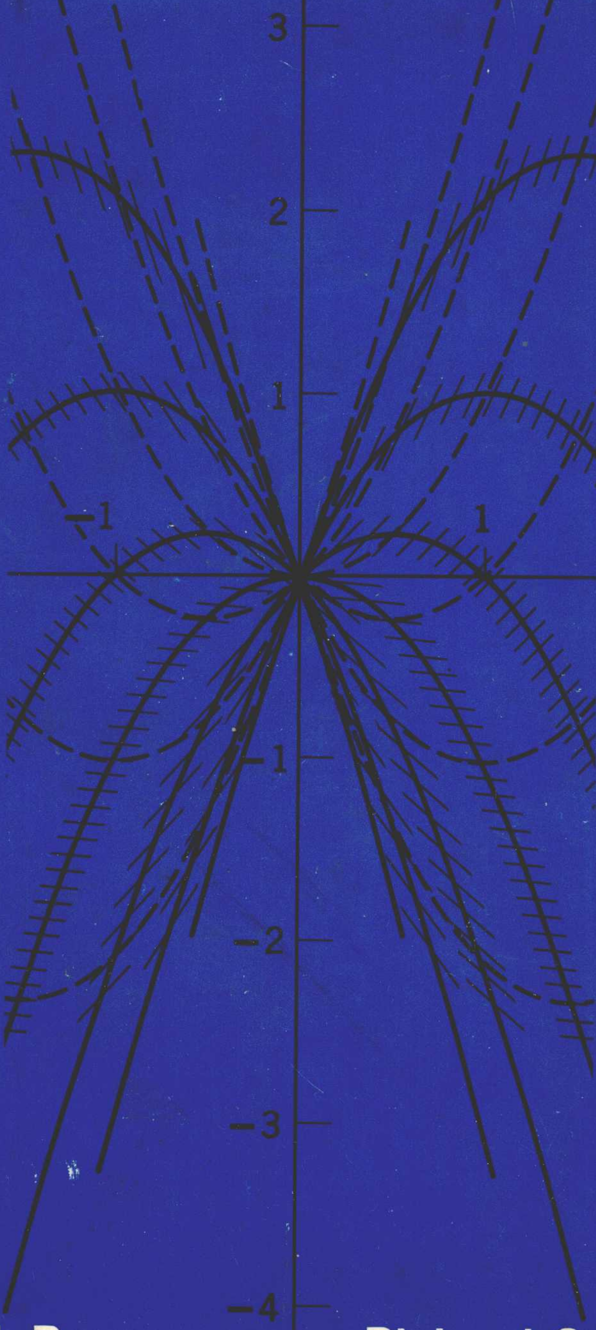
ELEMENTARY DIFFERENTIAL EQUATIONS AND BOUNDARY VALUE PROBLEMS



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Elementary Differential Equations and Boundary Value Problems

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Elementary Differential Equations and Boundary Value Problems

To My Mother, Ethel DiPrima,

and in Loving Memory

of My Father, Clyde DiPrima

Richard C. DiPrima

To My Mother, Marie S. Boyce,

and My Father, Edward G. Boyce

William E. Boyce

Preface

A course in elementary differential equations is an excellent vehicle through which to convey to the student a feeling for the interrelation between pure mathematics on the one hand, and the physical sciences or engineering on the other. Before the engineer can proceed confidently to use differential equations in his work, he must have at least a rudimentary knowledge of the basic theory, including some facts about the existence and uniqueness of solutions. Conversely, the student of pure mathematics often benefits greatly by a knowledge of some of the ways in which the necessity for solving specific problems has stimulated work of a more abstract nature.

This book is written from the intermediate viewpoint of the applied mathematician, whose interest in differential equations may at the same time be quite theoretical as well as intensely practical. We have therefore sought to combine a sound and accurate (but not particularly abstract) exposition of the elementary theory of differential equations with considerable material on methods of solution which have proven useful in a wide variety of applications. In keeping with this aim, several existence and uniqueness theorems have been carefully stated, and their meaning and significance discussed. In addition, we have stressed the distinctions between linear and nonlinear problems and between initial value and boundary value problems, describing in some detail the widely differing character of their solutions.

A beginning course, however, must usually be devoted primarily to methods of actually solving specific problems. We have given principal attention to those methods which are capable of broad application and which can be extended to problems beyond the range of our book. We emphasize that these methods have a systematic and orderly structure, and are not merely a miscellaneous collection of unrelated mathematical tricks.

Students frequently feel that courses and books on differential equations tend to be "cookbookish," and we have made a special effort to combat this tendency. Whenever possible, the student's prior knowledge is drawn upon to suggest a method of attack for a new type of problem. Moreover, the discussions of applications are almost always accompanied by an unusually detailed derivation of the relevant equations. In an elementary book we do not believe that terseness is a primary virtue, and in general we have striven for clarity rather than conciseness. We hope that whatever may have been lost in elegance has been regained in readability.

