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Preface

One of the main purposes of this monograph is to highlight the key role played by abstract analysis in simplifying and solving some fundamental problems in stochastic theory. This effort is analogous to a similar one published over a decade ago. The presentation is in a research-expository style, with essentially complete details, to make the treatment self-contained so as to be accessible to both graduate students seeking dissertation topics and other researchers desiring to work in this area. The aim is to give a unified and general account for a selected set of topics covering a large part of stochastic analysis. A central thread running through all the articles here is employment of functional analytic methods.

The work presented in the following six chapters is not only a unified treatment, but each one contains a substantial amount of new material appearing for the first time. They are devoted to both the random processes and fields, Gaussian as well as more general classes, with some serious applications and also several indications of them at several places. A more detailed synopsis of each chapter appears in the Introduction and Overview that follows immediately. All chapters have been reviewed.

It is hoped that these results will stimulate further research in these areas. In preparing this volume, I received considerable assistance from Dr. Y. Kakihara and Ms. Jan Patterson, as well as the authors. This is much appreciated. I also wish to thank CRC Press for their enthusiastic cooperation in publishing this book on schedule.

M. M. Rao

Contents

Introduction and overview

M. M. Rao

The material of the following chapters, presented in a research-expository style, consists of recent advances in some key areas of stochastic analysis wherein real (= functional) analysis methods and ideas play a prominent role. The topics discussed are detailed with numerous references bringing the reader to current research activity in the subject, and at the same time pointing out several problems that lead to promising investigations. This is particularly helpful for graduate students as well as other researchers who would pursue work in the areas covered in the following chapters. These appear in stochastic theory, and moreover interesting new functional analysis problems are suggested by the former. The current work complements the studies of a previous volume, published a decade ago, and concentrates on areas mostly considered since that time and includes certain topics that could not be treated then. Some important applications are also presented now and they motivate new areas of potential interest. Let us discuss the material of the chapters in some detail giving an overview of the topics to potential readers.

1. The first chapter, written by Carmona, is on stochastic modeling useful to several important problems, of interest in applications, along with an account of the underlying theory. Thus a comprehensive model that can be specialized to describe several stochastic flows, including Brownian, Jacobian and Manhatten flows as well as the mass transport, is discussed. This general model is defined by the following (nonlinear first order) stochastic differential equation:

$$dX_t = \vec{v}(t, X_t)dt + \sqrt{2\kappa}dB_t, \tag{1}$$

where $\{B_t, \geq 0\}$ is a d-dimensional Brownian motion, $\kappa \geq 0$, and $\{\vec{v}(t,x), t \geq 0, x \in \mathbb{R}^d\}$ is itself a Gaussian random field on $\mathbb{R}^+ \times \mathbb{R}^d$ of the following type. For each $t \geq 0$, $\vec{v}(t,\cdot) : \mathbb{R}^d \to \mathbb{R}^d$ is stationary and for each $x \in \mathbb{R}^d$, $\vec{v}(\cdot, x)$ has an extension to $\mathbb{R} \to \mathbb{R}^d$, which is again stationary so that the covariance function of $\vec{v}$, say Γ, satisfies:

$$\Gamma(s, x; t, y) = \tilde{\Gamma}(s - t, x - y), \tag{2}$$

a $d \times d$-positive definite matrix. The values of interest for the first variable are $s, t \geq 0$, to be identified as time and $x, y \in \mathbb{R}^d$ as space variables. Here $\tilde{\Gamma}(\cdot,\cdot)$ is assumed integrable on $\mathbb{R} \times \mathbb{R}^d$ so that it has (a.e. Leb.) a spectral density. Under various specializations, the author discusses the solutions of

(1). It is applied, with $\kappa = 0$ resulting in a Gaussian velocity field, to shear Brownian and Manhatten flows, and their transport properties. If $\vec{v}(t,x) = -ax$ and $2\kappa = b^2$, the solution of (1) is an Ornstein-Uhlenbeck process. This is described first for the one-dimensional case and then extended to finite dimensions with $2\kappa = B$ as a positive definite matrix and $a = A$ a square matrix. This motivates a study of an infinite dimensional O.U. process with B as a positive definite operator on the state space of the B_t-process (a Hilbert space $\mathcal{H}$) and A as a linear operator on it. If A is a partial differential operator, then one gets a representation of the solution of the stochastic PDE (or SPDE) as:

$$\langle x, X_t\rangle = \langle x, X_0\rangle - \int_0^t \langle Ax, X_s\rangle ds + W(\chi_{[0,t)}x), \tag{3}$$

for each $x \in dom(A)$, $W(\cdot)$ being a Brownian motion process.

