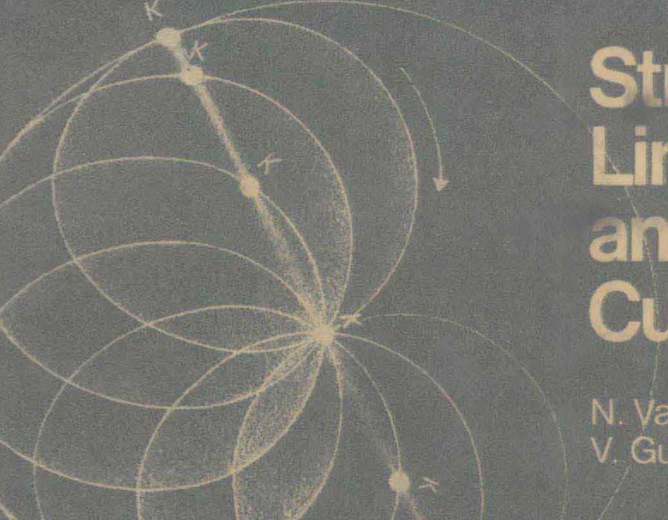

Straight Lines and Curves

N. Vasilyev
V. Gutenmacher

Mir Publishers
Moscow

ИЗДАТЕЛЬСТВО · МИР ·

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Прямые и кривые

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На английском языке

Contents

Preface

The main characters of this book are various geometric figures or, as they are frequently called here, "sets of points". The simplest figures in their different combinations appear first. They move, reveal new properties, intersect, combine, form entire families and change their appearance, sometimes to such an extent that they become unrecognizable. However, it is interesting to see old acquaintances in unusual situations surrounded by the new figures which appear at the end.

The book consists of approximately two hundred problems, most of them given with solutions or comments. There is a whole variety of problems, ranging from traditional problems in which one has to find and make use of some set of points, to simple investigations touching important mathematical concepts and theories (for instance "the cheese", "motor-boat", "bus" problems). Apart from ordinary geometric theorems on straight lines, circles and triangles, the book makes use of the method of coordinates, vectors and geometric transformations, and especially often the language of motion. A list of useful geometric facts and formulas is given in Appendices I and II. Some of the tedious finer points in the logic of the solutions are left to the reader. The symbol ⟨?⟩ replaces the words "Exercise", "Verify", "Is it clear to you?", "Think, why", etc., depending upon where it is. The beginning and the end of solutions are marked with the symbol □ while ↓ means that the solution or the answer to the problem is given at the end of the book. The problems at the beginning of each section are not usually difficult or else are analysed in detail in the book. The rest of the problems do not have to be solved in succession. One can, while reading the book, choose those which seem more attractive. It is useful to verify much of what is discussed in the problems through experiment: it is best to draw a diagram or—even better—several, with the figures in different positions. This experimental approach not only helps one to guess the answer and formulate a hypothesis but also often leads one to a mathematical proof. In drawing the diagrams in the margins the authors were convinced that almost behind every problem there is hidden an auxiliary problem of constructing the points or lines which are stated in the problem. The preliminary problem often appears to be more simple but it is no less interesting than the problem itself!

The authors are deeply thankful to I. M. Gelfand whose advice helped the entire work on the book, to I. M Yaglom, V. G. Boltyansky and J. M. Rabbot, who read the manuscript, for their significant remarks. Since the publication of the first edition (1970) of this book, it has been used in the work of the Moscow University Correspondence mathematics school. The experience which the teachers of this school shared with us and also the experience of our friends and colleagues has been taken into consideration in the detailed revision undertaken for the second edition.

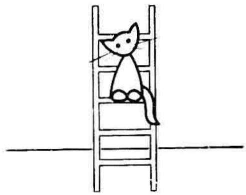
We thought it necessary to furnish the book with an additional appendix, Appendix III. This will assist in systematic study of the book, and will help to reveal relationships between different sections of the book which are not immediately apparent.

N. B. Vasilyev, V. L. Gutenmacher

Introduction

Introductory Problems

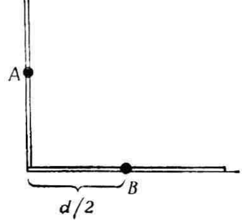
0.1. A ladder standing on a smooth floor against a wall slides down. Along what line does a cat sitting at the middle of the ladder move?



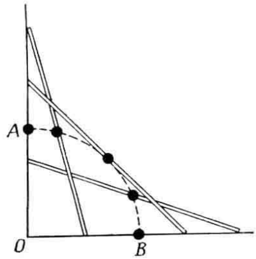
Let us suppose our cat is calm and sits quietly. Then. we can see behind this picturesque formulation the following mathematical problem.

A right angle is given. Find the midpoints of all the possible segments of given length d, which have their end-points lying on the sides of the given angle.

