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MATHEMATICS OF TWO-DIMENSIONAL TURBULENCE

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Mathematics of Two-Dimensional Turbulence

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and

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Preface

Equations and forces

Two-dimensional (2D) statistical hydrodynamics studies statistical properties of the velocity field $u(t,x)$ of a (imaginary) two-dimensional fluid satisfying the stochastic 2D Navier–Stokes equations

$$\begin{aligned} &\dot{u}(t,x) + \langle u, \nabla \rangle u - \nu \Delta u + \nabla p = f(t,x), \quad \operatorname{div} u = 0, \\ &u = (u^1, u^2), \quad x = (x_1, x_2). \end{aligned} \tag{1}$$

Here $\nu > 0$ is the kinematic viscosity, $u = u(t,x)$ is the velocity of the fluid, $p = p(t,x)$ is the pressure, and f is the density of an external force applied to the fluid. The space variable x belongs to a two-dimensional domain, which in this book is supposed to be bounded. Suitable boundary conditions are assumed. For example, one may consider the case when the domain is a rectangle $(0,a) \times (0,b)$, where a and b are positive numbers, and the equations are supplemented with periodic boundary conditions; that is to say, the space variable x belongs to the torus $\mathbb{R}^2/(a\mathbb{Z} \oplus b\mathbb{Z})$ (in the case of periodic boundary conditions we will assume that space-mean values of the force f and the solution u vanish). Equations (1) are *stochastic* in the sense that the initial condition $u_0 = u(0,x)$, or the force f, or both of them, are random, i.e., depend on a random parameter. So the solutions u are random vector fields. The task is to study various characteristics of u averaged in ensemble, or to study their properties which hold for most values of the random parameter. In this book, we assume that the force is random and refer the reader to [FMRT01] for a mathematical treatment of the Navier–Stokes equations with zero (or deterministic) force and random initial data.

The Reynolds number R of a random velocity field $u(t,x)$ is defined as

$$R = \frac{\langle \text{characteristic scale for } x \rangle \cdot \big(\mathbb{E}E(u)\big)^{1/2}}{\nu},$$

where $E(u) = \frac{1}{2}\int |u(x)|^{1/2}dx$ is the kinetic energy of the fluid and $\mathbb{E}$ denotes the average in ensemble. Since the forces we consider are smooth, then the solutions u of (1) are regular in x and their space-scale is of order one. So $R \sim \nu^{-1}(\mathbb{E}E(u))^{1/2}$. A velocity field u is called *turbulent* if $R \gg 1$. Turbulent solutions for (1) are of prime interest.

If the motion of a "physical" three-dimensional fluid is parallel to the (x_1, x_2)-plane and its velocity depends only on (x_1, x_2), i.e., $u = u(t, x_1, x_2)$ and $u^3 = 0$, then $(u^1, u^2)(t, x_1, x_2)$ satisfy (1). Such flows are called *two-dimensional*. Turbulent flows of real fluids are never two-dimensional (i.e., two-dimensional flows are never observed in experiments with high Reynolds number). Still, the 2D equations (1) and the 2D turbulence which they describe are now intensively studied by mathematicians, physicists, and engineers since, firstly, they appear in physics outside the realm of hydrodynamics (e.g., they describe flows of 2D films, see Figure 1 on p. xv), secondly, they provide a model[1] for the 3D Navier–Stokes equations and 3D turbulence, and, thirdly, the 3D statistical hydrodynamics in thin domains is approximately two-dimensional; see Section 6.1 of this book. Accordingly, two-dimensional statistical hydrodynamics is important for meteorology to model intermediate-scale flows in atmosphere (see Figures 6.1 and 6.2 in Chapter 6).

Statistical properties of the random force f are very important. It is natural and traditional to assume that

(a) the random field $f(t, x)$ is smooth in x, and
(b) it is stationary in t with rapidly decaying correlations.
If the space domain is unbounded, we should also assume that
(c) the space correlations of f decay rapidly.

However, (c) is not relevant for this book since we only consider flows in bounded domains.