Several applications of the solution process are considered. Numerical simulations are also discussed. Applications to mass transport are given using the theory of ODE (especially utilizing the Lyapounov exponents – indeed, for the ODE in Banach spaces Lyapounov and Bohl exponents play a key role as seen from Dalecky and Krein (1974), p.116 ff). A discussion of SPDE in this context is included, along with diffusion approximations, and a good collection of related papers are in the bibliography. Several unsolved problems are also pointed out at various places. In fact the paper is based on a series of lectures given by the author recently, and the freshness of the exposition is maintained.

2. One of the important reasons to separate the index of the process X in the above work as $\mathbb{R}^+ \times \mathbb{R}^d$ is to treat the first component as time and all the stochastic differentials (or integrals) are defined relative to this one-dimensional parameter, and the existing theory suffices. If it is multidimensional, there are several new problems and multiple stochastic integration and its properties are needed. The second chapter, written by Green, addresses this problem if the index is a two-dimensional parameter, and hence planar stochastic integrals are studied. Such integrals relative to a Brownian sheet have been developed in a fundamental paper by Cairoli and Walsh (1975), and for an extended analysis in SDEs these have to be obtained for more general integrators than Brownian sheets that include, for instance, (sub)martingales. To take into account all such cases, Green develops a planar integration relative to quasimartingales, using a generalized boundedness principle, originally due to Bochner (1955). An extension of the work of Cairoli and Walsh presents several problems involving conditions that are automatic in the Brownian case, but must be suitably formulated to have the desired generality.

The extended Bochner boundedness principle may be stated as follows. Let $X : [a,b] \to L^p(P)$, $p \geq 0$ be a (random) function, $\mathcal{O} \subset \mathcal{B}([a,b]) \otimes \Sigma$ a

σ-subalgebra, $\varphi_i : \mathbb{R} \to \bar{\mathbb{R}}^+, i = 1, 2$, increasing functions, and $\alpha : \mathcal{O} \to \bar{\mathbb{R}}^+$ a σ-finite measure. Then X is said to be *L^{φ_1,φ_2}-bounded relative to $\mathcal{O}$* and α, if there exists some constant $K(= K_{\varphi_1,\varphi_2,\mathcal{O},\alpha} > 0)$ such that for each simple function $f = \sum_{i=0}^{n-1} a_i \chi_{(t_i,t_{i+1}]}$, $a \le t_0 < t_1 < \cdots < t_n \le b$ one has:

$$E(\varphi_2(\tau(f))) \le K \int_{\Omega\times[a,b]} \varphi_1(f)d\alpha, \ \tau(f) = \sum_{i=0}^{n-1} a_i(X(t_{i+1}) - X(t_i)). \quad (4)$$

If $\varphi_1(x) = \varphi_2(x) = x^2$, then X is termed $L^{2,2}$-bounded. For a Brownian motion X, one can take $\mathcal{O} = \mathcal{B}([a,b]) \otimes (\emptyset, \Omega)$, and $\alpha = Leb. \otimes P$, so that X is $L^{2,2}$-bounded (with $K = 1$ and equality in (3)). It can also be shown that a square integrable submartingale is $L^{2,2}$-bounded with $\mathcal{O}$, a predictable σ-subalgebra and α a suitable σ-finite measure on $\mathcal{O}$, for (4). Thus an L^{φ_1,φ_2}-bounded X qualifies to be a stochastic integrator and a dominated convergence theorem is valid. Green considers planar integrals relative to such general integrators, namely quasimartingales, and extends most of the Cairoli-Walsh theory under a suitable condition called "cross-term domination". This is satisfied not only for Brownian sheets, but also for square integrable two parameter martingales as well as many quasimartingales. Then he develops stochastic line and surface integrals, a Fubini type theorem, and moreover a (stochastic) Green theorem. Further a definition of stochastic partial differential is given. These results will be of considerable interest in a study of SPDEs in accordance with Walsh's (1984) account. All the basic work with complete details for the latter study is thus contained in this somewhat long article.

