Soliton Management in Periodic Systems

Boris A. Malomed

Springer

SOLITON MANAGEMENT IN PERIODIC SYSTEMS

Boris A. Malomed
Tel Aviv University
Israel

Springer

Boris A. Malomed
Tel Aviv University, Israel

Soliton Management in Periodic Systems

Consulting Editor: D. R. Vij

Library of Congress Control Number 2005933255

ISBN 13 978-0-387-25635-1
ISBN 10 0-387-25635-0
ISBN 0-387-29334-5 (e-book)

Printed on acid-free paper.

Printed in the United States of America.

9 8 7 6 5 4 3 2 1 SPIN 11336105

springeronline.com

SOLITON MANAGEMENT IN PERIODIC SYSTEMS

Preface

This books makes an attempt to provide systematic description of recently accumulated results that shed new light on the well-known object – solitons, i.e., self-supporting solitary waves in nonlinear media. Traditionally, solitons are studied theoretically (in an analytical and/or numerical form) as one-, two-, or three-dimensional solutions of nonlinear partial differential equations, and experimentally – as pulses or beams in uniform media. Propagation of solitons in inhomogeneous media was considered too (chiefly, in a theoretical form), and a general conclusion (which could be easily expected) was that the soliton would suffer gradual decay in the case of weak inhomogeneity, and faster destruction in strongly inhomogeneous systems.

However, it was recently found, in sundry physical and mathematical settings, that a completely different, and much less obvious, situation is possible too – a soliton may remain a truly robust and intrinsically coherent object traveling long distances in periodic heterogeneous media, composed of layers with very different properties. A well-known example is *dispersion management* in fiber-optic telecommunications, i.e., the situation when a long fiber link consists of periodically alternating segments of fibers with opposite signs of the group-velocity dispersion. Such a structure of the link is necessary, as the dispersion must be compensated on average, which is provided by the alternation of negative- and positive-dispersion segments. In this case, a simple result is that localized pulses of light feature periodic internal pulsations but remain stable on average (do not demonstrate systematic degradation) in the absence of nonlinearity. A really nontrivial result is that optical solitons, i.e., *nonlinear* pulses of light, may also remain extremely stable propagating in such a periodically heterogeneous system. Moreover, under certain conditions, (quasi-) solitons may be robust even in a *random* dispersion-managed system, with randomly varying lengths of the dispersion-compensated cells (each cell is a pair of fiber segments with opposite signs of the dispersion).

While the dispersion management provides for the best known example of stability of solitons under "periodic management", examples of robust oscillating solitons in periodic heterogeneous systems were also found and investigated in some detail in a number of other settings. Essentially, they all belong to two areas – nonlinear optics and Bose-Einstein condensation (being altogether different physically, these fields have a lot in common as concerns their theoretical description). It should be said that the action of the periodic heterogeneity on a soliton may be realized in two different ways – as motion of the soliton through the inhomogeneous medium, or as strong periodic variation of system's parameter(s) in time, while the soliton does not move at all.

A very interesting example of the latter situation is the so-called *Feshbach-resonance management*, when the sign of the effective nonlinearity in a Bose-Einstein condensate periodically changes between self-attraction and self-repulsion. In the latter case, nontrivial examples of stable solitons have also been predicted.

The book aims to summarize results obtained in this field. In fact, a vast majority of results still have the form of theoretical predictions, as systematic experimental study of stability of solitons in periodic heterogeneous systems have only been performed in the context of the dispersion management in fiber optics. For this reason, the material collected in the book has a strong theoretical bias. A hope is that collecting the theoretical predictions in a systematic form may suggest directions for experimental investigation of solitons under the "periodic management". In particular, creation of solitons in Bose-Einstein condensates subjected to the Feshbach-resonance management, possibly in combination with spatially periodic potentials, provided by the so-called optical lattices, seems to be quite feasible in the real experiment, which would be especially interesting in two- and three-dimensional settings (creation of a three-dimensional soliton in a real experiment has never been reported in any field of physics, despite various theoretical predictions of this possibility).

