}{\partial y}\right)\\ & + F_x^{(3)} + \frac{3}{2}c_y^{(2)} + \frac{1}{2}\frac{\partial c_z^{(3)}}{\partial y} = m_3\ddot{u} + m_4\ddot{\theta}_x + m_5\ddot{\phi}_x + m_6\ddot{\psi}_x\end{aligned}$$

$$\begin{aligned}\delta\psi_y:\ & \frac{\partial M_{xy}^{(3)}}{\partial x} + \frac{\partial M_{yy}^{(3)}}{\partial y} - 3M_{yz}^{(2)} - \frac{3}{2}\left(\frac{\partial M_{xx}^{(2)}}{\partial x} + \frac{\partial M_{xy}^{(2)}}{\partial y} - \frac{\partial M_{zz}^{(2)}}{\partial x} - 2M_{xz}^{(1)}\right) - \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial M_{xz}^{(3)}}{\partial x} + \frac{\partial M_{yz}^{(3)}}{\partial y}\right)\\ & + F_y^{(3)} - \frac{3}{2}c_x^{(2)} - \frac{1}{2}\frac{\partial c_z^{(3)}}{\partial x} = m_3\ddot{v} + m_4\ddot{\theta}_y + m_5\ddot{\phi}_y + m_6\ddot{\psi}_y\end{aligned}$$

$$\delta\theta_z:\ \frac{\partial M_{xz}^{(1)}}{\partial x} + \frac{\partial M_{yz}^{(1)}}{\partial y} - M_{zz}^{(0)} - \frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial M_{xz}^{(1)}}{\partial x} + \frac{\partial M_{xy}^{(1)}}{\partial y}\right) + \frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial M_{yy}^{(1)}}{\partial y} + \frac{\partial M_{xy}^{(1)}}{\partial x}\right) + \frac{1}{2}\left(\frac{\partial M_{xz}^{(0)}}{\partial y} - \frac{\partial M_{yz}^{(0)}}{\partial x}\right)$$

$$+F_z^{(1)}+\frac{1}{2}\left(\frac{\partial c_y^{(1)}}{\partial x}-\frac{\partial c_x^{(1)}}{\partial y}\right)=m_1\ddot{w}+m_2\ddot{\theta}_z+m_3\ddot{\phi}_z$$

$$\delta\phi_z:\ \frac{\partial M_{xz}^{(2)}}{\partial x}+\frac{\partial M_{yz}^{(2)}}{\partial y}-2M_{zz}^{(1)}-\frac{1}{2}\frac{\partial}{\partial y}\left(\frac{\partial M_{xx}^{(2)}}{\partial x}+\frac{\partial M_{xy}^{(2)}}{\partial y}\right)+\frac{1}{2}\frac{\partial}{\partial x}\left(\frac{\partial M_{yy}^{(2)}}{\partial y}+\frac{\partial M_{xy}^{(2)}}{\partial x}\right)+\frac{\partial M_{xz}^{(1)}}{\partial y}-\frac{\partial M_{yz}^{(1)}}{\partial x}$$

$$+F_z^{(2)}+\frac{1}{2}\left(\frac{\partial c_y^{(2)}}{\partial x}-\frac{\partial c_x^{(2)}}{\partial y}\right)=m_2\ddot{w}+m_3\ddot{\theta}_z+m_4\ddot{\phi}_z \tag{19}$$

where the superposed dot on a variable indicates time derivative, for example, $\dot{u}=\partial u/\partial t$, $m_i(i=0,1,2,\cdots,6)$ are the mass moments of inertia

$$m_i=\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho(z)^i\,dz \tag{20}$$

$M_{ij}^{(k)}$ are the stress resultants

$$M_{ij}^{(k)}=\int_{-\frac{h}{2}}^{\frac{h}{2}}(z)^k\sigma_{ij}\,dz,\ \mathcal{M}_{ij}^{(k)}=\int_{-\frac{h}{2}}^{\frac{h}{2}}(z)^k m_{ij}\,dz,\ (k=0,1,2,3) \tag{21}$$

and $(\bar{f}_x, \bar{f}_y, \bar{f}_z)$ are the body forces (measured per unit volume), $(\bar{t}_x, \bar{t}_y, \bar{t}_z)$ the surface forces (measured per unit area) on S, and (q_x^t, q_y^t, q_z^t) the distributed forces (measured per unit area) on Ω^+, (q_x^b, q_y^b, q_z^b) the distributed forces (measured per unit area) on Ω^-, and $(\bar{c}_x, \bar{c}_y, \bar{c}_z)$ be the body couples (measured per unit volume) in the (x, y, z) coordinate directions. Additional details can be found in [15].