3. A different approach to stochastic modeling via probability metrics is the subject of Chapter 3 by Rachev, Haynatzka and Haynatzki. This is particularly useful for studies of large sample behavior and weak convergence of processes. These metrics are of two kinds, namely those depending on distributions of single random variables and those depending on joint distributions of two or more random variables. Typical examples are the classical Lévy and Fréchet metrics, i.e., if X, Y are the random variables with distributions F_X, F_Y then the Lévy metric is:

$$L(X,Y) = \inf\{\varepsilon > 0 : F_X(x-\varepsilon) - \varepsilon \le F_Y(x) \le F_X(x+\varepsilon), \, x \in \mathbb{R}\}, \quad (5)$$

and the Fréchet or convergence in probability metric is:

$$d(X,Y) = \int_\Omega \frac{|X-Y|}{1+|X-Y|} dP \ (= \int_{\mathbb{R}^2} \frac{|x-y|}{1+|x-y|} dF_{X,Y}(x,y)). \quad (6)$$

These two cases are generalized to wide classes of probability (semi)metrics. Keeping the applications in mind the triangle inequality is weakened. Thus if $\rho : S \times S \to \mathbb{R}^+$ is a mapping, let

$$(i)\rho(x,y) = 0 \Leftrightarrow (\Leftarrow)x = y, (ii)\rho(x,y) = \rho(y,x),$$
$$(iii)\rho(x,y) \le k[\rho(x,z) + \rho(y,z)],\ \forall x,y,z \in S, \text{ and some } k \ge 1.$$

This ρ is termed a *quasi-(semi)metric*. Further one can use some ideas of abstract analysis and define several classes of such metrics. For instance, if $\varphi : \mathbb{R} \to \mathbb{R}^+$ is a generalized Young function satisfying a Δ_2-condition for large values, then define for any random variables X, Y with values in a metric space (S, d):

$$\rho_\varphi(X,Y) = \int_\Omega \varphi(d(X,Y))dP \tag{7}$$

or

$$\tilde{\rho}_\varphi(X,Y) = \inf\{\varepsilon > 0 : \varphi(F_X(x) - F_Y(x+\varepsilon)) < \varepsilon, \\ \varphi(F_Y(x) - F_X(x+\varepsilon)) < \varepsilon,\ x \in \mathbb{R}\}. \tag{8}$$

All these and their generalizations play important roles in the (weak) convergence of stochastic processes. Analysis with many of these (quasi)metrics has been extensively carried out by many people, including V. M. Zolotarev and especially Rachev himself together with several collaborators. In this paper first a very readable account of probability metrics, their numerous applications, and an extensive bibliography are included. Also the authors make a novel application of these metrics in obtaining some limit theorems for two problems in the spread of AIDS. This is done first for discrete time, and then the results are extended to continuous time by an approximation where the model now consists of a system of stochastic differential equations depending on a parameter N, for a fixed number (here 4) of communities. Then as $N \to \infty$, they establish that the number of infectives in each class converges to the unique solution of a Liouville type SDE. All the assumptions of the model and the underlying methodology are presented. There is an extensive bibliography on related work.

4. Chapter 4, written by me, is devoted exclusively to higher order SDEs. First order equations starting with the Langevin's, have been generalized in the literature to nonlinear equations using the full development of the Itô calculus and its extension to square integrable martingales. However, the second and higher order equations, which similarly start with the motion of a simple harmonic oscillator, have not received a corresponding treatment. The linear constant coefficient case has been studied by Dym (1966) who noted that, with white noise process as driving force, they exhibit special

features that are not seen in the first order case. He discussed the existence of solutions of an equation of the form

$$a_0 \frac{d^n X(t)}{dt^n} + a_1 \frac{d^{n-1} X(t)}{dt^{n-1}} + \cdots + a_n X(t) = \varepsilon(t), \tag{9}$$

with a_i as constants, written symbolically, but may be interpreted rigorously in integrated form as:

$$\int_{\mathbf{R}} \varphi(t)(I_{.,} X(t))dt = \int_{\mathbf{R}} dB(t), \tag{10}$$

where $L_n = a_0 \frac{d^n}{dt_n} + \cdots + a_n$ is a differential opretor, φ a smooth function of compact support, and $dB(t) = \varepsilon(t)dt, B(t)$ being Brownian motion. The existence of a distributional solution can be established and shown to be Markovian. The sample path analysis uses this property through semigroup theory, but now the associated infinitesimal generator of the latter is found to be a degenerate elliptic differential operator. This is a characteristic feature of the higher order cases.

