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Daniel B. Shapiro

Compositions of Quadratic Forms

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Compositions of Quadratic Forms

by

Daniel B. Shapiro

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Dedicated to Amanda, Becky and Jacob

Introduction

This book addresses basic questions about compositions of quadratic forms in the sense of Hurwitz and Radon. The initial question is: For what dimensions can they exist? Subsequent questions involve classification and analysis of the quadratic forms which can occur in a composition.

This topic originated with the "1, 2, 4, 8 Theorem" concerning formulas for a product of two sums of squares. That theorem, proved by Adolf Hurwitz in 1898, was generalized in various ways during the folowing century, leading to the theories discussed here. This area is worth studying because it is so centrally located in mathematics: these compositions have close connections with mathematical history, algebra, combinatorics, geometry, and topology.

Compositions have deep historical roots: the 1, 2, 4, 8 Theorem settled a long standing question about the existence of "n-square identities" and exhibited some of the power of linear algebra. Compositions are also entwined with the nineteenth century development of quaternions, octonions and Clifford algebras.

Another attraction of this subject is its fascinating relationship with Clifford algebras and the algebraic theory of quadratic forms. A general composition formula involves arbitrary quadratic forms over a field, not just the classical sums of squares. Such compositions can be reformulated in terms of Clifford algebras and their involutions. There is also a close connection between the forms involved in compositions and the multiplicative quadratic forms introduced by Pfister in the 1960s.

All the known constructions of composition formulas for sums of squares can be achieved using integer coefficients. A composition formula with integer coefficients can be recast as a combinatorial object: a special sort of matrix of symbols and signs. These "intercalate" matrices have been studied intensively, leading to a classification of the integer compositions which involve at most 16 squares.

Finally this topic is connected with certain deep questions in geometry. For instance, composition formulas provide examples of vector bundles on projective spaces, of independent vector fields on spheres, of immersions of projective spaces into euclidean spaces, and of Hopf maps between euclidean spheres. The topological tools developed to analyze these topics also yield results about real compositions.

Let us now describe the original question with more precision: A composition formula of size $[r, s, n]$ is a sum of squares formula of the type

$$(x_1^2 + x_2^2 + \cdots + x_r^2) \cdot (y_1^2 + y_2^2 + \cdots + y_s^2) = z_1^2 + z_2^2 + \cdots + z_n^2$$

where $X = (x_1, x_2, \dots, x_r)$ and $Y = (y_1, y_2, \dots, y_s)$ are systems of indeterminates and each $z_k = z_k(X, Y)$ is a bilinear form in X and Y. Such a formula can be viewed in several different ways, with each version providing different insights and techniques. Hurwitz restated the formula as a system of r different $n \times s$ matrices. More geometrically (assuming that the z_k's have real coefficients), the formula becomes a bilinear pairing

$$f : \mathbb{R}^r \times \mathbb{R}^s \longrightarrow \mathbb{R}^n$$

which satisfies the norm condition: $|f(x, y)| = |x| \cdot |y|$ for $x \in \mathbb{R}^r$ and $y \in \mathbb{R}^s$. For example the usual multiplication of complex numbers provides a formula of size $[2, 2, 2]$. In the original sums-of-squares language, this bilinear pairing becomes the formula:

$$(x_1^2 + x_2^2)\cdot(y_1^2 + y_2^2) = z_1^2 + z_2^2 \quad \text{where } z_1 = x_1y_1 - x_2y_2 \text{ and } z_2 = x_1y_2 + x_2y_1.$$

The quaternion and octonion algebras, discovered in the 1840s, provide similar formulas of sizes $[4, 4, 4]$ and $[8, 8, 8]$. Using his matrix formulation Hurwitz (1898) proved that a formula of size $[n, n, n]$ exists if and only if n is 1, 2, 4 or 8. Hurwitz and Radon used similar techniques to determine exactly when formulas of size $[r, n, n]$ can exist. It is far more difficult to analyze compositions of sizes $[r, s, n]$ when $r, s < n$.

These ideas have been generalized in two main directions, determining the contents of the two parts of this book.

Part I: If the composition involves general quadratic forms over a field in place of the sums of squares, what can be said about those forms? Interesting results have been obtained for the classical sizes $[r, n, n]$.