As concerns theoretical results, virtually all of them are not rigorous ones, for an obvious reason – it is very difficult to rigorously prove the existence of stable oscillating localized solutions in models based on nonlinear partial differential equations with periodically varying coefficients, which provide for the theoretical description of the systems with periodic management. Therefore, theoretical results are either purely numerical ones, or, sometimes, they are known in a (semi-) analytical form, which is based (most frequently) on the variational approximation. Nevertheless, despite the lack of the rigorous theory, there is a possibility to summarize the results in a systematic and sufficiently consistent form. An attempt of that is done in this book. It should be said that the presentation of material in the book has a rather subjective character (which is, probably, inevitable in a book on such a topic), as emphasis is made on those issues and aspects which seem specially interesting or significant from the viewpoint of the author.

The subject of the periodic management of solitons is far from being completed. Not only the experimental results are very scarce, as said above, but also theoretical analysis (even in a non-rigorous form) of many important problems should be further advanced. However, although the field is in the state of development, a coherent description of its current status is quite possible.

Three distinct parts can be identified in the book. The first chapter (Introduction), which is, as a matter of fact, a separate part by itself, gives a possibly general overview of solitons, with an intention to briefly outline the most important theoretical models and results obtained in them, as well as most significant experimental achievements. Since the length of the introduction is limited, the outline was focused on models and settings related to the realms of nonlinear optics and Bose-Einstein condensation, as the concepts and techniques of the periodic soliton managements have been developed in these areas. The introduction also includes a brief description of the subject and particular objectives of the book. Then, two technical parts (one includes chapters 2 – 6, and the other chapters 7 – 10) report results, respectively, for one-dimensional and multidimensional solitons. Such separation is natural, as methods used for the study of one-dimensional settings, and the respective results, are very different from those which are relevant to multidimensional problems (nevertheless, chapter 10 includes some results for a one-dimensional situation too, which are closely related to the basic two-dimensional problem which is considered in that chapter).

Writing this book would not be possible without valuable collaborations and discussions with a large number of colleagues. It is my great pleasure to express the gratitude to F. Kh. Abdullaev, J. Atai, B. B. Baizakov, Y. B. Band, A. Berntson, J. G. Caputo, A. R. Champneys, P. Y. P. Chen, P. L. Chu, D. J. Frantzeskakis, B. V. Gisin, D. J. Kaup, P. G. Kevrekidis, Y. S. Kivshar, R. A. Kraenkel, T. Lakoba, U. Mahlab, D. Mihalache, V. Pérez-García, M. Salerno, M. Segev, N. Smyth, L. Torner, M. Trippenbach, F. Wise, and J. Yang. Special thanks are due to younger collaborators (some of them were my students or postdoc associates), including R. Driben, A. Gubeskys, M. Gutin, A. Kaplan, M. Matuszewski, T. Mayteevarunyoom, M. I. Merhasin, G. Theocharis, and I. Towers.

The work on particular projects that have generated essential results included in this book was supported, in various forms and parts, by grants No. 1999459 from the Binational (US-Israel) Science Foundation, and No. 8006/03 from the Israel Science Foundation. At a smaller scale, support was also provided by the European Office of Research and Development of the US Air Force, and Research Authority of the Tel Aviv University.

List of acronyms used in the text:

1D, 2D, 3D - one-dimensional, two-dimensional, three-dimensional
AWG - antiwaveguide
BEC - Bose-Einstein condensation/condensate
BG - Bragg grating
CW - continuous-wave (solution)
DM - dispersion management
DS - dark soliton
FF - fundamental-frequency (wave)
FP - fixed point
FR - Feshbach resonance
FRM - Feshbach-resonance management
FWHM - full width at half-maximum (of an optical pulse)
FWM - four-wave mixing
GPE - Gross-Pitaevskii equation
GS - gap soliton
GVD - group-velocity dispersion
GVM - group-velocity mismatch
HS - hot spot (a local perturbation switching a spatial soliton)
ISI - inter-symbol interference
IST - inverse-scattering transform
KdV - Korteweg - de Vries (equation)
ME - Mathieu equation
NLM - nonlinearity management
NLS - nonlinear Schrödinger (equation or soliton)
ODE - ordinary differential equation
OL - optical lattice
PAD - path-average dispersion
PCF - photonic-crystal fiber
PDE - partial differential equation
PR - parametric resonance
QPM - quasi-phase-matching
RI - refractive index
RZ - return-to-zero (signal)
SH - second harmonic
SHG - second-harmonic generation
SPM - self-phase modulation
SSM - split-step model
STS - spatiotemporal soliton
TF - Thomas-Fermi (approximation)
TS - Townes soliton
VA - variational approximation
WDM - wavelength-division multiplexing
WG - waveguide (when referred to in the context of the waveguding-antiwaveguiding model)
XPM - cross-phase modulation