Let us try to guess what sort of a set this is. Obviously, when the segment rotates with its end-points sliding along the sides of the angle, its centre describes a certain line. (This is obvious from the first picturesque statement of the problem.) First of all, let us determine where the end-points of this line lie. They correspond to the extreme positions of the segment when it is vertical or



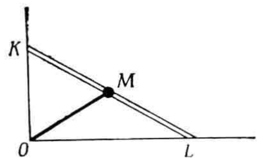
horizontal. This means that the endpoints A and B of the line lie on the sides of the angle at a distance $d/2$ from its vertex.



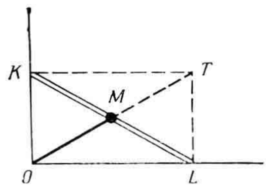
Let us plot a few intermediate points of this line. If you do this accurately enough, you will see that all of them lie at the same distance from the vertex O of the given angle. Thus, we can say that

The unknown line is an arc of a circle of radius $d/2$ with centre at O.

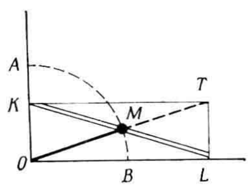
Now we must prove this.



☐ We shall first prove that the midpoint M of the given segment KL ($|KL| = d$) always lies at a distance $d/2$ from the point O. This follows from the fact that the length of the median OM of the right-angled triangle KOL is equal to half the length of the hypotenuse KL. (One can easily convince oneself of the validity of this fact by extending the triangle KOL up to the rectangle $KOLT$ and recalling that the diagonals KL and OT of the rectangle are equal in length and are bisected by the point of intersection M.)



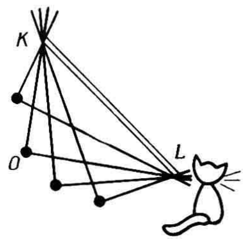
Thus, we have proved that the midpoint of the segment KL always lies on the arc $\overset{\frown}{AB}$ of a circle with centre O. This arc is the set of points we were looking for.



Strictly speaking, we have to prove also that an arbitrary point M of the arc $\widehat{AB}$ belongs to the unknown set. It is easy to do this. Through any point M of the arc $\widehat{AB}$ we may draw a ray OM, mark off the segment $|MT| = |OM|$ along it, drop perpendiculars TL and TK from the point T to the sides of the angle and the required segment KL with its midpoint at M is constructed. □

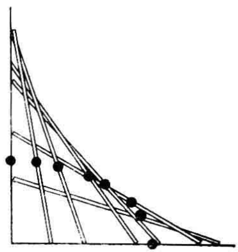
The second half of the proof might appear to be unnecessary: It is quite clear that the midpoint of the segment KL describes a "continuous line" with end-points A and B; it means that the point M passes through the whole of the arc $\widehat{AB}$ and not just through parts of it. This analysis is perfectly convincing, but it is not easy to give it a strict mathematical form.

Let us now consider the motion of the ladder (from problem **0.1**) from another point of view. Suppose that the segment KL (the "ladder") is fixed and the straight lines KO and LO ("the wall" and "the floor") rotate correspondingly about the points K and L so that the angle between them is always a right angle. The fact that the distance from the centre of the segment to the vertex O of the right angle always remains the same, reduces to a well-known theorem: *if two points K and L are given in a plane, then the set of points O for which the*



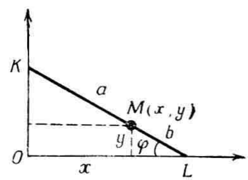
angle $\widehat{KOL}$ *equals* 90° *is a circle with diameter KL.* This theorem and also its generalization, which will be given in the proposition **E** of Sec. 2, will frequently help us in the solution of problems. Let us return to problem **0.1** and put a more general question.

0.2. Along what line does the cat move if it does not sit at the middle of the ladder?



In the figure a few points on one such line are plotted. It can be seen that it is neither a straight line nor a circle, i.e. it is a new curve for us. The coordinate method will help us to find out what sort of curve it is.

□ We introduce a coordinate system regarding the sides of the angle as the axes Ox and Oy. Suppose the cat sits at some point $M(x;\ y)$ at a distance a from the end-point K of the ladder and at a distance b from L ($a + b = d$). We shall find the equation connecting the x and y coordinates of the point M.

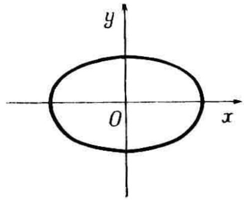


If the segment KL is inclined to the axis Ox at an angle φ, then $y = b \sin \varphi$ and $x = a \cos \varphi$; hence, for any arbitrary φ $\left(0 \leqslant \varphi \leqslant \frac{\pi}{2}\right)$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1. \qquad (1)$$

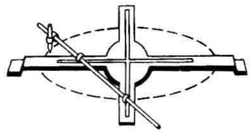
The set of points whose coordinates satisfy this equation is an *ellipse*.

Hence, the cat will move along an ellipse. □



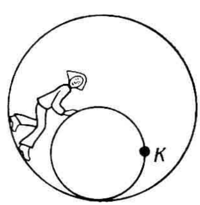
Note that when $a = b = d/2$, then if the cat sits as above at the middle of the ladder, and equation (1) becomes the equation of a circle $x^2 + y^2 = (d/2)^2$. Thus, we get one more solution of problem **0.1**, an analytical solution.