In mathematics, the point of view[2] that turbulence in dimensions 2 and 3 should be described by the Navier–Stokes equations with a random force satisfying (a)–(c) goes back to A. N. Kolmogorov; see in [VF88]. Also see that book for some results on stochastic Navier–Stokes equations in the whole space $\mathbb{R}^d$, $d = 2$ or 3, with a random force satisfying (a)–(c).

We consider three classes of random forces:

Kick forces. These are random fields of the form

$$f(t, x) = h(x) + \sum_{k\in\mathbb{Z}} \delta(t - \tau_k)\eta_k(x), \tag{2}$$

[1] This model is not perfect since it is well known that the Navier–Stokes equations in dimensions 2 and 3 are very different. Still, it may be the best available now. Another popular model for the 3D Navier–Stokes system is the Burgers equation; see the review [BK07] by Bec and Khanin. For the stochastic 1D Burgers equation, see [Bor12].

[2] which is not at all a unique insight on turbulence!

where h is a smooth deterministic function, $\tau_k = k\tau$ with some $\tau > 0$, and $\{\eta_k\}$ are independent identically distributed random vector functions, which we assume to be divergence-free. For $t \in (\tau_{k-1}, \tau_k)$ (i.e., between two consecutive kicks) a solution $u(t, x)$ for (1), (2) satisfies the deterministic equations $(1)_{f=h}$, and at the time τ_k, when the k^{th} kick $\eta_k(x)$ comes, it has an instant increment equal to that kick; see Section 2.3. The kick forces are singular in t and are not stationary in t, but statistically periodic (the difference between the two notions is not great if the time t is much longer than the period τ between the kicks). An advantage of this class of random forces is that the kicks η_k may have any statistics.

White in time forces. These are random fields of the form

$$f(t, x) = h(x) + \frac{d}{dt}\zeta(t, x), \tag{3}$$

where h is as above and $\zeta(t) = \zeta(t, \cdot)$ is a Wiener process in the space of smooth divergence-free vector functions. Such random fields are stationary and singular in t. A disadvantage is that they must be Gaussian; see Section 2.4.

Compound Poisson processes. These are kick forces (2) for which the periods $\tau_k - \tau_{k-1}$ between kicks are independent exponentially distributed random variables.

A big technical advantage of these three classes of random forces is that the corresponding solution $u(t, x)$, regarded as a random process $u(t, \cdot) =: u(t)$ in the space of vector fields, is a Markov process. At the time of writing it is not clear how to extend the results of this book to arbitrary smooth random forces f satisfying (a) and (b).

What is in this book?

We are concerned with basic problems and questions, interesting for physicists and engineers working in the theory of turbulence. Accordingly Chapters 3–5 (which form the main part of this book) end with sections where we explain the physical relevance of the obtained results. These sections also provide brief summaries of the corresponding chapters.

In Chapters 3 and 4, our main goal is to justify, for the 2D case, the statistical properties of a fluid's velocity field $u(t, x)$ which physicists assume in their work. We refer the reader to the books [Bat82; Fri95; Gal02], written in a sufficiently rigorous way and where the underlying assumptions are formulated in a clear manner.[3] The first postulate in the physical theory of turbulence is that

[3] Apart from a few pages at the end, the book [Bat82] is about 3D flows. But all discussions and most of the results may be literally translated to the 2D case.

statistical properties of a turbulent flow $u(t,x)$ converge, as time goes to infinity, to a statistical equilibrium independent of the initial data. Mathematically speaking, this means that a process $u(t,\cdot)$, defined by Eq. (1) in the space of vector fields, has a unique stationary measure, and every solution converges to this measure in distribution. That is, the law of the random field $x \mapsto u(t,x)$ (which is a time-dependent measure in a function space) converges, when $t \to \infty$, to the measure in question. Random processes possessing this property of "short-range memory" are said to be *mixing*.