Contents

Preface **ix**

1 Introduction **1**

1.1 An overview of the concept of solitons 1

1.1.1 Optical solitons 2

1.1.2 Solitons in Bose-Einstein condensates, and their counterparts in optics 13

1.2 The subject of the book: Solitons in Periodic Heterogeneous media ("Soliton management") 18

1.2.1 General description 18

1.2.2 One-dimensional optical solitons 20

1.2.3 Multidimensional optical solitons 23

1.2.4 Solitons in Bose-Einstein condensates 23

1.2.5 The objective of the book 24

2 Periodically modulated dispersion, and dispersion management: basic results for solitons **25**

2.1 Introduction to the topic 25

2.2 The model with the harmonic modulation of the local dispersion 27

2.2.1 Variational equations 27

2.2.2 Soliton dynamics in the model with the harmonic modulation 29

2.3 Solitons in the model with dispersion management 32

2.4 Random dispersion management 37

2.5 Dispersion-managed solitons in the system with loss, gain and filtering 40

2.5.1 Distributed-filtering approximation 40

2.5.2 The lumped-filtering system 43

2.6 Collisions between solitons and bound states of solitons in a two-channel dispersion-managed system 45

2.6.1 Effects of inter-channel collisions 45

2.6.2 Inter-channel bound states 48

2.6.3 Related problems 49

3 The split-step model **51**

3.1 Introduction to the model ... 51

3.2 Solitons in the split-step model ... 52

3.2.1 Formulation of the model ... 52

3.2.2 Variational approximation ... 54

3.2.3 Comparison with numerical results ... 56

3.2.4 Diagram of states for solitons and breathers in the split-step system ... 60

3.3 Random split-step system ... 64

3.4 A combined split-step dispersion-management system: Dynamics of single and paired pluses ... 66

3.4.1 The model ... 66

3.4.2 Transmission of an isolated pulse ... 67

3.4.3 Transmission of pulse pairs ... 69

4 Nonlinearity management for quadratic, cubic, and Bragg-grating solitons **73**

4.1 The tandem model and quasi-phase-matching ... 73

4.2 Nonlinearity management: Integration of cubic and quadratic nonlinearities with dispersion management ... 74

4.2.1 The mo del ... 75

4.2.2 Results: transmission of a single pulse ... 76

4.2.3 Co-propagation of a pair of pulses ... 78

4.3 Nonlinearity management for Bragg-grating and nonlinear-Schrödinger solitons ... 80

4.3.1 Introduction to the problem ... 80

4.3.2 Formulation of the model ... 80

4.3.3 Stability diagram for Bragg-grating solitons ... 82

4.3.4 Stability of solitons in the NLS equation with the periodic nonlinearity management ... 85

4.3.5 Interactions between solitons and generation of moving solitons ... 85

5 Resonant management of one-dimensional solitons in Bose-Einstein condensates **89**

5.1 Periodic nonlinearity management in the one-dimensional Gross-Pitaevskii equation ... 89

5.2 Resonant splitting of higher-order solitons under the Feshbach-resonance management 93
5.2.1 The model 93
5.2.2 Numerical results 94
5.2.3 Analytical results 96
5.3 Resonant oscillations of a fundamental soliton in a periodically modulated trap 97
5.3.1 Rapid periodic modulation 97
5.3.2 Resonances in oscillations of a soliton in a periodically modulated trap 98