We have used most of the material in this book in one form or another in teaching the sophomore course on elementary differential equations which is required of almost all engineering and science majors at Rensselaer Polytechnic Institute. These students have completed a three-term (11-semester-hour) calculus sequence; however, most of the material is accessible to students following a somewhat briefer study of the calculus. No specific knowledge of advanced calculus, matrix algebra, or complex variables (other than the algebra of complex numbers) is required, although peripheral allusions to these topics occasionally occur.

While the scope of the book can better be judged from the Table of Contents, it may be worthwhile to note here some of its principal features. Chapter 1 is a short introduction; the most important topic is a discussion of the difference between explicit and implicit forms of solutions. Chapter 2 is concerned with first order equations. It proceeds from the premise that it is vital to grasp the distinction between linear and nonlinear equations. The first three sections are devoted to an elaboration of this distinction. Thereafter come a discussion of the standard elementary integration methods and a consideration of several applications, including the straight line motion of a body with variable mass. The chapter closes with a proof of the Picard theorem as an illustration of the type of analysis required in a study of existence theory.

Chapters 3 through 7 deal with linear ordinary differential equations. In Chapter 3 we discuss linear second order equations. The concepts of fundamental sets of solutions, linear independence, and superposition are emphasized, together with methods of solution. Examples are drawn from the fields of mechanical vibrations and electrical networks. The ideas of this chapter are extended to higher order linear equations in Chapter 5. In Chapter 4, on power series solutions, we show why the classification of points as ordinary, regular singular, or irregular singular is necessary and natural, rather than arbitrary. We use the Euler equation as a model for handling more general equations having a regular singular point. The treatment of the cases in which the roots of the indicial equation are equal, or differ by an integer, is more thorough than is customary, and is illustrated by appropriate forms of the Bessel equation. In Chapter 6 the Laplace transform is introduced, and its usefulness in solving initial value problems having piecewise continuous or impulsive forcing terms is emphasized. Chapter 7 deals with systems of first order linear equations, both from the point of view of elimination processes and from that of the construction of fundamental sets of solutions. In the latter connection the close relationship with single linear equations is brought out by writing solutions of systems as vectors.

In Chapter 8 there is an unusually careful treatment of discrete numerical techniques for solving initial value problems. Procedures ranging from the Euler tangent line method to the Runge-Kutta and Milne predictor-corrector methods are discussed and compared. A great deal of emphasis is given to the underlying principles governing numerical procedures and their refinement, and to a consideration of the kinds, sources, and control of errors.

The remaining chapters are concerned with boundary value problems, Fourier series, and partial differential equations. The basic theory of linear two-point boundary value problems, both homogeneous and nonhomogeneous, is outlined in Chapter 9. This material, while of fundamental importance, is rarely found in elementary books. Chapter 10 treats not only the elementary manipulative aspects of Fourier series, but also introduces in an intuitive manner the concepts of mean convergence and completeness. The final chapter deals with the classical partial differential equations of heat conduction, wave motion, and potential theory, and their solution by the method of separation of variables.

For classroom use we believe that this book has more than average flexibility. Beginning with Chapter 4 the chapters are substantially independent of each other, the only major exception being that Chapter 10 is required for Chapter 11. Furthermore, in most chapters the principal ideas are developed in the first few sections; the remaining ones are devoted to extensions and applications. Thus the instructor has maximum control over the selection and arrangement of course material, and over the depth to which the various topics are to be explored.

At Rensselaer we have found it possible to cover approximately the following material in a three-hour one-semester course:

Sections 1.1–1.2;
Sections 2.1–2.7 plus parts of Sections 2.9–2.11;
Sections 3.1–3.6.2 plus parts of Sections 3.7–3.8;
Sections 4.1–4.5;
Sections 5.1–5.3;
Sections 7.1–7.3;
Sections 8.1–8.5;
Sections 9.1–9.4;
Sections 10.1–10.4;
Sections 11.1–11.2.2.

A variety of other arrangements are also possible. In particular, if the students have had some exposure to differential equations in their calculus sequence, then some of the material in Chapters 2 and 3 could be dispensed with, and some of the later chapters taken up more thoroughly.

There are a large number of illustrative problems worked in full detail as examples. In addition, most sections are followed by problem sets for the student, with all answers collected at the end of the book. Many of the problems involve nothing more than routine calculations, while others require somewhat more analytical ability. Finally, some problems, indicated by an asterisk, are either quite difficult or else suggest extensions of the theory beyond the point reached in the text.

Several sections are also designated by asterisks. These contain material of a somewhat more advanced or difficult nature than the main text; for example, the proof of Picard's theorem in Section 2.12. This policy has been adopted so as not to interrupt the text with intricacies beyond the grasp or interest of the average reader, while keeping this material available for the unusually able or ambitious student. We have found these starred sections, together with the starred problems, of particular value in honors sections.