In Chapter 3, we study the problem of convergence to a statistical equilibrium for Markov processes corresponding to equations with the three classes of random forces as above. We prove abstract theorems which establish the exponential mixing for certain classes of Markov processes. Next we show that these theorems apply to Eq. (1) if a random force f satisfies certain mild non-degeneracy assumptions. This establishes the convergence to a unique statistical equilibrium and proves that it is exponentially fast.

If the viscosity ν and the force f continuously depend on a parameter in such a way that the former stays positive and the latter stays non-degenerate, then the stationary measure continuously depends on this parameter. For any fixed initial data $u(0)$ the law of the corresponding solution $u(t)$ continuously depends on the parameter as well. In Section 4.3, we show that this continuity is uniform in time $t \geq 0$. That is, in two space dimensions the statistical hydrodynamics is stable, no matter how big the Reynolds number, whereas the "usual" hydrodynamics of large Reynolds numbers is very unstable.

The mixing has a number of important consequences, well-known in physics, but taken for granted there. Namely, consider any observable quantity $F(u)$, such as the first or second component of the velocity field $u = (u^1, u^2)$, or the energy $E = \frac{1}{2}\int |u|^2 dx$, or the enstrophy $\frac{1}{2}\int (\operatorname{curl} u)^2 dx$. Then $F(t) = F(u(t,\cdot))$ is an ergodic process. That is, *its time average converges to the ensemble average with respect to the stationary measure.* We show that the difference between the two mean values (in time and in ensemble) decays as $T^{-\gamma}$, where $\gamma < 1/2$ and T is the time of averaging; see Section 4.1.1. Next, if the ensemble average for an observable $F(u)$ vanishes, then the process $F(t)$ satisfies the central limit theorem: the law of the random variable

$$\frac{1}{\sqrt{T}} \int_0^T F(t)\, dt$$

converges, as $T \to \infty$, to a normal distribution $N(0,\sigma)$. For non-trivial observables F, the dispersion σ is strictly positive. In particular, for large T the random variables $T^{-1/2}\int_0^T u^j(t,x)dt$, $j = 1, 2$, are almost Gaussian. Physicists say that *on large time-scales a turbulent velocity field is approximately Gaussian.* These and some other related results are proved in Chapter 4.

In Chapter 5 we study velocity fields $u(t,x)$, corresponding to solutions of (1) with a force (2) or (3) where $h=0$, when the viscosity ν is small and the Reynolds number is large. There we only discuss stationary measures and stationary-in-time solutions u_ν (i.e., solutions $u_\nu(t,x)$ such that the law $\mathcal{D}(u_\nu(t))$ for each t equals the stationary measure). First we observe that for a limit of order one to exist as $\nu\to 0$, the force f should be proportional to $\sqrt{\nu}$; see Section 5.2.4. So the equations read as

$$\dot{u}(t,x)+\langle u,\nabla\rangle u-\nu\Delta u+\nabla p=\sqrt{\nu}\, f(t,x), \quad \operatorname{div} u=0,$$

where f is the force (2) or (3) with $h\equiv 0$. This is in sharp contrast with the 3D theory, where it is believed that a limit of order one exists for the original scaling (1), without the additional factor $\sqrt{\nu}$ on the right-hand side.[4] In that chapter we restrict ourselves to the case when the space domain is the square torus $\mathbb{T}^2=\mathbb{R}^2/2\pi\mathbb{Z}^2$. The results remain true for the non-square tori $\mathbb{R}^2/(a\mathbb{Z}\oplus b\mathbb{Z})$, but the argument does not apply to the equations in a bounded domain with the Dirichlet boundary condition.

