THEORY AND DECISION LIBRARY / VOLUME 31

PETER C. FISHBURN

THE FOUNDATIONS OF EXPECTED UTILITY

D. REIDEL PUBLISHING COMPANY

DORDRECHT : HOLLAND / BOSTON : U.S.A. / LONDON : ENGLAND

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Library of Congress Cataloging in Publication Data

Fishburn, Peter C.
The foundations of expected utility.

(Theory and decision library; v. 31)
Bibliography: p.
Includes indexes.
1. Utility theory. I. Title. II. Series.
HB201.F54 1982 330.15'7 82-9138
ISBN 90-277-1420-7 AACR2

Published by D. Reidel Publishing Company,
P.O. Box 17, 3300 AA Dordrecht, Holland.

Sold and distributed in the U.S.A. and Canada
by Kluwer Boston Inc.,
190 Old Derby Street, Hingham, MA 02043, U.S.A.

In all other countries, sold and distributed
by Kluwer Academic Publishers Group,
P.O. Box 322, 3300 AH Dordrecht, Holland.

D. Reidel Publishing Company is a member of the Kluwer Group.

Printed in The Netherlands

PREFACE

This book offers a unified treatment of my research in the foundations of expected utility theory from around 1965 to 1980. While parts are new, the presentation draws heavily on published articles and a few chapters in my 1970 monograph on utility theory. The diverse notations and styles of the sources have of course been reconciled here, and their topics arranged in a logical sequence.

The two parts of the book take their respective cues from the von Neumann–Morgenstern axiomatization of preferences between risky options and from Savage's foundational treatment of decision making under uncertainty. Both parts are studies in the axiomatics of preferences for decision situations and in numerical representations for preferences. Proofs of the representation and uniqueness theorems appear at the ends of the chapters so as not to impede the flow of the discussion.

A few warnings on notation are in order. The numbers for theorems cited within a chapter have no prefix if they appear in that chapter, but otherwise carry a chapter prefix (Theorem 3.2 is Theorem 2 in Chapter 3). All lower case Greek letters refer to numbers in the closed interval from 0 to 1. The same symbol in different chapters has essentially the same meaning with one major exception: $x, y, \ldots$ mean quite different things in different chapters.

I am indebted to many people for their help and encouragement. Werner Leinfellner's generous invitation to contribute to the series in which this book appears was essential and is deeply appreciated. Fred Roberts and Peter Farquhar shared ideas that led to jointly-authored papers I have relied on in Chapter 2, 6, and 7. Ed Zajac provided the moral and organizational support on behalf of Bell Laboratories' management that enabled the book to be written, and Janice Ivanitz did a truly superb job of typing the manuscript. My greatest debt is to Jimmie Savage, whose influence is beyond reckoning.

For the record, I would like to acknowledge the works I had a part in that served as source material for the book. Complete references are given here only for papers not cited later: F(xy) signifies Fishburn (19xy) in the References. Chapter 2 is based in part on F(70, Chapter 8) and Fishburn and Roberts (1978). Chapter 3 is based on F(67), F(70, Chapter 10), F(75a), and my 'Unbounded Utility Functions in Expected Utility Theory', *Quarterly Journal of Economic* **90** (1976), 163–168. Chapter 4 grew out of F(71a): its proofs have not appeared previously. Chapter 5 is based on F(71b); 'Alternative Axiomatizations of One-Way Expected Utility', *Annals of Mathematical Statistics* **43** (1972), 1648–1651; 'Bounded One-Way Expected Utility', *Econometrica* **43** (1975), 867–875; and 'A Note on Linear Utility', *Journal of Economic Theory* (1982). Chapter 6 relies on Fishburn and Farquhar (1979); 'Independence in Utility Theory with Whole Product Sets', *Operations Research* **13** (1965), 28–45; and 'Additive Representations of Real-Valued Functions on Subsets of Product Sets', *Journal of Mathematical Psychology* **8** (1971), 382–388. Chapters 7 and 8 devolved from F(76), Fishburn and Roberts (1978), and F(80).