6 MANAGEMENT FOR THE CHANNEL SOLITONS: A WAVEGUIDING-ANTIWAVEGUIDING SYSTEM 105
6.1 Introduction to the topic 105
6.2 The alternate waveguiding-antiwaveguiding structure 106
6.3 Analytical consideration of a spatial soliton trapped in a weak alternate structure 108
6.4 Numerical results 111
6.4.1 Beam propagation in the alternate structure 111
6.4.2 Switching of beams by the hot spot 111

7 STABILIZATION OF SPATIAL SOLITONS IN BULK KERR MEDIA WITH ALTERNATING NONLINEARITY 115
7.1 The model 115
7.2 Variational approximation 116
7.3 Numerical results 118

8 Stabilization of two-dimensional solitons in Bose-Einstein condensates under Feshbach-resonance management 121
8.1 The model and variational approximation 122
8.1.1 General consideration 122
8.1.2 The two-dimensional case 122
8.1.3 Variational approximation in the three-dimensional case... 124
8.2 Averaging of the Gross-Pitaevskii equation and Hamiltonian 127
8.3 Direct numerical results 128

9 Multidimensional dispersion management **131**
9.1 Models 131
9.2 Variational approximation 132
9.3 Numerical results 134
9.4 The three-dimensional case 140

10 Feshbach-resonance management in optical lattices **143**
10.1 Introduction to the topic 143
10.2 Stabilization of three-dimensional solitons by the Feshbach-resonance management in a quasi-one-dimensional lattice 144
10.2.1 The model and variational approximation 144
10.2.2 Numerical results 146
10.3 Alternate regular-gap solitons in one-and two-dimensional lattices under the ac Feshbach-resonance drive 148
10.3.1 The model 148
10.3.2 Alternate solitons in two dimensions 150
10.3.3 Dynamics of one-dimensional solitons under the Feshbach-resonance management 155

Chapter 1

Introduction

1.1 An overview of the concept of solitons

The concept of *solitons* (solitary waves) plays a profoundly important role in modern physics and applied mathematics, extending beyond the bounds of these disciplines. It was introduced in 1965 by Zabusky and Kruskal who numerically simulated collisions between solitary waves (pulses) in the Korteweg - de Vries (KdV) equation, and discovered that these pulses not only are stable in isolation, but also completely recover their shapes after collisions [175]; this observation was an incentive which had soon led to the discovery of the *inverse scattering transform* (IST) and the very concept of integrable nonlinear partial differential equations (PDEs) [72]. The next principally important step in this direction was made by Zakharov and Shabat, who had demonstrated that the integrability is not a peculiarity specific to a single (KdV) equation, but is also featured by another equation which finds very important applications in physics, *viz.*, the nonlinear Schrödinger (NLS) equation [177]. Integrability of the sine-Gordon equation, which was actually known, in terms of the Bäcklund transformation, since the 19th century, was also naturally incorporated into the IST technique (the sine-Gordon equation finds its most important physical realization in superconductivity, as a dynamical model of a long *Josephson junction*, i.e., a thin layer of an insulator sandwiched between two bulk superconductors [170]). Further development of the studies in this field has produced a body of results which have become a classical contribution to several core areas of physics and mathematics. The IST technique and results produced by it were summarized in several well-known books written by the very same people who had produced these results [176, 11, 133].

Parallel to the theoretical developments, great progress has been achieved in experimental studies of solitons. The very first published report of observation of a soliton is due to John Scott Russell, who spotted a stable localized elevation running on the surface of water in a canal in Edinburgh, and pursued it on horseback. In retrospective, the most astonishing feature of this report, published in 1844 [149], is the very fact that J. S. Russell was able to instantaneously understand the significance of the phenomenon.