Denote by μ_ν the unique stationary measure. We show that the set of measures $\{\mu_\nu, 0<\nu\le 1\}$ is tight (i.e., relatively compact) and that any limit point $\mu_0=\lim_{\nu_j\to 0}\mu_{\nu_j}$ is a non-trivial invariant measure for the Euler system

$$\dot{u}(t,x)+\langle u,\nabla\rangle u+\nabla p=0, \quad \operatorname{div} u=0.$$

It is supported by the set of divergence-free vector fields from the Sobolev space H^2 of order two. This result agrees well with the popular belief that *the Euler equation is "responsible" for 2D turbulence*. We do not know if a limiting measure μ_0 is unique, i.e., if $\mu_0=\lim_{\nu\to 0}\mu_\nu$. But we know that the measures μ_ν satisfy, uniformly in $\nu>0$, infinitely many algebraical relations, called the *balance relations*. These relations depend only on two scalar characteristics of the force f. This indicates some *universality features of 2D turbulence*. Such universality is another physical belief. In Section 5.1.3, we use the balance relations to prove that for any t and x the random variables $u_\nu(t,x)$ and $\operatorname{curl} u_\nu(t,x)$ have finite exponential moments uniformly in $\nu\ge 0$. In Section 5.2, we study further properties of the limiting measures μ_0. In particular, we establish that any μ_0 has no atoms and that its support is an infinite-dimensional set.

The results of Chapter 5 provide a foundation of the mathematical theory of space-periodic 2D turbulence. In Section 5.3, we discuss the relation of these results with the existing heuristic theory of 2D turbulence, originated by Batchelor and Kraichnan.

[4] Note that for the small-viscosity Burgers equation the right scaling of the force is also trivial, i.e., without any additional factor; see [BK07; Bor12].

The difference between 2D turbulence and real physical 3D turbulence is very great. In Chapter 6, we discuss a few rigorous results on 3D turbulence, related to the 2D theory presented in the preceding sections. Namely, in Section 6.1 we discuss (without proof) the convergence of the statistical characteristics of a flow in a thin 3D layer, corresponding to the 3D Navier–Stokes system with a random kick force, to those of a 2D flow in the limiting 2D surface. In contrast with similar deterministic results, the convergence holds uniformly in time. So a class of *anisotropic 3D turbulent flows may be approximated by 2D flows* like those which we consider in our book. Section 6.2 contains a discussion of results due to Da Prato, Debussche, and Odasso, and Flandoli and Romito, showing that weak solutions of the stochastic 3D Navier–Stokes system perturbed by a white-in-time random force (which a priori are non-unique) may be arranged as a Markov process. This process is mixing if the force is rough as a function of the space variable. Finally, in Section 6.3, we invoke the methods of control theory to study further properties of stationary measures for Eqs. (1), (3).

Other equations

The abstract theorems from Chapters 3 and 4 and the methods developed there to study the solutions of Eq. (1) apply to many other stochastic equations. For instance, one can consider the stochastic complex Ginzburg–Landau equation with a conservative nonlinearity,

$$\dot{u} + i\Delta u - i|u|^{2m}u = \Delta u - u + f(t,x), \tag{4}$$

where $x \in \mathbb{T}^d$, $d \le 3$. If $d = 1$ or 2, then $m \ge 0$, while if $d = 3$, then one can take, say, $m \in [0, 1]$. Such equations describe *optical turbulence*. If f is a bounded kick force, then direct analogues of the theorems in Chapters 3 and 4 remain true for (4) with the same proof.

However, if the force f is white in time, then the methods of Chapters 3 and 4 apply only to Eq. (4) with $m = 1$ if $d = 1$ and $m < 1$ if $d \ge 2$ (while the equation defines a good Markov process for any m as above). That is, for some deep reason, the arguments developed to treat the stochastic Navier–Stokes equations (1) with white-in-time forces apply only to PDEs with conservative nonlinearities of degree ≤ 3;[5] see Section 3.5.5.

Readers of this book

The book is aimed at mathematicians and physicists with some background in PDEs and in stochastic methods. Standard university courses on these subjects are sufficient since the book is provided with preliminaries on function

[5] But the method applies to Eq. (4) with $m > 1$ if we add a strong nonlinear damping $-|u|^{2m'}u$, $m' \ge m$, on the right-hand side.