In Part II, Chapter 9 is based on F(70, Chapter 13); 'Preference-Based Definitions of Subjective Probability', *Annals of Mathematical Statistics* **38** (1967), 1605–1617; and 'Additivity in Utility Theory with Denumerable Product Sets', *Econometrica* **34** (1966), 500–503. Chapter 10 is also based on F(70, Chapter 13) as well as 'A General Theory of Subjective Probabilities and Expected Utilities', *Annals of Mathematical Statistics* **40** (1969), 1419–1429, and 'Subjective Expected Utility with Mixture Sets and Boolean Algebras', *Annals of Mathematical Statistics* **43** (1972), 917–927. Material in the first part of Chapter 11 was adapted from F(75b), and Chapter 12 was developed from F(73) and F(74).

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July 1981

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CHAPTER 1

INTRODUCTION

Early in the Eighteenth Century, the mathematicians Daniel Bernoulli and Gabriel Cramer (Bernoulli, 1738) argued that the maximization of expected profit or wealth could not adequately describe the choices of reasonable individuals among risky monetary options. Consider, for example, an individual who can invest a sum of money in one of two options, A and B. Option A is riskless and guarantees \$1000 profit, whereas B is a risky venture that yields either a \$2000 loss or a \$4200 profit, each with probability $\frac{1}{2}$. Despite the fact that B has a larger expected profit, a prudent individual may well prefer A to B. Based on related examples, Bernoulli and Cramer proposed that risky monetary options be evaluated not by their expected returns but rather by the expectations of the utilities of their returns. Although utility of money could be expected to increase in the amount, there is no compelling reason why it should be linear in the amount. In particular, if an individual's utility of wealth increases at a decreasing rate, then he will prefer some options to others that have higher expected returns but are also perceived to involve more risk.

Despite its early beginning, expected utility lay in relative obscurity until John von Neumann and Oskar Morgenstern axiomatized it for their theory of games more than two hundred years after Bernoulli's paper was published (von Neumann and Morgenstern, 1944). Several years earlier, Frank P. Ramsey outlined a theory of subjective probability and expected utility (Ramsey, 1931), but this went virtually unnoticed until the appearance of Leonard J. Savage's classic on the foundations of statistics (Savage, 1954). Drawing on Ramsey as well as von Neumann and Morgenstern for expected utility and de Finetti (1937) for subjective probability, Savage presented the first complete axiomatization of subjective expected utility, in which the notion of personal or subjective probability is integrated with expected utility.

Part I of the present work is devoted to the von Neumann–Morgenstern theory and to generalizations and extensions of their basic idea. Part II then considers subjective expected utility, showing how aspects of the theory in Part I can be used to derive representations of preferences that involve subjective probability. Further introductory comments and previews of the two parts of the book are provided in the remainder of this chapter.

1.1. PART I: EXPECTED UTILITY

The importance of the von Neumann–Morgenstern contribution lies in its derivation of a linear utility representation for preferences from simple, appealing axioms for a qualitative preference relation on a set of objects that is closed under an operation that resembles convex combinations. Although their formulation seems far removed from the description given above for Bernoulli and Cramer, I shall note shortly how the expected-utility form arises from the linear utility representation derived by von Neumann and Morgenstern.

Their axioms, which are presented in a slightly different form in Chapter 2, apply a binary relation $\succ$ ('is preferred to') to a set $\mathcal{M}$ that is closed under an operation on triples $(\lambda, x, y) \in [0, 1] \times \mathcal{M} \times \mathcal{M}$. We denote the element in $\mathcal{M}$ that results from the operation on (λ, x, y) by $\lambda x \oplus (1 - \lambda)y$. Appropriate assumptions about $\oplus$, along with the preference axioms for $\succ$ on $\mathcal{M}$, imply the existence of a real-valued function u on $\mathcal{M}$ that preserves $\succ$ and is linear:

$$x \succ y \quad \text{iff} \quad u(x) > u(y),$$

$$u(\lambda x \oplus (1 - \lambda)y) = \lambda u(x) + (1 - \lambda)u(y),$$

for all $x, y \in \mathcal{M}$ and all $\lambda \in [0, 1]$. The latter property, which says that u is linear in $\oplus$, should not be confused with the notion of a utility function on a real variable (such as money) that is a linear function of the variable. Although the abstract theory can be applied to cases in which $\mathcal{M}$ is a real variable and $\lambda x \oplus (1 - \lambda)y$ is the convex combination $\lambda x + (1 - \lambda)y$, interesting applications endow $\mathcal{M}$ with considerably more structure.

