Y. K. Cheung
A. Y. T. Leung

Finite Element Methods in Dynamics

SCIENCE PRESS
BEIJING/NEW YORK/HONG KONG
KLUWER ACADEMIC PUBLISHERS
DORDRECHT/BOSTON/LONDON

Finite Element Methods in Dynamics

by

Y.K. Cheung and A.Y.T. Leung

Science Press
BEIJING / NEW YORK / HONG KONG

Kluwer Academic Publishers
DORDRECHT / BOSTON / LONDON

Library of Congress Cataloging in Publication Data

Cheung. Y.K.
Finite element methods in dynamics/by Y.K. Cheung, A.Y. T. Leung.
p. cm.
Translated from Chinese.
Includes bibliographical references and index.
ISBN 0-7923-1313-5
1. Structural dynamics. 2. Finite element method.
I. Leung, A.Y.T. II. Title.
TA654.C5215 1992
624.1'7 –dc20 91-19401

ISBN 7-03-002730-2/O·514 (Science Press)
ISBN 0-7923-1313-5 (Kluwer Academic Publishers)

Kluwer Academic Publishers incorporates
the publishing programmes of
D. Reidel, Martinus Nijhoff, Dr W. Junk and MTP Press.

Distributors for People's Republic of China:
Science Press, Beijing.

Sold and distributed in the U.S.A. and Canada
by Kluwer Academic Publishers,
101 Philip Drive, Norwell, MA 02061, U.S.A.

In all other countries, sold and distributed
by Kluwer Academic Publishers Group,
P.O. Box 322, 3300 AH Dordrecht, The Netherlands.

Printed in Hong Kong

PREFACE

Structural dynamics has been receiving ever increasing attention in the past few decades. This is mainly due to human needs, such as adventure in outer space, oil extraction in angry seas, tall buildings in populated cities, long span bridges across wide channels, etc. This is also due to the flexibility and accuracy of new analytical methods, such as finite element method, and the power of new engineering tools, such as digital computers. Indeed, if economically permissible, the engineering profession, which is well equipped with the recognized analytical methods, solution methods and computation tools, can design safe structures in dynamic environments within any predetermined confidence level. This book presents the latest developments in structural dynamics with particular emphasis on the formulation of equations of motion by finite element methods and the solutions of those equations using micro-computers. The authors believe that it is the first book which deals with frequency-dependent shape functions for realistic finite element modelling of dynamic problems and which points out that the conventional shape functions are only approximations. As the computation and application share equal importance, a complete listing of a natural vibration finite element package is given. A feature of this book in handling the forced vibration problem is to separate the solution into two parts, steady state and transient in connection with the complementary function, and enables an engineer to design each distinct part independently. Some advanced topics such as substructure and synthesis which are viewed in a modern unified manner are also included.

We are grateful to Associate Professor Fu Zizhi for his suggestion that we should write a book for the Science Press. We are obliged to our

graduate students for carrying out the computation of some of the worked examples, and in particular to Mr. Chen Shuhui for checking the manuscripts. Finally, we are indebted to our Departmental staff, Mrs. Brenda Ng and Mrs. K. Cheung, for typing part of the manuscripts and drawing the diagrams.

Y. K. Cheung

A. Y. T. Leung

Hong Kong

July 1990

CONTENTS

CHAPTER			PAGE
1.	FINITE ELEMENTS AND STRUCTURAL DYNAMICS		1
	1.1	INTRODUCTION	1
	1.2	VIBRATION OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM	4
	1.3	VIBRATION OF MULTI-DEGREE-OF-FREEDOM SYSTEMS	9
		1.3.1 Introduction	9
		1.3.2 Natural Modes	11
		1.3.3 Modal Analysis	14
	1.4	FINITE ELEMENT PROCEDURE IN DYNAMIC ANALYSIS	16
	1.5	DISPLACEMENT SHAPE FUNCTION	22
		1.5.1 Introduction	24
		1.5.2 Choice of Shape Functions	24
		1.5.3 Simple Polynomial	25
		1.5.4 Lagrange Polynomial	25
		1.5.5 Hermite's Formula	28
		1.5.6 Area Coordinates	31
		1.5.7 Volume Coordinates	33
		1.5.8 Higher Order 'Triangular' Elements	34
		1.5.9 Serendipity Rectangular Shape Functions	38
		1.5.10 Mapping	43
	1.6	DYNAMIC STIFFNESS METHOD	43
		REFERENCES	45
2.	ELEMENT MATRICES		46
	2.1	INTRODUCTION	46
	2.2	BAR ELEMENT	47
		2.2.1 Stiffness Matrix	47
		2.2.2 Consistent Mass Matrix Formulation	47
		2.2.3 Continuous Mass Matrix Formulation	49
		2.2.4 Other Mass Matrix Models	50
		2.2.5 Torsion	51
	2.3	BEAM ELEMENTS	52
		2.3.1 Thin Uniform Beam Element	53