The general third-order theory developed herein contains all of the existing plate theories but some of them have not been extended to contain the microstructure parameters and the vonKarman nonlinearity. They are summarized in the recent paper by Reddy and Kim [15].

CONCLUSIONS

A general third-order theory of functionally graded plates with microstructure-dependent length scale parameter and the von-Karman nonlinearity is presented. The theory accounts for temperature dependent properties of the constituents in the functionally graded material, and modified couple stress theory is used to bring a microstructural length scale parameter. The equations of motions and associated force boundary conditions are derived using Hamilton's principle. The theory developed contains 11 generalized displacements. The existing plate theories, namely, a third-order theory with vanishing surface tractions, the Reddy third-order plate theory [7], the first-order plate theory, and the classical plate theory can be obtained as special cases of the developed general third-order plate theory. Three-dimensional constitutive relations must be used, consistent with the three-dimensional strain field, to develop plate constitutive relations. More complete development is given in the forthcoming paper [15].

The general third-order theory and its special cases developed herein can be used to construct finite element models of functionally graded plates with geometric nonlinearity and microstructure dependent length scale parameter. For the general case, the finite element models allow C^0-approximation of all 11 generalized displacements. The third-order plate theories with vanishing surface tractions require C^0 interpolation of $(u, v, \theta_x, \theta_y)$ and Hermite interpolation of w, θ_z, and ϕ_z. Computational models and their applications of some of the theories presented here are yet to appear. Also, analytical (e. g., Navier) solutions based on the linear theories may be obtained.

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REFERENCES

[1] Yang. F., Chong, A. C. M., Lam, D. C. C., and Tong, P., "Couple stress based strain gradient theory for elasticity." *International Journal of Solids and Structures*, **39**, 2002, 2731-2743.

[2] Park, S.K. and Gao, X-L., "Bernoulli-Euler beam model based on a modified couple stress theory." *Journal of Micromechanics and Microengineering*, **16**, 2006, 2355-2359.

[3] Ma, H.M., Gao, X-L., and Reddy, J.N., "A Microstructure-dependent Timoshenko beam model based on a modified couple stress theory." *Journal of the Mechanics and Physics of Solids*, **56**, 2008, 3379-3391.

[4] Ma, H.M., Gao, X-L., and Reddy, J.N., "A nonclassical Reddy-Levinson beam model based on a modified couple stress theory." *International Journal for Multiscale Computational Engineering*, **8**(2), 2010, 167-180.

[5] Ma, H.M., Gao, X-L., and Reddy, J.N., "A non-classical Mindlin plate model based on a modified couple stress theory." *Acta Mechanica* 2011, DOI 10.1007/s00707-011-0480-4.

[6] Reddy, J.N., "Microstructure-dependent couple stress theories of functionally graded beams." *Journal of Mechanics and Physics of Solids*, 59, 2011, 2382-2399.

[7] Reddy, J.N., "A simple higher-order theory for laminated composite plates." *J Appl Mech*, **51**, 1984, 745-752.

[8] Reddy, J.N., "A refined nonlinear theory of plates with transverse shear deformation." *Int J Solids Struct*, 20, 1984, 881-896.

[9] Reddy, J.N., "A small strain and moderate rotation theory of laminated anisotropic plates." *J Appl Mech*, **54**, 1987, 623-626.

[10] Reddy, J.N., "A general non-linear third-order theory of plates with moderate thickness." *Int J Non-Linear Mech*, **25**(6), 1990, 677-686.

[11] Reddy, J.N., "A general nonlinear third-order theory of functionally graded plates." *Int. J. Aerospace and Lightweight Structures*, **1**(1), 2011, 1-21.

[12] Bose, P. and Reddy, J.N., "Analysis of composite plates using various plate theories, part 1: formulation and analytical results." *Struct Engng Mech*, **6**(6), 1998, 583-612.

[13] Praveen, G.N. and Reddy, J.N., "Nonlinear transient thermoelastic analysis of functionally graded ceramic-metal plates." *Journal of Solids and Structures*, **35**(33), 1998, 4457-4476.