A case in point arises by taking $\mathcal{M}$ as the set $\mathscr{P}_0(\mathscr{C})$ of all simple probability measures on a set $\mathscr{C}$ of consequences or outcomes. By definition, $p \in \mathscr{P}_0(\mathscr{C})$ iff p maps $\mathscr{C}$ into $[0, 1]$ such that $p(c) = 0$ for all but a finite number of $c \in \mathscr{C}$, and $\Sigma p(c) = 1$. Let $\lambda p \oplus (1 - \lambda)q$ be the convex combination $\lambda p + (1 - \lambda)q$ of measures $p, q \in \mathscr{P}_0(\mathscr{C})$, so that this combination is the simple measure that assigns probability $\lambda p(c) + (1 - \lambda)q(c)$ to each $c \in \mathscr{C}$. Given the foregoing linear utility representation for $\succ$ on $\mathscr{P}_0(\mathscr{C})$, extend u from $\mathscr{P}_0(\mathscr{C})$ to $\mathscr{C}$ by defining the utility of consequence c to be the utility of the measure that assigns probability 1 to c:

$$u(c) = u(p) \quad \text{when} \quad p(c) = 1.$$

Then the linearity property $u(\lambda p + (1-\lambda)q) = \lambda u(p) + (1-\lambda)u(q)$ leads, by a simple inductive argument, to the expected-utility form

$$u(p) = \sum_{\mathscr{C}} p(c)u(c)$$

for all $p \in \mathscr{P}_0(\mathscr{C})$. Thus, this application of the basic theory yields a utility function on $\mathscr{C}$ such that p is preferred to q iff the expected utility of p exceeds the expected utility of q.

Although consequences in $\mathscr{C}$ are profits or wealths in the Bernoulli–Cramer context, elements in $\mathscr{C}$ could be anything. They might be real vectors, qualitative descriptions of the future, pure strategies or n-tuples of pure strategies in a game context, Savage acts, and so forth. Many of our later developments will be based on specialized $\mathscr{C}$ sets.

Preview

The first chapter in Part I presents two sets of axioms for the linear utility representation of von Neumann and Morgenstern. Both assume that $\mathscr{M}$ is a mixture set as defined by Herstein and Milnor (1953), but use somewhat different axioms for $\succ$ on $\mathscr{M}$. It is then shown how the basic theory can be generalized by replacing the equality relation in the mixture-set axioms by the symmetric complement $\sim$ of $\succ$, where $x \sim y$ means that neither $x \succ y$ nor $y \succ x$. The relation $\sim$ is often referred to as an indifference relation.

Chapter 3 extends the expected-utility form for simple probability measures to more general probability measures, with

$$u(p) = \int_{\mathscr{C}} u(c)\,\mathrm{d}p(c)$$

for all p in a set $\mathscr{P}$ of measures that includes $\mathscr{P}_0(\mathscr{C})$. New axioms, involving closure properties for $\mathscr{P}$ and dominance axioms for $\succ$ on $\mathscr{P}$, are used in the extension. Both finitely additive and countably additive measures are considered. The question of whether u on $\mathscr{C}$ must be bounded is also examined.

The axioms in Chapter 2 are ordering, independence and continuity conditions. Independence is primarily responsible for linearity, whereas the continuity or Archimedeam axiom ensures that utilities will be real numbers. Chapter 4 investigates the structure of preferences on $\mathcal{M}$ when the Archimedean axiom is omitted. A special condition on preference hierarchies leads to a quasilinear utility representation in which real-valued utilities $u(x)$ are replaced by utility vectors $(u_1(x), \ldots, u_n(x))$ whose lexicographic ordering preserves preference:

$$x \succ y \quad \text{iff} \quad u_1(x) > u_1(y) \quad \text{or} \quad [u_1(x) = u_1(y), u_2(x) > u_2(y)]$$
$$\text{or} \ldots \text{or} \quad [u_1(x) = u_1(y), \ldots, u_{n-1}(x) = u_{n-1}(y),$$
$$u_n(x) > u_n(y)].$$