2.3.2 Moderately Thick Uniform Beam (Timoshenko Beam) Element 56
2.3.3 Effects of Axial Force on a Thin Uniform Beam 62
2.3.4 Continuous Mass Model 63
2.3.5 Higher Order Elements 67
2.4 2D AND 3D LAGRANGE FAMILY ELEMENTS 68
2.5 AREA COORDINATE ELEMENTS 72
2.5.1 Displacement Interpolation Method 75
2.5.2 Strain Interpolation Method 89
2.6 PLATE BENDING ELEMENTS 92
2.6.1 Introduction 92
2.6.2 12-DOF Rectangular Elements 93
2.6.3 Hermite Family Elements 96
2.6.4 Triangular Plate Element 99
2.7 ISOPARAMETRIC ELEMENTS 109
APPENDIX 115
REFERENCES 126
3. SYSTEM MATRICES 128
3.1 COORDINATE TRANSFORMATION 128
3.2 INCIDENCE NUMBER METHOD OF ASSEMBLING 134
3.3 STORAGE TECHNIQUES 139
3.4 SOLUTION OF SYSTEM MATRICES 141
3.5 EXAMPLE 143
4. SOLUTION OF SYSTEM MATRICES 146
4.1 INTRODUCTION 146
4.2 MATRIX TRANSFORMATION AND SPECIAL MATRICES 148
4.3 GAUSS ELIMINATION AND STURM SEQUENCE 151
4.4 POWER ITERATION AND INVERSE ITERATION 156
4.5 TRANSFORMATION METHODS 162
4.5.1 Jacobi Method 163
4.5.2 Generalized Jacobi Method 167
4.5.3 Givens Method 168
4.5.4 Householder Method 173
4.5.5 Fast Plane Rotations 175

4.5.6 LR, QR, QL and Cholesky Algorithms for Solving Tridiagonal Matrices 182
4.5.7 Extensions of LR, QR, QL and Cholesky Algorithms 188
4.5.8 Generalized Eigenvalue Problem 190
4.6 SUBSPACE ITERATION 192
4.7 LANCZOS METHOD 198
5. DYNAMIC RESPONSE AND ADVANCED TOPICS 206
5.1 INTRODUCTION 206
5.2 LAGRANGIAN EQUATIONS 207
5.3 TIME STEP INTEGRATION 211
5.4 MODAL ANALYSIS 212
5.5 STEADY STATE RESPONSE 215
5.6 HEAVILY DAMPED SYSTEM 219
5.7 DYNAMIC SUBSTRUCTURE METHODS 223
5.7.1 Theory 223
5.7.2 Solution of the Nonlinear Eigenvalue Problem 225
5.7.3 Methods of Guyan and Irons 225
5.7.4 Ritz Vector Method 225
5.7.5 Substructure Synthesis and Dynamic Substructures 226
5.7.6 Numerical Examples 227
REFERENCES 233
6. NATVIB - A COMPUTER PACKAGE 235
6.1 INTRODUCTION 235
6.2 STRUCTURE IDENTIFICATION 235
6.3 DATA FILE 236
6.4 EXTENSIONS 239
6.5 ORGANIZATION OF INPUT AND OUTPUT DATA 243
6.6 PROGRAMME LISTINGS 251
6.7 EXAMPLES 273
6.8 REPEATED INCIDENCE METHOD FOR TALL BUILDINGS . . . 273
REFERENCES 289
Index 290

CHAPTER 1

FINITE ELEMENTS AND STRUCTURAL DYNAMICS

1.1. INTRODUCTION

Structural dynamics is the study of the relations between applied forces and deformation and stress responses of structures as functions of time. It is a rapidly expanding area of applications of the now well-known finite element method in the fields of aeronautical, civil, mechanical and offshore engineering. Because of the additional time variable, many dynamic problems cannot be solved by analytical methods as effectively as their counterparts in statics. It can be claimed that a common numerical method such as the finite element method is a more valuable tool in dynamics than in statics.