[14] Reddy, J.N., *Energy Principles and Variational Methods in Applied Mechanics*, 2nd ed, John Wiley & Sons, New York, 2002.

[15] Reddy, J.N. and Kim, J., "A nonlinear modified couple stress-based third-order theory of functionally graded plates." *Composite Structures*, **94**, 2012, 1128-1143.

SOME CONCEPTUAL ISSUES IN THE MODELLING OF CRACKED BEAMS FOR LATERAL-TORSIONAL BUCKLING ANALYSIS

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KEYWORDS

Lateral-torsional buckling, kirchhoff-clebsch theory, connection, crack, non-conservative loading, stability, uniform moment, variational and energy method, spring models

ABSTRACT

This paper is focused on the lateral-torsional buckling of cracked or weakened elastic beams. The crack is modelled with a generalized elastic connection law, whose equivalent stiffness parameters can be derived from fracture mechanics considerations. The same type of generalised spring model can be used for beams with semi-rigid connections, typically in the field of steel or timber engineering. As the basis for the present investigation, we consider a strip beam with fork end supports and exhibiting a single vertical edge crack, subjected to uniform bending in the plane of greatest flexural rigidity. The effect of prebuckling deformation is taken into consideration within the framework of the Kirchhoff-Clebsch theory. First, the three-dimensional elastic connection law adopted is a direct extension of the planar case, but this leads to a paradoxical conclusion: the critical moment is not affected by the presence of the crack, regardless of its location. It is shown that the above paradox is due to the non-conservative nature of the connection model adopted. Simple alternatives to this cracked-section constitutive law are proposed, based on conservative moment-rotation laws (quasi-tangential and semi-tangential) and consistent variational arguments.

INTRODUCTION

The numerous investigations devoted to the buckling of cracked elastic structures have so far focused mainly on the flexural buckling behaviour of columns -*e.g.*, [1, 2]. For such an in-plane analysis, the crack may be reasonably modelled by a simple elastic rotational spring with an "equivalent stiffness", as suggested by Okamura *et al.* [1]. The fundamental constitutive law of the cracked cross-section is therefore expressed as

$$M = k\Delta\theta, \tag{1}$$

where M is the bending moment acting at the cracked section, k is the equivalent stiffness and $\Delta\theta$ is the relative rotation (slope difference) occurring at the cracked cross-section.

In order to tackle out-of-plane buckling problems (*e.g.*, the lateral-torsional buckling of beams— see Figure 1), it is necessary to generalise the above constitutive law. The most straightforward generalisation corresponds to the diagonal relation

$$\begin{bmatrix} M_1 \\ M_2 \\ M_3 \end{bmatrix} = \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \begin{bmatrix} \Delta\theta_1 \\ \Delta\theta_2 \\ \Delta\theta_3 \end{bmatrix}, \tag{2}$$

where M_1 is the torsional moment, M_2 is the out-of-plane (minor-axis) bending moment, M_3 is the in-plane (major-axis) bending moment, the $\Delta\theta_i$ $(i=1, 2, 3)$ are the relative rotations, associated with each direction, occurring at the cracked cross-section and each k_i is the stiffness relating a relative rotation with the corresponding moment. The cracked cross-section constitutive law can be further generalised by considering off-diagonal (coupling) terms, making it possible to take into account the effects of anisotropy and crack orientation—see, for instance, Wang *et al.* [3] who address the closely related problem of beam vibration.

The amount of work dealing with the lateral-torsional buckling of cracked beams is rather scarce. Carloni *et al.* [4] investigated the lateral-torsional buckling of cracked I-beams under uniform bending, but restricted the constitutive law to the torsional term, *i.e.*,

$$M_1 = k_1\Delta\theta_1. \tag{3}$$

Karaagac *et al.* [5] studied the lateral-torsional buckling of a cracked cantilever beam submitted to a concentrated force, adopting Eq. (2) to describe the cracked cross-section constitutive behaviour—the buckling problem was solved by means of the finite element method. Finally, the authors are not aware of the publication of any closed-form solution concerning the lateral-torsional buckling behaviour of cracked beams.

The same type of spring model can be used for beams with semi-rigid connections. In the field of steel structures, the effect of semi-rigid connections on the out-of-plane behaviour of I-beams has been numerically assessed by several authors, such as Krenk and Damkilde [6] or more recently Basaglia *et al.* [7]. In order to include the effects of the warping restraint, the constitutive law (2) may be augmented to