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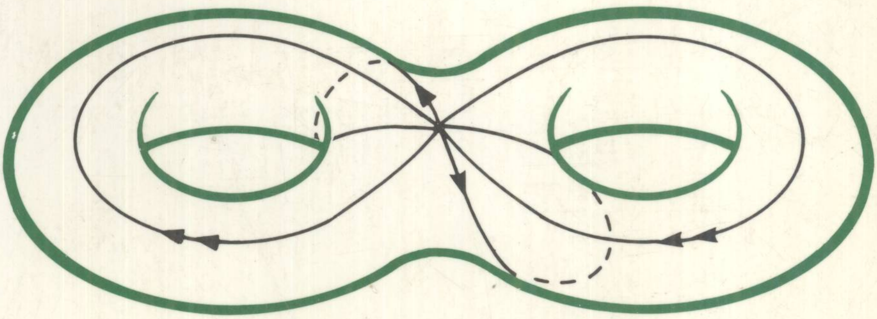

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GRADUATE STUDENT SERIES IN PHYSICS

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GEOMETRY, TOPOLOGY AND PHYSICS

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To Yoko

PREFACE

This book is a considerable expansion of lectures I gave at the School of Mathematical and Physical Sciences, University of Sussex during the winter term of 1986. The audience included postgraduate students and faculty members working in particle physics, condensed matter physics and general relativity. The lectures were quite informal and I have tried to keep this informality as much as possible in this book. The proof of a theorem is given only when it is instructive and not very technical; otherwise examples will make the theorem plausible. Many figures will help the reader to obtain concrete images of the subjects.

In spite of the extensive use of the concepts of topology, differential geometry and other areas of contemporary mathematics in recent developments in theoretical physics, it is rather difficult to find a self-contained book that is easily accessible to postgraduate students in physics. This book is meant to fill the gap between highly advanced books or research papers and the many excellent introductory books. As a reader, I imagined a first-year postgraduate student in theoretical physics who has some familiarity with quantum field theory and relativity. In this book, the reader will find many examples from physics, in which topological and geometrical notions are very important. These examples are eclectic collections from particle physics, general relativity and condensed matter physics. Readers should feel free to skip examples that are out of their direct concern. However, I believe these examples should be the *theoretical minima* to students in theoretical physics. Mathematicians who are interested in the application of their discipline to theoretical physics will also find this book interesting.

The book is largely divided into four parts. Chapters 1 and 2 deal with the preliminary concepts in physics and mathematics respectively. In Chapter 1, a brief summary of the physics treated in this book is given. The subjects covered are path integrals, gauge theories (including monopoles and instantons), defects in condensed matter physics, general relativity, Berry's phase in quantum mechanics and strings. Most of the subjects are subsequently explained in detail from the topological and geometrical viewpoints. Chapter 2 supplements the undergraduate mathematics that the average physicist has studied. If readers are quite familiar with sets, maps and general topology, they may skip this chapter and proceed to the next.

Chapters 3 to 8 are devoted to the basics of algebraic topology and differential geometry. In Chapters 3 and 4, the idea of the classification of spaces with homology groups and homotopy groups is introduced. In Chapter 5, we define a manifold, which is one of the central concepts in modern theoretical physics. Differential forms defined there play very important roles throughout this book. Differential forms allow us to define the dual of the homology group called the de Rham cohomology group in Chapter 6. Chapter 7 deals with a manifold endowed with a metric. With the metric, we may define such geometrical concepts as connection, covariant derivative, curvature, torsion and many more. In Chapter 8, a complex manifold is defined as a special manifold on which there exists a natural complex structure.

Chapters 9 to 12 are devoted to the unification of topology and geometry. In Chapter 9, we define a fibre bundle and show that this is a natural setting for many physical phenomena. The connection defined in Chapter 7 is naturally generalised to that on fibre bundles in Chapter 10. Characteristic classes defined in Chapter 11 enable us to classify fibre bundles using various cohomology classes. Characteristic classes are particularly important in the Atiyah–Singer index theorem in Chapter 12. We do not prove this, one of the most important theorems in contemporary mathematics, but simply write down the special forms of the theorem so that we may use them in practical applications in physics.

Chapters 13 and 14 are devoted to the most fascinating applications of topology and geometry in contemporary physics. In Chapter 13, we apply the theory of fibre bundles, characteristic classes and index theorems to the study of anomalies in gauge theories. In Chapter 14, Polyakov's bosonic string theory is analysed from the geometrical point of view. We give an explicit computation of the one-loop amplitude.