Unlike the other chapters in Part I, Chapter 5 does not assume that the indifference relation $\sim$ is transitive, but it does presume that $\succ$ or its transitive closure is a partial order. Suitable independence and Archimedean axioms yield a 'one-way' linear utility representation in which $u(x) > u(y)$ whenever $x \succ y$, but not conversely. A lexicographic one-way representation arises when the Archimedean axiom is omitted.

The final three chapters of Part I involve specializations with Cartesian product sets. Chapter 6 begins with a linear u on $\mathcal{P}_0(\mathcal{C})$ and shows first that a simple marginal indfference condition is necessary and sufficient for the additive representation

$$u(c_1, c_2, \ldots, c_n) = \sum_{i=1}^{n} u_i(c_i)$$

for all $(c_1, \ldots, c_n) \in \mathcal{C}$ whenever $\mathcal{C}$ is a subset of a product set $\mathcal{C}_1 \times \mathcal{C}_2 \times \cdots \times \mathcal{C}_n$. We then consider $\mathcal{C} = \mathcal{D} \times \mathcal{E}$ and identify a necessary and sufficient condition for the multiadditive form

$$u(d, e) = \sum_{j=1}^{n} f_j(d) g_j(e) + h(d),$$

where the f_j and h are real-valued functions on $\mathcal{D}$, and the g_j are real-valued functions on $\mathcal{E}$.

Chapters 7 and 8 are concerned with a preference relation $\succ$ defined on a product of mixture sets $\mathcal{M}_1 \times \mathcal{M}_2 \times \cdots \times \mathcal{M}_n$ rather than on a single mixture set. This formulation applies directly to n-person games when $\mathcal{M}_i$ is the set of mixed strategies for player i and $\succ$ is the preference relation of a designated player. Chapter 7 shows how the axioms of Chapter 2 can be generalized to yield a multilinear utility function u on

$\mathcal{M}_1 \times \cdots \times \mathcal{M}_n$ that preserves $\succ$, where u is multilinear if

$$u(x_1, \ldots, x_{i-1}, \lambda x_i \oplus (1-\lambda)y_i, x_{i+1}, \ldots, x_n)$$
$$= \lambda u(x_1, \ldots, x_i, \ldots, x_n) + (1-\lambda)u(x_1, \ldots, y_i, \ldots, x_n)$$

whenever $i \in \{1, \ldots, n\}$, $x_j \in \mathcal{M}_j$ for all $j \neq i$, and $x_i, y_i \in \mathcal{M}_i$. Chapter 8 discusses the extension of this form to the multilinear expected-utility representation

$$u(p_1, \ldots, p_n) = \int_{\mathscr{C}} u(c_1, \ldots, c_n)\,\mathrm{d}p_n(c_n) \ldots \mathrm{d}p_1(c_1)$$

when $\mathcal{M}_i$ is a set $\mathscr{P}_i$ of probability measures on $\mathscr{C}_i$, $p_i \in \mathscr{P}_i$ for $i = 1, \ldots, n$, and $\mathscr{C} = \mathscr{C}_1 \times \cdots \times \mathscr{C}_n$.

Uniqueness properties for the utility functions involved in the various representations given above will be established when we encounter these representations later. Readers who may wish to scan ensuing chapters should also be advised that the sometimes cumbersome notation $\lambda x \oplus \oplus (1-\lambda)y$ will be written as $x \lambda y$, and that $\lambda p + (1-\lambda)q$ will always denote the literal convex combination of real-valued functions p and q.

1.2. PART II: SUBJECTIVE EXPECTED UTILITY

We have already noted that Savage (1954) presented the first complete axiomatization of subjective expected utility. A thorough account of Savage's theory is given in Chapter 14 of Fishburn (1970), and I shall therefore provide only a brief sketch of his ideas here.