As this book is intended to be an introductory level text, only linear problems are considered. The topics covered include free and forced vibrations. In particular, the finite element method is used to formulate the discretised equations of motion and then, various computational methods are introduced to solve such equations. Attention is also paid to the practical and large scale analysis of structures in dynamic environment.

The distribution of the displacements in a structure can be described by a displacement function,

$$\{\mathbf{u}(x,y,z,t)\} = [u(x,y,z,t), v(x,y,z,t), w(x,y,z,t)]^T$$

Sometimes the spatial and time variables of the displacement function are separable. It can then be expressed as the sum of products of time and space parameters, i.e.

$$u(x,y,z,t) = q_1(t)u_1(x,y,z) + q_2(t)u_2(x,y,z) + \ldots + q_n(t)u_n(x,y,z) \tag{1.1.1}$$

in which $q_i(t)$ are generalized coordinates to be determined and $u_i(x,y,z)$ are displacement functions, which are usually prescribed or computed beforehand by the finite element method.

Very often, the conditions of being separable in time and spatial variables cannot be satisfied exactly. However, in engineering practice, the conditions can always be satisfied approximately when a sufficiently large number of terms are taken in Eq.(1.1.1).

By discretising the space domain into finite elements, the complex time-space problem can be reduced to a time-dependent problem alone at some discrete points x_k, y_k, z_k (or nodes). Eq.(1.1.1) can then be written as

$$u(t) = q_1(t)u_1 + q_2(t)u_2 + \ldots + q_n(t)u_n \tag{1.1.2}$$

Where $u_i = u_i(x_k, y_k, z_k)$ is a finite element nodal parameter.

The generalized coordinates q_i are coupled in general, and cannot be solved one by one. However, there is a class of displacement functions which will uncouple the generalized coordinates such that q_i can be solved individually. These are the so-called eigenfunctions, or when discretised, eigenvectors. They are also often known as natural modes. Fortunately, the natural modes can be determined from the simplest equation of motion of undamped harmonic vibration, in which the time variable has been eliminated. Therefore the normal procedures of solving a structural dynamic problem are as follows:

(i) formulate the equations of motion in time variable alone at nodal points with assumed displacement functions;

(ii) solve for the natural modes;

(iii) uncouple the terms in the equations of motion by using the natural modes; and

(iv) solve the uncoupled sets of equations of motion one by one for the generalized coordinates q_i and finally compute the displacement response.

The above procedures are based on the modal analysis method. However, there are other methods in which the coupled equations of motion are integrated directly without reference to natural modes, and they are called direct integration methods. In this text only the modal method will be discussed extensively because it has wide applications in linear structural mechanics.

In this chapter, a spring system with a single mass will be used as an introduction to the vibration phenomena, and the method of modal analysis will be illustrated by a multi-degree-of-freedom system. Energy theorems and finite element procedure in dynamics will also be included in the presentation.