I would like to express deep gratitude to my teachers, friends and students. Special thanks are due to Tetsuya Asai, David Bailin, Hiroshi Khono, David Lancaster, Sigeki Matsutani, Hiroyuki Nagashima, David Pattarini, Felix E A Pirani, Kenichi Tamano, David Waxman and David Wong. The basic concepts in Chapter 5 owe very much to the lectures by F E A Pirani at King's College, University of London. The evaluation of the string Laplacian in Chapter 14 using the Eisenstein series and the Krönecker limiting formula was suggested by T Asai. I would like to thank Euan Squires, David Bailin and Hiroshi Khono for useful comments and suggestions. David Bailin suggested that I should write this book. He also advised Professor Douglas F Brewer to include this book is his series. I would like to thank the Science and Engineering Research Council of the United Kingdom, which made my stay at Sussex possible. It is a pity that I have no secretary to thank for the beautiful typing. Word processing has been carried out by myself on two NEC PC9801 computers. Jim A Revill of Adam Hilger helped me in

many ways while preparing the manuscript. His indulgence over my failure to meet deadlines is also acknowledged. Many musicians have filled my office with beautiful music during the preparation of the manuscript: I am grateful to J S Bach, Ryuichi Sakamoto, Ravi Shankar and Erik Satie. Finally I am greatly indebted to my wife Yoko, to whom this book is dedicated, for her encouragement and moral support.

Mikio Nakahara
Shizuoka, February 1989

CONTENTS

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1

BACKGROUND IN PHYSICS

We assume that the reader is familiar with elementary quantum field theory and elementary relativity. In the present chapter, we outline the physics which we shall be concerned with in this book. This chapter is intended to establish notations and conventions and also to give enough background of selected topics with which many students may not be very familiar. Most of the topics are subsequently analysed in detail from topological and geometrical viewpoints.

We put c (the speed of light) $= \hbar$ (Planck's constant$/2\pi$) $= k$ (Boltzmann's constant) $= 1$, unless written explicitly. We employ the **Einstein summation convention**: if the same index appears twice, once as a superscript and once as a subscript, then the index is summed over all possible values. For example, if μ runs from 1 to m, we have

$$A^\mu B_\mu = \sum_{\mu=1}^{m} A^\mu B_\mu.$$

The Euclidean metric is $g_{\mu\nu} = \delta_{\mu\nu} = \mathrm{diag}(+1, \ldots, +1)$ while the Minkowski metric is $g_{\mu\nu} = \eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$.

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ denote the sets of natural numbers, integers, real numbers and complex numbers, respectively. $\mathbb{H}$ denotes the set of quaternions. Let $(1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k})$ be a basis such that $\boldsymbol{i}\cdot\boldsymbol{j} = -\boldsymbol{j}\cdot\boldsymbol{i} = \boldsymbol{k}$, $\boldsymbol{j}\cdot\boldsymbol{k} = -\boldsymbol{k}\cdot\boldsymbol{j} = \boldsymbol{i}$, $\boldsymbol{k}\cdot\boldsymbol{i} = -\boldsymbol{i}\cdot\boldsymbol{k} = \boldsymbol{j}$, $\boldsymbol{i}^2 = \boldsymbol{j}^2 = \boldsymbol{k}^2 = -1$. Then

$$\mathbb{H} = \{a + \boldsymbol{i}b + \boldsymbol{j}c + \boldsymbol{k}d \,|\, a, b, c, d \in \mathbb{R}\}.$$

Note that $\boldsymbol{i}$, $\boldsymbol{j}$ and $\boldsymbol{k}$ have the 2×2 matrix representations, $\boldsymbol{i} = \mathrm{i}\sigma_3$, $\boldsymbol{j} = \mathrm{i}\sigma_2$, $\boldsymbol{k} = \mathrm{i}\sigma_1$, where the σ_i are the Pauli spin matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \sigma_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

The symbol ■ denotes the end of a proof.

1.1 Path integral and quantum field theories

Quantum field theory (QFT) has achieved great success in particle physics as well as in condensed matter physics. We cannot find any evidence against QFT when applied to metals, superconductors, superfluids, quantum electrodynamics (QED), quantum chromodynamics

(QCD), electroweak theory and grand unified theories (GUTS). So far we have not established a QFT for gravity. Superstring theory seems to be a good candidate for the *Theory of Everything* (TOE), including gravity. Although superstring theory deals with one-dimensional objects rather than particles, the basic tool to describe it is QFT. We start our exposition with a short review of the standard QFT in the path integral formalism. Relevant references are Bailin and Love (1986), Cheng and Li (1984) and Ramond (1981). Huang (1982) and Ryder (1985) contain a good introduction to topological methods in QFT. Federbush (1987) is a survey of QFT written by a mathematician.