Savage's basic primitives are a set $\mathscr{C}$ of consequences, a set S of states of the world, and a preference relation $\succ$ on the set $\mathscr{C}^S$ of all functions $f, g, \ldots$ from S into $\mathscr{C}$. The functions in $\mathscr{C}^S$ are Savage's acts: if the individual does f and state $s \in S$ obtains – or is the true state – then he will experience consequence $f(s)$ in $\mathscr{C}$. The individual is presumed to be uncertain about the state that obtains, or will obtain. In Savage's representation, this uncertainty is reflected by a finitely additive probability measure P on the set $\mathscr{S}$ of all subsets of S. An element $A \in \mathscr{S}$ is called an event, and $P(A)$ is a quantitative measure of the individual's degree of belief that event A obtains, i.e. that some state $s \in A$ obtains. Hence $P(A)$ is the individual's personal or subjective probability for event A.

Savage uses seven axioms for $\succ$ on $\mathscr{C}^S$. These include a typical ordering axiom, several independence conditions, a continuity axiom, and a dominance postulate. His axioms imply that there exists a bounded real-valued function u on $\mathscr{C}$ and a finitely additive probability measure P on $\mathscr{S}$ such that expected utilities preserve $\succ$:

$$f \succ g \quad \text{iff} \quad \int_S u(f(s))\,\mathrm{d}P(s) > \int_S u(g(s))\,\mathrm{d}P(s).$$

In addition, P is uniquely determined, and u is unique up to a positive affine transformation, i.e. v on $\mathscr{C}$ satisfies the representation in place of u if and only if there are real numbers a and b with $a > 0$ such that $v(c) = au(c) + b$ for all $c \in \mathscr{C}$. His axioms also imply that events in $\mathscr{S}$ are continuously divisible in the sense that, for any $A \in \mathscr{S}$ and any $\lambda \in [0, 1]$, there is a $B \subseteq A$ such that

$$P(B) = \lambda P(A).$$

Although this forces S to be uncountably infinite, $\mathscr{C}$ can have as few as two members.

The influence of de Finetti (1937) and von Neumann and Morgenstern (1944) on Savage is evident in the proof of his representation theorem. Let $\succ^*$ be a binary relation on $\mathscr{S}$, with $A \succ^* B$ interpreted as "A is more probable than B". Formally, $A \succ^* B$ holds if and only if $f \succ g$ whenever c and d are consequences such that c is preferred to d, $f(s) = c$ for all $s \in A$, $f(s) = d$ for all $s \in S \backslash A$, $g(s) = c$ for all $s \in B$, and $g(s) = d$ for all $s \in S \backslash B$. In other words, $A \succ^* B$ if the individual would rather take his chances on A than B to obtain a preferred consequence.

Following de Finetti's lead, Savage proves that his axioms imply that there is a unique finitely additive probability measure P on $\mathscr{S}$ for which

$$A \succ^* B \quad \text{iff} \quad P(A) > P(B),$$

for all $A, B \in \mathscr{S}$. He then uses P to construct simple probability measures on $\mathscr{C}$ from specialized acts, and shows that his axioms imply those of von Neumann and Morgenstern for preferences on the simple measures. This yields the expected-utility representation for 'simple acts', and the representation for more general acts then follows from Savage's dominance postulate.

A number of other writers, including Suppes (1956), Anscombe and Aumann (1963), Pratt *et al.* (1964, 1965), Pfanzagl (1968), Bolker (1967), and Luce and Krantz (1971), have devised other axiomatizations for representations of subjective expected utility. These were motivated in part by a desire to generalize certain aspects of Savage's system, including his continuously divisible events and the very rich structure of his act set. A detailed review of these and related theories is given in Fishburn (1981).

Preview

My own work in subjective expected utility, which is closely allied with the approach taken by Anscombe and Aumann (1963) and Pratt *et al.* (1964, 1965), was also motivated by a desire to weaken some of the strong structural presumptions in Savage's theory. At the same time, it employs other structures that are not used by Savage. In part, these additional structures make direct use of concepts developed in Part I, so that Part II of the book can be viewed as a natural sequel to Part I.