1.1.1 Optical solitons

Qualitative consideration

In the modern experimental and theoretical studies of solitons, the most significant progress has been achieved in optics and, most recently, in Bose-Einstein condensates (BECs). A milestone achievement was the creation of bright temporal solitons in nonlinear optical fibers in 1980 [127], after this possibility had been predicted seven years earlier [79]. In the realm of nonlinear optics, this was followed by the creation of dark solitons in fibers [60, 98, 172], bright spatial solitons in planar nonlinear waveguides [118, 18], and *gap solitons* (GSs) in fiber Bragg gratings [57]. In all these cases, the soliton is supported by interplay between the chromatic dispersion (in the temporal domain) or diffraction (for spatial solitons) of the electromagnetic wave and cubic self-focusing nonlinearity, induced by the Kerr effect. The latter may be realized as an effective positive correction, $\Delta n(I)$, to the local refractive index (RI) of the material medium, which is proportional to the local intensity, I, of that very electromagnetic wave on which the RI acts, i.e., $\Delta n(I) = n_2 I$ with a positive coefficient n_2. Besides the *self-focusing* sign of the Kerr effect ($\Delta n(I) > 0$), its essential property in normal optical materials is the instantaneous character (no temporal delay between $\Delta n(I)$ and I). In view of the fundamental importance of the temporal and spatial optical solitons supported by this mechanism, it is relevant to present a short quantitative explanation for it here.

In the course of the propagation in the nonlinear medium, the light pulse accumulates a phase shift that, through the correction $n_2 I$ to the RI, mimics the temporal shape of the pulse, $I = I(t)$. To understand this feature in a more accurate form, one may start from the normalized wave equation for the electric field E,

$$E_{zz} + E_{xx} + E_{yy} - \left(n^2 E\right)_{tt} = 0, \tag{1.1}$$

where the subscripts stand for the partial derivative, z is the propagation distance, x and y are transverse coordinates, t is time, and n is the above-mentioned RI (detailed derivation of the wave equation can be found, e.g., in book [15]). A solution to Eq. (1.1) for a one-dimensional wave, which must be a real function, is looked for as

$$E(z,t) = u(z)e^{ik_0 z - i\omega_0 t} + u^*(z)e^{-ik_0 z + i\omega_0 t}, \tag{1.2}$$

where $\exp\left(ik_0 z - i\omega_0 t\right)$ represents a rapidly oscillating carrier wave, the asterisk stands for the complex conjugation, and $u(z,t)$ is a slowly varying complex local amplitude. Substituting this in Eq. (1.1), in the lowest approximation one obtains the dispersion relation between the propagation constant (wave number) k and frequency ω, $k_0^2 = \left(n_0\omega_0\right)^2$, with n_0 the RI in the linear approximation. The next-order approximation, which takes into regard the above correction to the RI, $n = n_0 + n_2 I$, yields an evolution equation for the amplitude,

$$i\frac{du}{dz} + \frac{n_0 n_2}{k_0}\omega_0^2 I u = 0. \tag{1.3}$$

Actually, this equation is a nonlinear one, as the intensity is tantamount to the squared amplitude, $I = |u|^2$. A solution to Eq. (1.3) is simply $\Delta\phi = \left(n_0 n_2\right)\left(\omega_0^2/k_0\right) I z$, where $\Delta\phi$ is a nonlinear contribution to the wave's phase (the accumulation of the nonlinear phase is usually called self-phase modulation, SPM). The corresponding SPM-induced frequency shift being $\Delta\omega = -\partial\Delta\phi/\partial t$, one obtains an expression for it,

$$\Delta\omega = -n_0 n_2 \frac{\omega_0^2}{k_0}\frac{dI}{dt}z. \tag{1.4}$$

It follows from Eq. (1.4) that the lower-frequency components of the pulse, with $\Delta\omega < 0$, develop near its front, where $dI/dt > 0$ (the intensity grows with time), while higher frequencies, with $\Delta\omega > 0$, develop close to the rear of the pulse, where $dI/dt < 0$.

On the other hand, the dielectric response of the material medium is not strictly instantaneous, featuring a finite temporal delay. This implies that the linear part, $\epsilon \equiv n_0^2$, of the multiplier n^2 in the wave equation (1.1) (the dynamic dielectric permeability) is, as a matter of fact, a linear operator, rather than simply a multiplier. The accordingly modified form of the linear term $(\epsilon E)_{tt}$ in Eq. (1.1) becomes $\left(\int_0^\infty \epsilon(\tau)E(t-\tau)d\tau\right)_{tt}$, where τ is the delay time. Finally, approximating this nonlocal-in-time expression by a quasi-local expansion, $\epsilon_0 E_{tt} + \epsilon_2 E_{tttt} + ...$, which is justified when the actual delay in the dielectric response is very small, gives rise to second- and higher-order group-velocity-dispersion (GVD), alias chromatic-dispersion, terms in the eventual propagation equation, which can be translated into the corresponding linear dispersion relation, $k = k(\omega)$ [15].