1.1.1 Path integral formulation of quantum mechanics

Let $\hat{q}$ be a position operator in the Schrödinger picture and let $|q\rangle$ be its eigenvector with eigenvalue q:

$$\hat{q}|q\rangle = q|q\rangle. \tag{1.1}$$

$\hat{q}$ is independent of time and so is the eigenvector $|q\rangle$. A state $|\psi(t)\rangle$ satisfies the Schrödinger equation

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi(t)\rangle = H|\psi(t)\rangle \tag{1.2}$$

whose formal solution is $|\psi(t)\rangle = \exp(-\mathrm{i}Ht)|\psi(0)\rangle$. If the coordinate is diagonalised, the state is represented as

$$\psi(q, t) = \langle q|\psi(t)\rangle. \tag{1.3}$$

Let $\hat{q}(t)$ be a position operator in the Heisenberg picture and let $|q, t\rangle$ be an *instantaneous* eigenvector of $\hat{q}(t)$:

$$\hat{q}(t)|q, t\rangle = q|q, t\rangle. \tag{1.4}$$

Since $\hat{q}(t)$ depends on time, $|q, t\rangle$ may not be an eigenvector of $\hat{q}(t')$ for $t' \neq t$. The dynamics of $\hat{q}(t)$ is dictated by the Heisenberg equation of motion, with the formal solution

$$\hat{q}(t) = \mathrm{e}^{\mathrm{i}Ht}\hat{q}\mathrm{e}^{-\mathrm{i}Ht} \tag{1.5}$$

from which we find

$$|q, t\rangle = \mathrm{e}^{\mathrm{i}Ht}|q\rangle. \tag{1.6}$$

The wavefunction is $\psi(q, t) = \langle q, t|\psi\rangle$.

Let us consider a process in which a particle starting at q at time t is found at q' at later time t'. By the fundamental assumption of quantum mechanics, the probability amplitude associated with this process is

$$\langle q', t'|q, t\rangle = \langle q'|\mathrm{e}^{-\mathrm{i}H(t'-t)}|q\rangle. \tag{1.7}$$

We show that this amplitude is evaluated by summing over all possible paths that connect (q, t) and (q', t'). Inserting the identity

$$\int \mathrm{d}q |q, t\rangle\langle q, t| = 1$$

into (1.7), we have

$$\langle q', t'|q, t\rangle = \int \mathrm{d}q_1 \ldots \mathrm{d}q_n \langle q', t'|q_n, t_n\rangle \ldots \langle q_2, t_2|q_1, t_1\rangle\langle q_1, t_1|q, t\rangle \tag{1.8}$$

where we have divided the interval $t' - t$ into $n + 1$ pieces,

$$t_{i+1} - t_i = \varepsilon \qquad t_0 = t \qquad t_{n+1} = t'. \tag{1.9}$$

Each inner product is

$$\langle q_{i+1}, t_{i+1}|q_i, t_i\rangle = \langle q_{i+1}|\mathrm{e}^{-\mathrm{i}H\varepsilon}|q_i\rangle \simeq \langle q_{i+1}|q_i\rangle - \mathrm{i}\varepsilon\langle q_{i+1}|H|q_i\rangle. \tag{1.10}$$

Suppose the Hamiltonian is of the form

$$H = \hat{p}^2/2m + V(\hat{q}). \tag{1.11}$$

Noting that $\langle q_{i+1}|q_i\rangle = \delta(q_{i+1} - q_i)$ and $\langle q_i|p_i\rangle = \mathrm{e}^{ipq}$, we find

$$\begin{aligned}\langle q_{i+1}|H(\hat{p}, \hat{q})|q_i\rangle &\simeq \langle q_{i+1}|\hat{p}^2/2m|q_i\rangle + V\left(\frac{q_i + q_{i+1}}{2}\right)\delta(q_{i+1} - q_i)\\ &= \int \frac{\mathrm{d}p}{2\pi} H\left(p, \frac{q_i + q_{i+1}}{2}\right)\mathrm{e}^{\mathrm{i}(q_{i+1}-q_i)p}\end{aligned} \tag{1.12}$$

where use has been made of the completeness

$$\int \frac{\mathrm{d}p}{2\pi} |p, t\rangle\langle p, t| = 1.$$

Substituting this result into (1.10), we have

$$\begin{aligned}\langle q_{i+1}, t_{i+1}|q_i, t_i\rangle &\simeq \int \frac{\mathrm{d}p}{2\pi}\left[1 - \mathrm{i}\varepsilon H\left(p, \frac{q_i + q_{i+1}}{2}\right)\right]\mathrm{e}^{\mathrm{i}(q_{i+1}-q_i)p}\\ &\simeq \int \frac{\mathrm{d}p}{2\pi} \mathrm{e}^{\mathrm{i}p(q_{i+1}-q_i)} \exp\left[-\mathrm{i}\varepsilon H\left(p, \frac{q_i + q_{i+1}}{2}\right)\right].\end{aligned} \tag{1.13}$$