Part II: What sizes r, s, n are possible in the general case? Does the answer depend on the field of coefficients? Many partial results have been obtained using methods of algebraic topology, combinatorics, linear algebra and geometry.

Further descriptions of the historical background and the contents of this work appear in Chapter 0 and in the Introduction to Part II.

Readers of this work are expected to have knowledge of some abstract algebra. The first two chapters assume familiarity with only the basic properties of linear algebra and inner product spaces. The next five chapters require quadratic forms, Clifford algebras, central simple algebras and involutions, although many of those concepts are developed in the text. For example, Clifford algebras are defined and their basic properties are established in Chapter 3. Later chapters assume further background. For example Chapter 11 uses algebraic number theory and Chapter 12 employs algebraic topology.

Each chapter begins with a brief statement of its content and ends with some exercises, usually involving alternative methods or related results. In fact many related topics and open questions have been converted to exercises. This practice lengthens the exercise sections, but adds some further depth to the book. The Notes at the end of each chapter provide additional comments, historical remarks and references. At the end of the book there is a fairly extensive bibliography, arranged alphabetically by first author.

Most of the material described in this book has already appeared in the mathematical literature, usually in research papers. However there are many items that have not been previously published. These include:

- an improved version of the Eigenspace Lemma (2.10);
- a discussion of anti-commuting skew-symmetric matrices, Exercise 2.13;
- the trace methods used to analyze (2, 2)-families, Chapter 5;
- the treatment of composition algebras, Chapter 1.A (due to Conway);
- the analysis of "minimal" pairs, Chapter 7;
- properties of the topological space of all compositions, Chapter 8;
- monotopies and isotopies, Chapter 8 (due to Conway);
- the matrix approach to Pfaffians, Chapter 10;
- Hasse principle for divisibility, Chapter 11.A (due to Wadsworth);
- general monomial compositions, Chapter 13.B;
- the characterization of all compositions of codimension 2, (14.18);
- nonsingular and surjective bilinear pairings over fields, Exercises 14.16–19.

This book evolved over many years, starting from series of lectures I gave on this subject at the Universität Regensburg (Germany) in 1977, at the Universidad de Chile in 1981, at the University of California-Berkeley in 1983, at the Universität Dortmund (Germany) in 1991, at the Universidad de Talca (Chile) in 1999 and several times at the Ohio State University. I am grateful to these institutions, to the National Science Foundation, to the Alexander von Humboldt Stiftung and to the Fundación Andes for their generous support. It is also a pleasure to thank many friends and colleagues for their interest in this work and their encouragement over the years. Special thanks are due to several colleagues who have made observations directly affecting this book. These include J. Adem, R. Baeza, E. Becker, A. Geramita, J. Hsia, I. Kaplansky, M. Knebusch, K. Y. Lam, T. Y. Lam, D. Leep, T. Smith, M. Szyjewski, J.-P. Tignol, A. Wadsworth, P. Yiu, and S. Yuzvinsky. Extra thanks are due to Adrian Wadsworth for providing great help and support in the early years of my mathematical career.

I am also grateful to those colleagues and students who have proofread sections of this book, finding errors and making worthwhile suggestions. However I take full responsibility for the remaining grammatical and mathematical errors, the incorrect cross references, the inconsistencies of notation and the gaps in understanding.

As mentioned above, this book has been in progress for many years. In fact it is hard for me to believe how long it has been. The writing was finally finished in 1998, barely in time to celebrate the centennial of the Hurwitz 1, 2, 4, 8 Theorem.

Table of Contents

Part II. Compositions of Size $[r, s, n]$

Chapter 0

Historical Background

The theory of composition of quadratic forms over fields had its start in the 19th century with the search for n-square identities of the type

$$(x_1^2 + x_2^2 + \cdots + x_n^2) \cdot (y_1^2 + y_2^2 + \cdots + y_n^2) = z_1^2 + z_2^2 + \cdots + z_n^2$$

where $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ are systems of indeterminates and each $z_k = z_k(X, Y)$ is a bilinear form in X and Y. For example when $n = 2$ there is the ancient identity

$$(x_1^2 + x_2^2) \cdot (y_1^2 + y_2^2) = (x_1y_1 + x_2y_2)^2 + (x_1y_2 - x_2y_1)^2.$$

In this example $z_1 = x_1y_1 + x_2y_2$ and $z_2 = x_1y_2 - x_2y_1$ are bilinear forms in X, Y with integer coefficients. This formula for $n = 2$ can be interpreted as the "law of moduli" for complex numbers: $|\alpha| \cdot |\beta| = |\alpha\beta|$ where $\alpha = x_1 - ix_2$ and $\beta = y_1 + iy_2$.