In particular, the normal (positive) GVD (which means that waves with a higher frequency have a smaller group velocity, as expressed by the condition that the second-order-dispersion coefficient is positive, $\beta_2 \equiv d^2k/d\omega^2 > 0$) reinforces the above (nonlinearity-induced) trend to the temporal separation between the low- and high-frequency components of the pulse, contributing to its rapid spread. On the contrary, anomalous (negative) GVD ($\beta_2 < 0$), which also occurs in real materials, may *compensate* the nonlinearity-induced spreading. With the magnitudes of the dispersion and intensity properly matched, the balance may be perfect, giving rise to very robust pulses, i.e., solitons.

Nonlinear Schrödinger equation and solitons

Putting all the above ingredients together, and assuming that the amplitude u in Eq. (1.2) is a slowly varying function of z and "reduced time", $\tau \equiv t - k'_\omega z$ (here and below, the value of the derivative k'_ω is calculated at the carrier-wave's frequency, $\omega = \omega_0$), one arrives at the *nonlinear Schrödinger* (NLS) equation which governs the evolution of $u(z,\tau)$,

$$iu_z - \frac{1}{2}\beta u_{\tau\tau} + \gamma|u|^2 u = 0, \tag{1.5}$$

where β replaces β_2 (the replacement will not lead to confusion, as higher-order dispersion, which is different from β_2, is not dealt with below), and $\gamma \equiv n_2\sqrt{\epsilon_0}\omega_0^2/k_0$. The introduction of τ instead of t is necessary to eliminate a term with the first derivative in t (the group-velocity term), thus casting the NLS equation in the simplest possible form, namely, the one given by Eq. (1.5).

Below, a number of models will be considered that may be viewed as various generalizations of the NLS equation (1.5) – two-component systems, equations with different nonlinearities, multidimensional systems, etc. A very recent succinct review of equations of the NLS type can be found in article [105].

An elementary property of the NLS equation is its *Galilean invariance*: any given solution $u(z,\tau)$ automatically generates a family of moving solutions by means of the *Galilean boost* that depends on an arbitrary real parameter c (it is an inverse-velocity shift, relative to the inverse group velocity, k'_ω, of the carrier wave):

$$u(z,t;c) = u\left(z,\tau - cz\right)\exp\left(\frac{ic^2}{2\beta}z - \frac{ic}{\beta}\tau\right). \tag{1.6}$$

Another simple property of Eq. (1.5) is the *modulational instability* of CW (continuous-wave) solutions, $u_{\text{CW}} = A_0 \exp\left(i\gamma A_0^2 z\right)$ with an arbitrary amplitude A_0: although the CW solution does not contain the GVD coefficient β, it is stable in the case of $\beta\gamma < 0$, and unstable (against τ-dependent perturbations) in the opposite case.

The NLS equation has natural Lagrangian and Hamiltonian representations. The former one will be considered below (see Eq. (2.7)), while the latter takes the form

$$iu_z = \frac{\delta H}{\delta u^*}, \tag{1.7}$$

where $\delta/\delta u^*$ is the functional derivative, the asterisk stands for the complex conjugation, and the Hamiltonian,

$$H = -\frac{1}{2}\int_{-\infty}^{+\infty}\left(\beta\left|u_\tau\right|^2 + \gamma|u|^4\right)d\tau, \tag{1.8}$$

is considered as a functional of two formally independent arguments, $u(\tau)$ and $\left(u(\tau)\right)^*$. The Hamiltonian is a dynamical invariant of Eq. (1.5), i.e., $dH/dz = 0$. Two other straightforward dynamical invariants of the NLS equation are energy E, alias norm of the solution (in the context of fiber optics, the energy is different from the Hamiltonian), and momentum P,

$$E \equiv \frac{1}{2}\int_{-\infty}^{+\infty}\left|u(\tau)\right|^2 d\tau, \tag{1.9}$$