This becomes exact when $\varepsilon \to 0$, that is when $n \to \infty$. The amplitude (1.8) is now given by

$$\begin{aligned}\langle q', t'|q, t\rangle &\simeq \lim_{n\to\infty} \int \frac{\mathrm{d}p_0}{2\pi} \ldots \frac{\mathrm{d}p_n}{2\pi} \int \mathrm{d}q_1 \ldots \mathrm{d}q_n\\ &\quad\times \exp\left\{\mathrm{i}\varepsilon \sum_{i=0}^{n}\left[p_i \frac{q_{i+1} - q_i}{\varepsilon} - H\left(p_i, \frac{q_i + q_{i+1}}{2}\right)\right]\right\}.\end{aligned} \tag{1.14}$$

This is symbolically written as

$$\int \mathscr{D}p\,\mathscr{D}q \exp\left(\mathrm{i}\int_t^{t'} \mathrm{d}t\,[p\dot{q} - H(p, q)]\right) \tag{1.15}$$

which is called the **path integral** for the transition amplitude. It is clear from this construction that we have summed over all paths satisfying the boundary condition.

Example 1.1 Consider a free particle with $H = \hat{p}^2/2m$. (1.14) is

$$\langle q', t'|q, t\rangle = \lim_{n\to\infty} \int \prod_{i=0}^{n} \frac{\mathrm{d}p_i}{2\pi} \int \prod_{i=1}^{n} \mathrm{d}q_i \exp\left[\mathrm{i}\sum_{i=0}^{n}\left(p_i(q_{i+1} - q_i) - \varepsilon\frac{p_i^2}{2m}\right)\right].$$

To integrate over q, we rewrite the exponent as

$$-p_0 q + p_n q' + \sum_{i=1}^{n} q_i(p_{i-1} - p_i) - \varepsilon\sum_{i=1}^{n}\frac{p_i^2}{2m}.$$

q-integrations yield an infinite product of δ-functions

$$(2\pi)^n \prod_{i=1}^{n} \delta(p_i - p_{i-1})$$

which states that the momentum is conserved at each stage of the evolution. The amplitude becomes

$$\langle q', t'|q, t\rangle = \int \frac{\mathrm{d}p_0}{2\pi} \exp\left[\mathrm{i}\left(p_0(q' - q) - \frac{p_0^2}{2m}(t' - t)\right)\right]. \tag{1.16}$$

To evaluate this amplitude, we note the formula

$$\int_{-\infty}^{\infty} \frac{\mathrm{d}p}{2\pi} \exp(-ap^2 - bp) = \frac{1}{(4\pi a)^{1/2}} \exp\left(\frac{b^2}{4a}\right) \qquad (a > 0). \tag{1.17}$$

We uncritically think that $\mathrm{i}(t' - t)/2m$ is a positive real number. (We may introduce a new variable $\tau \equiv \mathrm{i}t$ (**Wick rotation**) so that $(\tau' - \tau)/2m > 0$.) We finally have

$$\langle q', t'|q, t\rangle = \left(\frac{m}{2\pi\mathrm{i}(t' - t)}\right)^{1/2} \exp\left(\frac{\mathrm{i}m(q' - q)^2}{2(t' - t)}\right). \tag{1.18}$$

When the kinetic energy is of the form (1.11), we may execute p-integrations in (1.14). If we Wick rotate the time so that $\tau = \mathrm{i}t$ and replace $\mathrm{i}\varepsilon$ by ε, we have

$$\int \frac{\mathrm{d}p_i}{2\pi} \exp\left(\mathrm{i}p_i(q_{i+1} - q_i) - \varepsilon\frac{p_i^2}{2m}\right) = \left(\frac{m}{2\pi\varepsilon}\right)^{1/2} \exp\left(\frac{m(q_{i+1} - q_i)^2}{-2\varepsilon}\right)$$

where use has been made of (1.17). Now the amplitude is given in terms of q-integrations only,

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