A similar 4-square identity was found by Euler (1748) in his attempt to prove Fermat's conjecture that every positive integer is a sum of four integer squares. This identity is often attributed to Lagrange, who used it (1770) in his proof of that conjecture of Fermat. Here is Euler's formula, in our notation:

$$(x_1^2 + x_2^2 + x_3^2 + x_4^2) \cdot (y_1^2 + y_2^2 + y_3^2 + y_4^2) = z_1^2 + z_2^2 + z_3^2 + z_4^2$$

where

$$z_1 = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$$

$$z_2 = x_1y_2 - x_2y_1 + x_3y_4 - x_4y_3$$

$$z_3 = x_1y_3 - x_2y_4 - x_3y_1 + x_4y_2$$

$$z_4 = x_1y_4 + x_2y_3 - x_3y_2 - x_4y_1.$$

After Hamilton's discovery of the quaternions (1843) this 4-square formula was interpreted as the law of moduli for quaternions. Hamilton's discovery came only after he spent years searching for a way to multiply "triplets" (i.e. triples of numbers) so that the law of moduli holds. Such a product would yield a 3-square identity. Already in his *Théorie des Nombres* (1830), Legendre showed the impossibility of such an identity. He noted that 3 and 21 can be expressed as sums of three squares of rational numbers, but that $3 \times 21 = 63$ cannot be represented in this way. It follows that a 3-square identity is impossible (at least when the bilinear forms have rational coefficients). If Hamilton had known of this remark by Legendre he might have given up the search to multiply triplets! Hamilton's great insight was to move on to four dimensions and to allow a non-commutative multiplication.

Hamilton wrote to John Graves about the discovery of quaternions in October 1843 and within two months Graves wrote to Hamilton about his discovery of an algebra of "octaves" having 8 basis elements. The multiplication satisfies the law of moduli, but is neither commutative nor associative. Graves published his discovery in 1848, but Cayley independently discovered this algebra and published his results in 1845. Many authors refer to elements of this algebra as "Cayley numbers". In this book we use the term "octonions".

The multiplication of octonions provides an 8-square identity. Such an identity had already been found in 1818 by Degen in Russia, but his work was not widely read. After the 1840s a number of authors attempted to find 16-square identities with little success. It was soon realized that no 16-square identity with integral coefficients is possible, but the arguments at the time were incomplete. These "proofs" were combinatorial in nature, attempting to insert $+$ and $-$ signs in the entries of a 16×16 Latin square to make the rows orthogonal.

In 1898 Hurwitz published the definitive paper on these identities. He proved that there exists an n-square identity with complex coefficients if and only if $n = 1, 2, 4$ or 8. His proof involves elementary linear algebra, but these uses of matrices and linear independence were not widely known in 1898. At the end of that paper Hurwitz posed the general problem: For which positive integers r, s, n does there exist a "composition formula":

$$(x_1^2 + x_2^2 + \cdots + x_r^2) \cdot (y_1^2 + y_2^2 + \cdots + y_s^2) = z_1^2 + z_2^2 + \cdots + z_n^2$$

where $X = (x_1, x_2, \ldots, x_r)$ and $Y = (y_1, y_2, \ldots, y_s)$ are systems of indeterminates and each $z_k = z_k(X, Y)$ is a bilinear form in X and Y?