To consider the corresponding nonlinear problem, it is necessary to define the derivative of X in the sense of mean and present conditions in order to interpret it in the pointwise sense as well. Only then the higher order SDEs can be studied. This and a generalization of the linear time dependent coefficient case (i.e., the a_i of (9) are $a_i(t)$ now) are treated in some detail in this chapter. Considering the second order case, for simplicity, the nonlinear form of (9) is studied:

$$d\dot{X}(t) = q(t, X(t), \dot{X}(t))dt + \sigma(t, X(t), \dot{X}(t))dB(t), \tag{11}$$

where q, σ are coefficients satisfying certain Lipschitz type conditions. The desired solution is an absolutely continuous process, for the driving force $\{B(t), t \geq 0\}$ which is now taken as an $L^{2,2}$-bounded process in the sense of (3) above. In this form it is shown that (11) has a unique absolutely continuous solution, and if moreover the $B(\cdot)$-process has independent increments then the solution is also Markovian. Next the $B(\cdot)$ is specialized to Brownian motion and an analysis of the associated semigroup is presented. This time one has to consider weighted continuous function space as the domain of this semigroup which however is not strongly continuous. The infinitesimal operator is again a degenerate elliptic operator with coefficients as functions of t, x, y. Analysis of these PDEs is difficult and many problems are pointed out for future investigations. A few results on the path behavior are then given. This work leads to multiparameter analogs (the solutions being random fields) and the SPDEs wherein the ideas and results of Chapter 2 can be utilized. But much of this remains to be explored.

5. An analysis of random processes and fields, which need not be Brownian but are square integrable, is considered in Chapter 5 by Swift. Most of the work here is related to several generalizations of (weakly) stationary processes and fields using the Fourier analysis methods and ideas. The central class is what is known as the harmonizable family. Thus a family $\{X_{\mathbf{t}}, \mathbf{t} \in \mathbb{R}^n\}$ is a harmonizable random field if it can be expressed as:

$$X_{\mathbf{t}} = \int_{\mathbf{R}^n} e^{\mathbf{t}\cdot\lambda} dZ(\lambda), \tag{12}$$

where $Z(\cdot)$ is a random measure on (the Borel σ-ring of) $\mathbb{R}^n$ with values in $L^2(P)$. Here one takes $E(X_{\mathbf{t}}) = 0 = E(Z(\lambda))$ for simplicity. If $F(\lambda, \lambda') = E(Z(\lambda)\bar{Z}(\lambda'))$, then F defines a bimeasure. If F has finite Vitali variation, then $X_{\mathbf{t}}$ is called ***strongly harmonizable***, and ***weakly harmonizable*** field otherwise. If moreover the covariance $r(\mathbf{s}, \mathbf{t}) = E(X_{\mathbf{s}}\bar{X}_{\mathbf{t}})$ is unchanged under rotations then $X_{\mathbf{t}}$ is *isotropic*. It is also possible to consider the increments of the process (field) to have the harmonizability and isotropy properties. In fact most of the questions investigated in the stationary case can be asked for the harmonizable families (cf., e.g., Yaglom (1987)). It should be noted that the covariance representation

$$r(\mathbf{s}, \mathbf{t}) = \iint_{\mathbf{R}^n \times \mathbf{R}^n} e^{i\mathbf{s}\cdot\lambda - i\mathbf{t}\cdot\lambda'} dF(\lambda, \lambda') \tag{13}$$

is in the Lebesgue sense only when F has finite Vitali variation. Otherwise one has to use a weaker Morse-Transue integration, suitably modified. The former is used in the strongly harmonizable case, and the latter for the weakly harmonizable ones. Now the structure of both these classes is discussed in detail in this chapter. Also treated are periodically correlated processes, local continuity, and almost periodicity of the sample paths. A recent general account of the basic theory of such processes, in multidimensions, is given by Kakihara (1997), and the current chapter essentially complements it. A large part on fields generalizes the work in Yadrenko (1983).

The analyticity of harmonizable random fields, and sampling theorems as well as the Cramér classes are treated here. Also discussed are integral (spectral) representations of fields with m^{th}-order increments of this class. Further harmonizable spacially isotropic random fields are treated. Some multidimensional extensions are briefly described. Several problems in the area that await future investigation are pointed out at various places. The account gives a comprehensive view of the area along with an extended bibliography for related studies.

6. The final chapter is devoted to the Gaussian dichotomy problem by Vakhania and Tarieladze. There are several proofs of the dichotomy theorem

in the literature. Here the authors present a simpler argument than before, avoiding any usage of conditioning which appears at least in invoking the martingale convergence before. An interesting aspect now is that the result is reduced to Kakutani's (1948) theorem on infinite product measures. The authors reformulated this result, for uncountable sets of probability measures, to use it in their dichotomy proof. Some refinements of others' works are also obtained from these ideas.