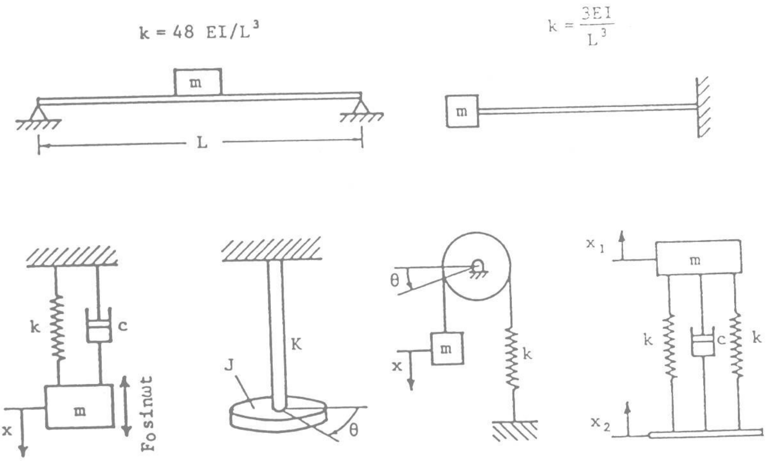
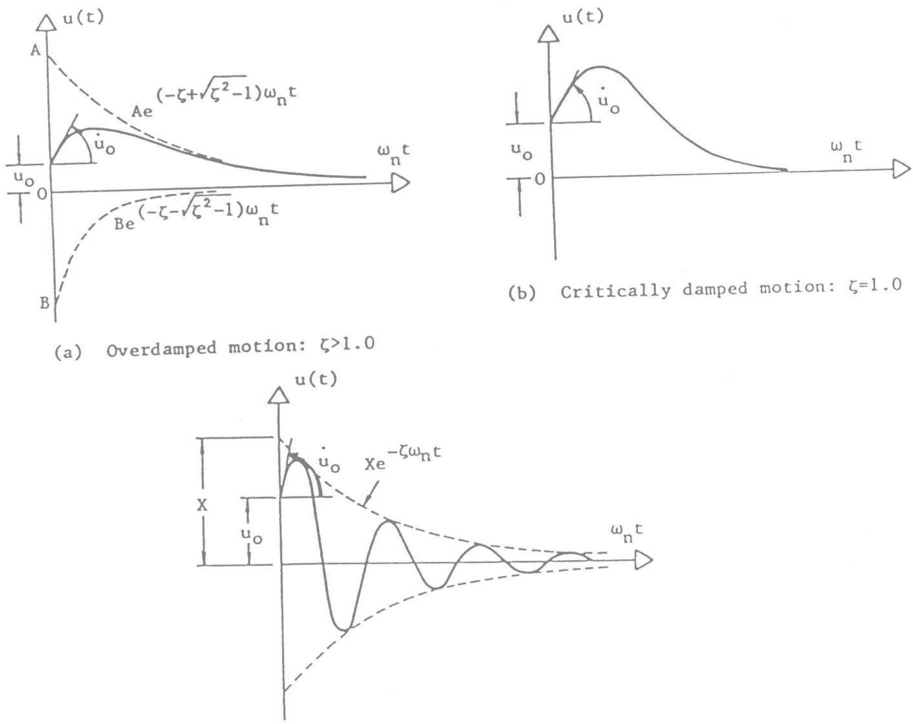


Fig. 1.2.1 Single-degree-of-freedom system



(a) Overdamped motion: $\zeta > 1.0$

(b) Critically damped motion: $\zeta = 1.0$

(c) Under-damped oscillation: $\zeta < 1$

Fig. 1.2.2 Damped free responses with initial conditions
A, B, X are integration constants

1.2. VIBRATION OF A SINGLE-DEGREE-OF-FREEDOM SYSTEM

The number of degrees of freedom of a vibrating system corresponds to the number of displacement parameters which are required to describe the vibrating configuration of the system adequately. Each system in Fig. 1.2.1 can be represented adequately by a single-degree-of-freedom system. The essential features include:

(i) the mass m, producing inertia force $-m\ddot{u}$;

(ii) the stiffness k, producing restoring force $-ku$;

(iii) the damping mechanism with coefficient c, producing viscous damping force $-c\dot{u}$; and

(iv) the applied force f.

Strictly speaking, the damping force is not necessarily proportional to the velocity. However, such a simplification serves the purpose of illustrating the vibration phenomena adequately. In the state of dynamic equilibrium,

$$m\ddot{u} + c\dot{u} + ku = f(t) \qquad (1.2.1)$$

This equation is to be solved for u(t) satisfying the initial conditions:

$$u(0) = u_0, \quad \dot{u}(0) = \dot{u}_0 \qquad (1.2.2)$$

Eq.(1.2.1) is often divided by m to give the following standard form

$$\ddot{u} + 2\zeta\omega_n\dot{u} + \omega_n^2 u = f(t)/m = F(t) \qquad (1.2.3)$$

where, by comparing term by term, $\zeta = c/2m\omega_n$, which is called damping ratio (or damping factor) and $\omega_n = \sqrt{k/m}$, which is called natural frequency. When $\zeta = 1$, the system is critically damped. Therefore, 100% x ζ is the percentage of critical damping. Damping can also be defined by the logarithmic decrement δ, which is the natural logarithm of the ratio of any two successive amplitudes. The relation between ζ and δ is given by $\zeta = 2\delta / \sqrt{(1 - \zeta^2)}$.