The initial chapter of Part II considers $\succ$ on the product $\mathcal{M}_1 \times \cdots \times \mathcal{M}_n$ of a finite number of mixture sets, as in Chapter 7. However, instead of using the axioms in Chapter 7, it applies axioms like those in Chapter 2 to $\succ$ on $\mathcal{M}_1 \times \cdots \times \mathcal{M}_n$ and shows that these lead to additive linear utilities of the form

$$U(x_1, \ldots, x_n) = \sum_{i=1}^{n} u_i(x_i),$$

where u_i on $\mathcal{M}_i$ is linear for each i. In the context of decision making under uncertainty, we can suppose that i indexes a finite set of states and that $\mathcal{M}_i$ applies to state i. If $\mathcal{M}_i = \mathcal{P}_0(\mathcal{C}_i)$, where $\mathcal{C}_i$ is the set of relevant consequences for state i, then the probabilities used in the simple measures in $\mathcal{P}_0(\mathcal{C}_i)$ can be viewed as 'extraneous scaling probabilities' that are generated by random mechanisms not directly associated with the states in S.

When minimal structural overlap among the $\mathcal{M}_i$ is presumed along with an interstate monotonicity axiom, it is shown that the u_i in the preceding expression can be aligned on a common scale so that U can be written as

$$U(x_1, \ldots, x_n) = \sum_{i=1}^{n} \rho_i u(x_i),$$

where u is linear on each $\mathcal{M}_i$ and the ρ_i are nonnegative numbers that sum to unity. In the states context, ρ_i is interpreted as the individual's subjective probability for state i. When $\mathcal{M}_i = \mathcal{P}_0(\mathcal{C}_i)$ for each i, the preceding form gives $\rho_1 u(c_1) + \cdots + \rho_n u(c_n)$ as the subjective expected utility of the act that assigns consequence c_i to state i for $i = 1, \ldots, n$.

This finite-states approach is then generalized to accommodate arbitrary state sets in Chapter 10. Rather than using a different mixture set for each state, Chapter 10 adopts the same mixture set $\mathcal{M}$ for all states, and views acts as mappings from S into $\mathcal{M}$. It also views the set of events as an arbitrary Boolean algebra $\mathcal{S}$ of subsets of S. Suitable axioms are then used to imply the existence of a finitely additive probability measure P on $\mathcal{S}$ and a linear function u on $\mathcal{M}$ such that

$$f \succ g \quad \text{iff} \quad \int_S u(f(s))\,\mathrm{d}P(s) > \int_S u(g(s))\,\mathrm{d}P(s),$$

for 'most' functions f and g from S into $\mathcal{M}$. Special considerations that arise from the generality of the formulation used in Chapter 10 are noted.

Chapter 11 examines a one-way version of the subjective expected utility model in which the indifference relation $\sim$ is not assumed to be transitive. It is based on the formulation of Chapter 10 in much the same way that Chapter 5 relates to Chapter 2.

The final chapter of the book considers a formulation for subjective expected utility based on conditional preference comparisons. It applies $\succ$ to $\mathcal{M} \times \mathcal{S}'$, where $\mathcal{M}$ is a mixture set (e.g., the set of simple probability measures defined over a set of Savage-type acts) and $\mathcal{S}'$ is a Boolean algebra of subsets of S with the empty event $\emptyset$ removed. The ordered pair $xA \in \mathcal{M} \times \mathcal{S}'$ is to be thought of as 'act' x under the supposition that event A obtains. The axioms of Chapter 12 lead to a quasi-conditional utility representation of the form

$$u(xA) = \sum_{i=1}^{n} P_A(A_i) u(xA_i)$$

when $\{A_1, \ldots, A_n\}$ is a partition of A. Here P_A is a (conditional) probability measure on $\{A \cap B : B \in \mathcal{S}\}$. These measures satisfy the chain rule

$$P_C(A) = P_C(B) P_B(A) \quad \text{when} \quad A \subseteq B \subseteq C.$$

Chapter 12 also considers the extension of the preceding representation to the general integral form

$$u(xA) = \int_A u(xs)\,\mathrm{d}P_A(s).$$

PART I

EXPECTED UTILITY