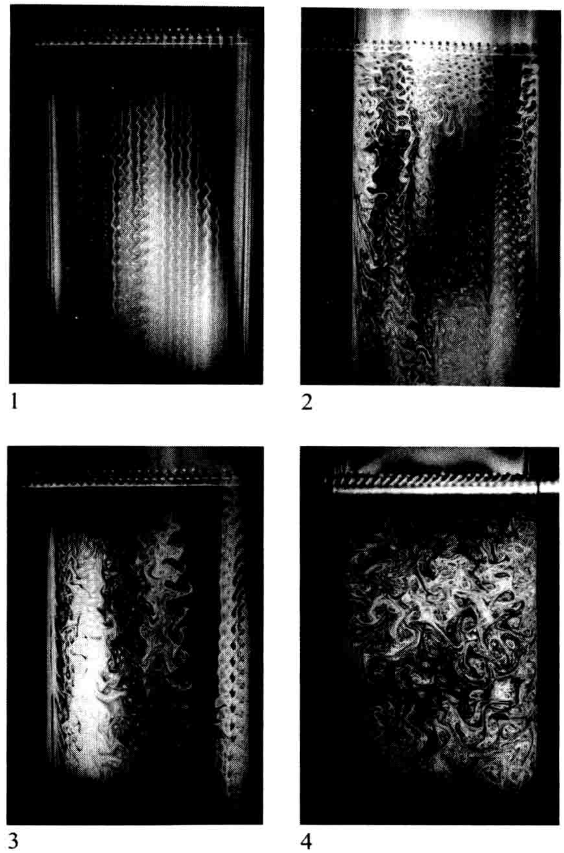

Figure 1 The onset of 2D turbulence. Panels 1–4 represent the down-motion of a soap film, punctured by a comb at the top. The Reynolds number is increasing from panel to panel. This is 2D turbulent motion described by the 2D Navier–Stokes system (1). Reprinted with permission from [Rut96]. Copyright 1996, American Institute of Physics.

spaces (Section 1.1), on the 2D Navier–Stokes equations (Chapter 2), and on stochastics (Sections 1.2 and 1.3). There the reader will find all the needed non-standard results.

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Contents

1

Preliminaries

1.1 Function spaces

1.1.1 Functions of the space variables

Let Q be a domain in $\mathbb{R}^d$ (i.e., a connected open subset of $\mathbb{R}^d$) or the torus $\mathbb{T}^d = \mathbb{R}^d/2\pi\mathbb{Z}^d$. We shall say that a domain Q is *Lipschitz* if its boundary ∂Q is locally Lipschitz.[1] We shall need *Lebesgue* and *Sobolev* spaces on Q and some embedding and interpolation theorems.

Lebesgue spaces

We denote by $L^p(Q;\mathbb{R}^n)$, $1 \le p \le \infty$, the usual Lebesgue space of vector-valued functions and abbreviate $L^p(Q;\mathbb{R}) = L^p(Q)$. We write $\langle \cdot, \cdot \rangle$ for the L_2 scalar product and $|\cdot|_p$ for the standard norm in $L^p(Q;\mathbb{R}^n)$.

Sobolev spaces

We denote by $C_0^\infty(Q;\mathbb{R}^n)$ the space of infinitely smooth functions $\varphi : Q \to \mathbb{R}^n$ with compact support. Let u and v be two locally integrable scalar functions on Q and let $\alpha = (\alpha_1, \dots, \alpha_d)$ be a multi-index. We say that *v is the α^{th} weak partial derivative of u* if

$$\int_Q u D^\alpha \varphi \, dx = (-1)^{|\alpha|} \int_Q v\varphi \, dx \quad \text{for all } \varphi \in C_0^\infty(Q;\mathbb{R}),$$

where $|\alpha| := \alpha_1 + \cdots + \alpha_d$ and $D^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$. In this case, we write $D^\alpha u = v$.

Let $m \ge 0$ be an integer. The space $H^m(Q, \mathbb{R}^n)$ consists of all locally integrable functions $u : Q \to \mathbb{R}^n$ such that the derivative $D^\alpha u$ exists in the weak

[1] This means that ∂Q can be represented locally as the graph of a Lipschitz function.