Here is an outline of Hurwitz's ideas, given without all the details. Suppose there is a composition formula of size $[r, s, n]$ as above. View X, Y and Z as column vectors. Then, for example, $z_1^2 + z_2^2 + \cdots + z_n^2 = Z^\top \cdot Z$, where the superscript $\top$ denotes the transpose. The bilinearity condition becomes $Z = AY$ where A is an $n \times s$ matrix whose entries are linear forms in X. The given composition formula can then be written as

$$(x_1^2 + x_2^2 + \cdots + x_r^2)Y^\top \cdot Y = Z^\top \cdot Z = Y^\top A^\top AY.$$

Since Y consists of indeterminates this equation is equivalent to

$$A^\top \cdot A = (x_1^2 + x_2^2 + \cdots + x_r^2)I_s,$$

where A is an $n \times s$ matrix whose entries are linear forms in X.

Of course I_s here denotes the $s \times s$ identity matrix. Since the entries of A are linear forms we can express $A = x_1A_1 + x_2A_2 + \cdots + x_rA_r$ where each A_i is an $n \times s$ matrix with constant entries. After substituting this expression into the equation and canceling like terms, we find:

> There are $n \times s$ matrices $A_1, A_2, \ldots, A_r$ over F satisfying
>
> $$A_i^\top \cdot A_i = I_s \quad \text{for } 1 \le i \le r,$$
> $$A_i^\top \cdot A_j + A_j^\top \cdot A_i = 0 \quad \text{for } 1 \le i, j \le r \text{ and } i \ne j.$$

This system is known as the "Hurwitz Matrix Equations". Such matrices exist if and only if there is a composition formula of size $[r, s, n]$.

Hurwitz considered these matrices to have complex entries, but his ideas work just as well using any field of coefficients, provided that the characteristic is not 2. Those matrices are square when $s = n$. In that special case the system of equations can be greatly simplified by defining the $n \times n$ matrices $B_i = A_1^{-1}A_i$ for $1 \le i \le r$. Then $B_1, \ldots, B_r$ satisfy the Hurwitz Matrix Equations and $B_1 = I_n$. It follows that:

> There are $n \times n$ matrices $B_2, \ldots, B_r$ over F satisfying:
>
> $$B_i^\top = -B_i, \quad B_i^2 = -I_n, \quad \text{for } 2 \le i \le r;$$
> $$B_iB_j = -B_jB_i \quad \text{whenever } i \ne j.$$

Such a system of $n \times n$ matrices exists if and only if there is a composition formula of size $[r, n, n]$. Hurwitz proved that the 2^{r-2} matrices $B_{i_1}B_{i_2}\ldots B_{i_k}$ for $2 \le i_1 \le \cdots \le i_k \le r-1$ are linearly independent. This shows that $2^{r-2} \le n^2$ and in the case of n-square identities (when $r = n$) quickly leads to the "1, 2, 4, 8 Theorem".

In 1922 Radon determined the exact conditions on r and n for such a system of matrices to exist over the real field $\mathbb{R}$. This condition had been found independently by Hurwitz for formulas over the complex field $\mathbb{C}$ and was published posthumously in 1923. They proved that:

A formula of size $[r, n, n]$ exists if and only if $r \le \rho(n)$,

where the "Hurwitz–Radon function" $\rho(n)$ is defined as follows: if $n = 2^{4a+b}n_0$ where n_0 is odd and $0 \le b \le 3$, then $\rho(n) = 8a + 2^b$. There are several different ways this function can be described. The following one is the most convenient for our purposes:

$$\text{If } n = 2^m n_0 \text{ where } n_0 \text{ is odd then } \rho(n) = \begin{cases} 2m+1 & \text{if } m \equiv 0, \\ 2m & \text{if } m \equiv 1, \\ 2m & \text{if } m \equiv 2, \\ 2m+2 & \text{if } m \equiv 3 \end{cases} \quad \text{(mod 4)}.$$

For example, $\rho(n) = n$ if and only if $n = 1, 2, 4$ or 8, as expected from the earlier theorem of Hurwitz. Also $\rho(16) = 9$, $\rho(32) = 10$, $\rho(64) = 12$ and generally $\rho(16n) = 8 + \rho(n)$. New proofs of the Hurwitz–Radon Theorem for compositions of size $[r, n, n]$ were found in the 1940s. Eckmann (1943b) applied the representation theory of certain finite groups to prove the theorem over $\mathbb{R}$, and Lee (1948) modified Eckmann's ideas to prove the result using representations of Clifford algebras. Independently, Albert (1942a) generalized the 1, 2, 4, 8 Theorem to quadratic forms over arbitrary fields, and Dubisch (1946) used Clifford algebras to prove the Hurwitz–Radon Theorem for quadratic forms over $\mathbb{R}$ (allowing indefinite forms). Motivated by a problem in geometry, Wong (1961) analyzed the Hurwitz–Radon Theorem using matrix methods and classified the types of solutions over $\mathbb{R}$. In the 1970s Shapiro proved the Hurwitz–Radon Theorem for arbitrary (regular) quadratic forms over any field where $2 \neq 0$, and investigated the quadratic forms which admit compositions. One goal of our presentation is to explain the curious periodicity property of the Hurwitz–Radon function $\rho(n)$:

$$\text{Why does } \rho(2^m) \text{ depend only on } m \pmod 4?$$

The explanation comes from the shifting properties of (s, t)-families as explained in Chapter 2.

Here are some of the questions which have motivated much of the work done in Part I of this book. Suppose σ and q are regular quadratic forms over the field F, where $\dim \sigma = s$ and $\dim q = n$. Then σ and q "admit a composition" if there is a formula

$$\sigma(X)q(Y) = q(Z),$$

where as usual $X = (x_1, x_2, \dots, x_s)$ and $Y = (y_1, y_2, \dots, y_n)$ are systems of indeterminates and each z_k is a bilinear form in X and Y, with coefficients in F. The quadratic forms involved in these compositions are related to Pfister forms.

In the 1960s Pfister found that for every m there do exist 2^m-square identities, provided some denominators are allowed. He generalized these identities to a wider class: a quadratic form is a *Pfister form* if it expressible as a tensor product of binary quadratic forms of the type $\langle 1, a\rangle$. In particular its dimension is 2^m for some m. Here we use the notation $\langle a_1, \dots, a_n\rangle$ to stand for the n-dimensional quadratic form $a_1x_1^2 + \cdots + a_nx_n^2$.

Theorem (Pfister). *If φ is a Pfister form and X, Y are systems of indeterminates, then there is a multiplication formula*

$$\varphi(X)\varphi(Y) = \varphi(Z),$$

where each component $z_k = z_k(X, Y)$ is a linear form in Y with coefficients in the rational function field $F(X)$. Conversely if φ is an anisotropic quadratic form over F satisfying such a multiplication formula, then φ must be a Pfister form.

The theory of Pfister forms is described in the textbooks by Lam (1973) and Scharlau (1985). When $\dim \varphi = 1, 2, 4$ or 8, such a multiplication formula exists using no denominators, since the Pfister forms of those sizes are exactly the norm forms of composition algebras. But if $\dim \varphi = 2^m > 8$, Hurwitz's theorem implies that any such formula must involve denominators. Examples of such formulas can be written out explicitly (see Exercise 5).

The quadratic forms appearing in the Hurwitz–Radon composition formulas have a close relationship to Pfister forms. For any Pfister form φ of dimension 2^m there is an explicit construction showing that φ admits a composition with some form σ having the maximal dimension $\rho(2^m)$. The converse is an interesting open question.

Pfister Factor Conjecture. Suppose q is a quadratic form of dimension 2^m, and q admits a composition with some form of the maximal dimension $\rho(2^m)$. Then q is a scalar multiple of a Pfister form.

This conjecture is one of the central themes driving the topics chosen for the first part of the book. In Chapter 9 it is proved true when $m \leq 5$, and for larger values of m over special classes of fields.

The second part of this book focuses on the more general compositions of size $[r, s, n]$. In 1898 Hurwitz already posed the question: Which sizes are possible? The cases where $s = n$ were settled by Hurwitz and Radon in the 1920s. Further progress was made around 1940 when Stiefel and Hopf applied techniques of algebraic topology to the problem, (for compositions over the field of real numbers). In Part II we discuss these topological arguments and their generalizations, as well as considering the question for more general fields of coefficients. Further details are described in the Introduction to Part II.

Exercises for Chapter 0

Note: For the exercises in this book, most of the declarative statements are to be proved. This avoids writing "prove that" in every problem.

1. In any (bilinear) 4-square identity, if $z_1 = x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4$ then z_2, z_3, z_4 must be skew-symmetric. (Compare 4-square identity of Euler above.)