$$P \equiv i\int_{-\infty}^{+\infty} uu_\tau^* d\tau. \tag{1.10}$$

Due to the fact that the NLS equation is exactly integrable by means of the IST, it has an infinite set of higher-order dynamical invariants, in addition to E, P, and H [176]. In particular, the first two higher-order invariants are

$$I_4 = \frac{1}{2}\int_{-\infty}^{+\infty}\left(-\beta uu_{\tau\tau\tau}^* + 3\gamma|u|^2 uu_\tau^*\right)d\tau, \tag{1.11}$$

$$I_5 = \frac{1}{4}\int_{-\infty}^{+\infty}\left[\beta^2\left|u_{\tau\tau}\right|^2 + 2\gamma^2|u|^6 + \gamma\beta\left(\left(|u|^2\right)_\tau\right)^2 + 6\gamma\beta\left|u_\tau\right|^2 u^2\right]d\tau \tag{1.12}$$

(the subscripts 4 and 5 imply that they follow the first three elementary dynamical invariants, E, P, and H). These higher-order invariants do not have a straightforward physical interpretation, and are seldom used in applications. Nevertheless, an example of a physical application of the invariants (1.11) and (1.12) will be presented in this book, when analyzing splitting of higher - order solitons in the model based on Eq. (5.5), see subsection 5.2.3.

In the case of the anomalous GVD, $\beta < 0$ (it is assumed that γ is positive), i.e., when the CW solutions are unstable, a commonly known family of soliton solutions to Eq. (1.5) is

$$u_{\rm sol}(z,\tau) = \frac{\eta}{\sqrt{\gamma}}\operatorname{sech}\left(\eta\left(\frac{\tau}{\sqrt{|\beta|}} - cz\right)\right)\exp\left(i\left[\frac{c\tau}{\sqrt{|\beta|}} + \frac{1}{2}\left(\eta^2 - c^2\right)z\right]\right), \tag{1.13}$$

where η and c are arbitrary real parameters, that determine the soliton's amplitude and the above-mentioned inverse-velocity shift. The function sech (hyperbolic secant) in this solution provides for the localization of the soliton. In the experiment, the temporal soliton is observed as a localized pulse running along the fiber with the velocity $V = 1/\left(k'_\omega + c\sqrt{|\beta|}\right)$. The entire soliton family (1.13) is stable against small perturbations.

The application of the IST yields exact solutions of the NLS equation more complex than the fundamental soliton (1.13). In particular, the initial condition (in the case of $\beta < 0$)

$$u_0(\tau) = n\frac{\eta}{\sqrt{\gamma}}\operatorname{sech}\left(\frac{\eta}{\sqrt{|\beta|}}\tau\right) \tag{1.14}$$

with integer n and arbitrary η, that generates the fundamental soliton for $n = 1$, gives rise to higher-order "n-solitons" for $n \geq 2$ [154]. Analytical expressions for these solitons with $n \geq 3$ are cumbersome. A relatively simple analytical solution describes the 2-soliton,

$$u_{2\rm sol} = \frac{4\eta}{\sqrt{\gamma}}\frac{\cosh\left(3\eta\tau/\sqrt{|\beta|}\right) + 3\exp\left(4i\eta^2 z\right)\cosh\left(3\eta\tau/\sqrt{|\beta|}\right)}{\cosh\left(4\eta\tau/\sqrt{|\beta|}\right) + 4\cosh\left(2\eta\tau/\sqrt{|\beta|}\right) + 3\cos\left(4\eta^2 z\right)}\exp\left(\frac{i}{2}\eta^2 z\right). \tag{1.15}$$

As seen from this expression, the shape of the 2-soliton, i.e., the distribution of the power in the soliton, $|u(z,\tau)|^2$, oscillates in z with the period

$$z_{\rm sol} = \frac{\pi}{2\eta^2}, \tag{1.16}$$

which is called the *soliton period*. It can be demonstrated that all the exact n-soliton solutions generated by the initial condition (1.14) with $N \geq 2$ oscillate with exactly the same period (1.16), irrespective of the integer value of n. In fact, $z_{\rm sol}$ is also an