Finally, a word about the physical layout of the book may be in order. Sections are numbered in decimal form. Theorems, figures, etc., are numbered consecutively in each chapter. Thus, Theorem 3.7 is the seventh theorem in Chapter 3, but is not necessarily in Section 3.7. An author's name, followed by a number in square brackets, refers to the corresponding entry in the references at the end of the book.

It is a pleasure to express our grateful appreciation to a large number of people who have assisted us in various ways in bringing this manuscript into final form. We would particularly like to thank Professors Bernard Friedman, Peter Henrici, David Moskovitz, Lee Segel, Wolfgang Wasow, and Henry Zatskis, all of whom critically read the manuscript in draft form. Of these our special thanks go to Professor Moskovitz, who provided a remarkably detailed list of suggested improvements. Mr. Charles Haines also checked the manuscript in various stages and prepared answers to many of the problems, and Mr. Paul McGloin of the R.P.I Computing Laboratory assisted in carrying out the numerical computations involved in Chapter 8. Professor Roland Lichtenstein suggested the general form of the Appendix to Chapter 2. Any errors which remain despite the efforts of these individuals are, of course, the sole responsibility of the authors.

The formidable task of typing the manuscript was expertly carried out by Miss Elizabeth Doocey, with equally expert assistance from Mrs. Ursula Bischofberger and Mrs. Jean Ciccarelli. We are also grateful to the editorial and production staff of John Wiley and Sons, Inc. who ably met the requests put to them by not one, but two, authors. Finally, our

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Elementary Differential Equations
and Boundary Value Problems

CONTENTS

CHAPTER 1

Introduction

Many important and significant problems in engineering, the physical sciences, and the social sciences, when formulated in mathematical terms, require the determination of a function satisfying an equation containing derivatives of the unknown function. Such equations are called *differential equations*. Perhaps the most familiar example is Newton's law

$$m\frac{d^2x}{dt^2} = F \tag{1}$$

for the position $x(t)$ of a particle acted on by a force F. In general F will be a function of time t, the position x, and the velocity dx/dt. To determine the motion of a particle acted on by a given force F it is necessary to find a function $x(t)$ satisfying Eq. (1). If the force is that due to gravity, then $F = -mg$ and

$$m\frac{d^2x}{dt^2} = -mg. \tag{2}$$

On integrating Eq. (2) we have

$$\frac{dx}{dt} = -gt + c_1,$$

$$x(t) = -\tfrac{1}{2}gt^2 + c_1t + c_2, \tag{3}$$

where c_1 and c_2 are constants. To determine $x(t)$ completely it is necessary to specify two additional conditions, such as the position and velocity of the particle at some instant of time. These conditions can be used to determine the constants c_1 and c_2.

In developing the theory of differential equations in a systematic manner it is helpful to classify different types of equations. One of the more obvious classifications is based on whether the unknown function depends on a single independent variable or on several independent variables. In the first case only ordinary derivatives appear in the differential equation and it is said to be an *ordinary differential equation*. In the second case the derivatives are partial derivatives and the equation is called a *partial differential equation.*

Two examples of ordinary differential equations, in addition to Eq. (1), are

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = E(t), \qquad Q = Q(t), \tag{4}$$

for the charge Q on a condenser in a circuit with capacitance C, resistance R, inductance L, and impressed voltage $E(t)$, and the equation governing the decay with time of an amount R of a radioactive substance, such as radium,

$$\frac{dR}{dt} = -kR, \qquad R = R(t), \tag{5}$$

where k is a known constant. Typical examples of partial differential equations are Laplace's (1749–1827) or the potential equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \qquad u = u(x, y); \tag{6}$$

the diffusion or heat equation

$$\alpha^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}, \qquad u = u(x, t); \tag{7}$$

and the wave equation

$$a^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}, \qquad u = u(x, t). \tag{8}$$

Here α^2 and a^2 are certain constants. The potential equation, the diffusion equation, and the wave equation arise in a variety of problems in the fields of electricity and magnetism, elasticity, and fluid mechanics. Each is typical of distinct physical phenomena (note the names), and each is representative of a large class of partial differential equations. While we will primarily be concerned with ordinary differential equations we will also consider partial differential equations, in particular the three important equations just mentioned.

1.1 ORDINARY DIFFERENTIAL EQUATIONS

The *order* of an ordinary differential equation is the order of the highest derivative that appears in the equation. Thus Eqs. (1) and (4) of the previous section are second order ordinary differential equations, and Eq. (5) is a first order ordinary differential equation. More generally, the equation

$$F(x, y, y', y'', \ldots, y^{(n)}) = 0 \tag{1}$$