The result of problem **0.2** explains the construction of a mechanism for drawing ellipses. This mechanism shown in the figure is called *Leonardo da Vinci's ellipsograph.*

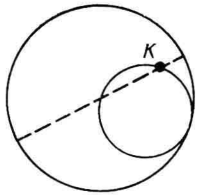


Copernicus' Theorem

0.3. Inside a stationary circle, another circle whose diameter is half the diameter of the first circle and which touches it from inside rolls without sliding. What line does the point K of the moving circle describe?

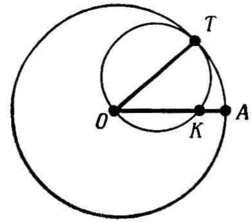


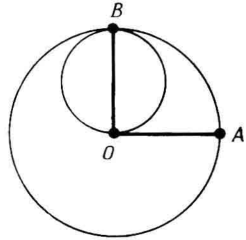
The answer to the problem is astonishingly simple: the point K moves along a *straight* line—more correctly along the diameter of the stationary circle. This result is called *Copernicus' theorem.*



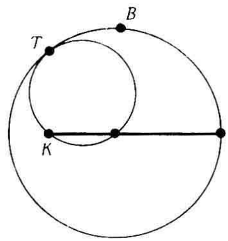
Try to convince yourself of the validity of this theorem by experiment. (It is important here that the inner circle rolls without sliding, i.e. the lengths of the arcs rolling against each other are equal). It is not difficult to prove, we need only to recall the theorem on the inscribed angle.

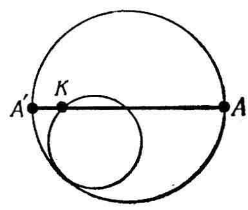
□ Suppose that the point of the moving circle, which occupies position A on the stationary circle at the initial instant, has come to the position K, and that T is the point of contact of the circle at the present moment of time. Since the lengths of the arcs $\widehat{KT}$ and $\widehat{AT}$ are equal and the radius of the movable circle is half as large, the angular size of the arc $\widehat{KT}$ in degrees is double that of the arc $\widehat{AT}$. Therefore, if O is the centre of the stationary circle, then according to the *theorem on the inscribed angle* (see p. 24), $\widehat{AOT} = \widehat{KOT}$. Hence, the point K lies on the radius AO.





This argument holds until the moment when the moving circle has rolled around one quarter of the bigger circle (the circles then touch at the point B of the bigger circle, for which $\widehat{BOA} = 90°$ and K coincides with O). After this, the motion will be continued in exactly the same way—the whole picture will be simply reflected symmetrically about the straight line BO and then, after the point K reaches the opposite end A' of the diameter AA', the circle will roll along the lower half of the stationary

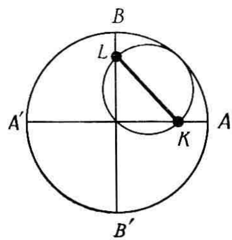




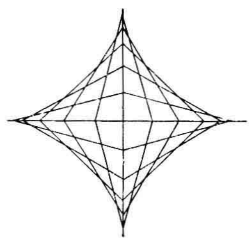
circle and the point K will return to A. □

Let us compare the results of problems **0.1** and **0.3.** They are attractive probably for the following reason. Both problems deal with the motion of figures (the first with the motion of a segment, the second with the motion of a circle). The motion is quite complicated, but the paths of certain points appear to be unexpectedly simple. These two problems turn out to be not only related in appearance, but the motions themselves, discussed in the problems also coincide with each other.

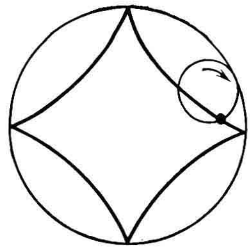
Indeed, suppose a circle of radius $d/2$ rolls along the inside of another circle of radius d, and suppose KL is a diameter of the moving circle rigidly fixed to it. According to Copernicus' theorem the points K and L move along stationary straight lines (along the diameters AA' and BB' of the bigger circle, respectively). Thus, the diameter KL slides with its endpoints along two mutually perpendicular straight lines, i.e. it moves just like the segment in the problem **0.1.**



One more interesting problem connected with the motion of the segment KL: what set of points is covered by this segment, or what is the union of all the possible positions of the segment KL during its motion? The curve bounding this set is called the



astroid. It is possible to construct this curve in the following way: make a circle of diameter $d/2$ roll inside another circle of diameter $2d$ and draw the trajectory of any particular point of the rolling circle. This trajectory will be the astroid. We shall discuss this curve and its close relatives in Sec. 7 of our book where the reader will make a more detailed acquaintance with the interconnection between the problems which we have discussed.



However, before discussing such intricate problems and curves, let us pay thorough attention to the problems dealing with straight lines and circles. Other types of lines will not appear in the first five paragraphs.