The authors then proceeded to present simple proofs of the assertions of the following type for measures on general locally convex topological vector spaces (LCTVSs). For instance, let $\mathcal{X}$ be a real LCTVS and $G \subset \mathcal{X}^*$ be a set of continuous linear functionals that separate points of $\mathcal{X}$. If $\sigma(G)$ is the smallest σ-algebra of $\mathcal{X}^*$ relative to which all elements of G are measurable, then a pair of Gaussian measures on $\sigma(G)$ (i.e., $\mu \circ f^{-1}, \nu \circ f^{-1}$ are Gaussian probabilities on $\mathbb{R}$ for each $f \in G$) satisfy the dichotomy theorem. Thereafter conditions in terms of mean and covariance operators for equivalence of μ, ν are presented. This work gives a fresh approach to an old problem, and harvests some consequences.

It is thus evident that real analysis methods play a fundamental role in all the works presented here. Moreover, there are places where the stochastic theory raises new questions of abstract analysis such as for SPDEs, not strongly continuous semigroup study, and degenerate elliptic operator theory itself. One expects that this interaction will help advance both subject areas as well as their applications.

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Chapter 1

Transport Properties of Gaussian Velocity Fields

René A. Carmona[1]

Abstract

The purpose of these lecture notes is to describe several mathematical problems which arise in the study of the statistical properties of the solutions of the equation:

$$dX_t = \vec{\mathbf{v}}(t, X_t)dt + \sqrt{2\kappa}dB_t$$

when $\{\vec{\mathbf{v}}(t,\mathbf{x});\ \mathbf{t} \geq 0, \mathbf{x} \in \mathbf{R}^{\mathbf{d}}\}$ is a mean zero stationary and homogeneous Gaussian field and $\{B_t;\ t \geq 0\}$ a process of Brownian motion. We are mostly interested in velocity fields with spectra of the Kolmogorov type. The study is motivated by problems of transport of passive tracer particles at the surface of a two-dimensional medium. We are mostly concerned with the mathematical analysis of problems from oceanography and we think of the surface of the ocean as a physical medium to which our modeling efforts could apply.

1.1 Introduction

The purpose of these notes is to present in a more or less informal manner a set of mathematical problems which arise in the study of the statistical properties of the solutions of the equation:

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$$dX_t = \vec{v}(t, X_t)dt + \sqrt{2\kappa}dB_t \tag{1.1}$$

when $\{\vec{\mathbf{v}}(t,\mathbf{x});\ \mathbf{t} \geq 0, \mathbf{x} \in \mathbf{R}^{\mathbf{d}}\}$ is a mean zero stationary and homogeneous Gaussian field, $\kappa \geq 0$ and $\{B_t;\ t \geq 0\}$ is a d-dimensional process of Brownian motion. Except for the last section, in which we discuss stochastic partial differential equations (SPDE for short), we shall restrict ourselves to the case $\kappa = 0$. Moreover, most of our efforts will be devoted to the case of velocity fields with spectra of the Kolmogorov type. This assumption is motivated by problems of fluid mechanics. Instead of considering velocity fields which are solutions of the Navier-Stokes equation, we use the dynamical approach and assume from the beginning that $\{\vec{\mathbf{v}}(t,\mathbf{x})\}$ is a stationary and homogeneous random field. According to Kolmogorov's theory of well-developed turbulence (i.e., for systems with high Reynolds numbers), this assumption is well founded. See, for example, Chapters 6 and 7 in [36] for an excellent account of this theory in a modern perspective. We shall add the assumption that the velocity field is Gaussian. As proven by the results of many wind velocity measurements, this is a very reasonable assumption.

Our main concern is the analysis of the transport of passive tracers at the surface of a two-dimensional medium. We are mostly interested in the mathematical modeling of problems from oceanography and we think of the surface of the ocean as a physical medium to which our modeling efforts could apply. For this reason we shall sometimes use the terminology *drifters* or *floats* for the passive tracers.

Transport properties of time-independent velocity fields have been studied both from a theoretical point of view and via computer simulations. The results have been reported in many publications. See, for example, the recent works [34] or [35]. The latter cannot be compared to ours because the time-independence of the velocity field drastically changes the nature of the simulation algorithms and the typical properties of the tracers. At this stage it is important to emphasize the differences between the two approaches. It is very often the case that the terminology *disordered systems* or *random media* is used for models in which the randomness is autonomous (i.e., time independent). The analysis of such systems is most naturally done by first studying the properties of the randomness and exhibiting properties of the random parameters of the model, which almost surely hold. Then the mathematical analysis is performed in a very classical manner, by fixing the values of the environment and then studying the system as if it were deterministic. This approach is rarely possible for time-dependent random models. Indeed, probabilistic-like arguments are needed throughout the analysis.

Closer to our point of view are the theoretical results obtained and the numerical simulations performed in the case of Brownian flows. See [47], [7],