Eq.(1.2.3) subject to the initial conditions in Eq.(1.2.2) can be solved by taking Laplace transforms [1,2] on the following terms

$$L\{u(t)\} = \bar{u}(s)$$
$$L\{\dot{u}(t)\} = s\bar{u}(s) - u(0)$$
$$L\{\ddot{u}(t)\} = s^2\bar{u}(s) - su(0) - \dot{u}(0)$$
$$L\{F(t)\} = \bar{F}(s),$$

Eq.(1.2.3) becomes

$$[s^2\bar{u}(s) - su(0) - \dot{u}(0)] + 2\zeta\omega_n[s\bar{u}(s) - u(0)] + \omega_n^2\bar{u}(s) = \bar{F}(s)$$

or

$$\bar{u}(s) = [\bar{F}(s) + (s + 2\zeta\omega_n)u_0 + \dot{u}_0]/\Delta$$

where $u_0 = u(0)$, $\dot{u}_0 = \dot{u}(0)$,

and $\Delta = s^2 + 2\zeta\omega_n s + \omega_n^2 = (s + \zeta\omega_n)^2 + \omega_d^2$

in which ω_d is the damped natural frequency which is equal to $\omega_n\sqrt{1 - \zeta^2}$. By virtue of inverse tranforms,

$$L^{-1}\{1/\Delta\} = \exp(-\zeta\omega_n t)\sin(\omega_d t)/\omega_d$$

$$L^{-1}\{(s + \zeta\omega_n)/\Delta\} = \exp(-\zeta\omega_n t)\cos\omega_d t$$

and $$L^{-1}\{\bar{F}(s)/\Delta\} = \frac{1}{\omega_d}\int_0^t F(\tau)\exp[-\zeta\omega_n(t - \tau)]\sin\omega_d(t - \tau)d\tau$$

The solution can be expressed as

$$u(t) = \int_0^t F(\tau)h(t-\tau)d\tau + g(t)u_0 + h(t)\dot{u}_0 \qquad (1.2.4)$$

$$h(t) = \omega_d^{-1}\exp(-\zeta\omega_n t)\sin\omega_d t$$

$$g(t) = \exp(-\zeta\omega_n t)\cos\omega_d t + \zeta\omega_n\sin\omega_d t$$

The first term is usually called the Duhamel's integral or convolution integral, while the terms g(t) and h(t) are called transients because of the presence of $\exp(-\zeta\omega_n t)$ which is a decaying function of time. It should be noted that in Eq.(1.2.4) the condition for oscillatory solution is $\zeta < 1$, (Fig.1.2.2c). The critically-damped cases, i.e. $\zeta \geq 1$, will not be considered here (Figs. 1.2.2a and b).

The following are basic examples in vibration, and the readers are strongly recommended to derive the solutions as an exercise.

(i) Undamped harmonic vibration:

Forcing function, $f = f_0 \sin \nu t$

Damping coefficient, $c = 0$

The solution is given by

$$u = \frac{u_s\omega_n^2}{\omega_n^2 - \nu^2}\sin\nu t = \bar{u}\sin\nu t$$

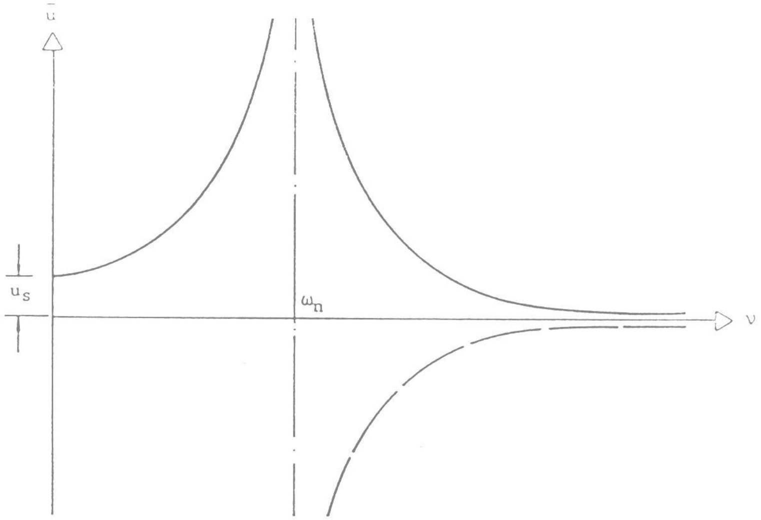


Fig. 1.2.3 Response amplitude against forcing frequency

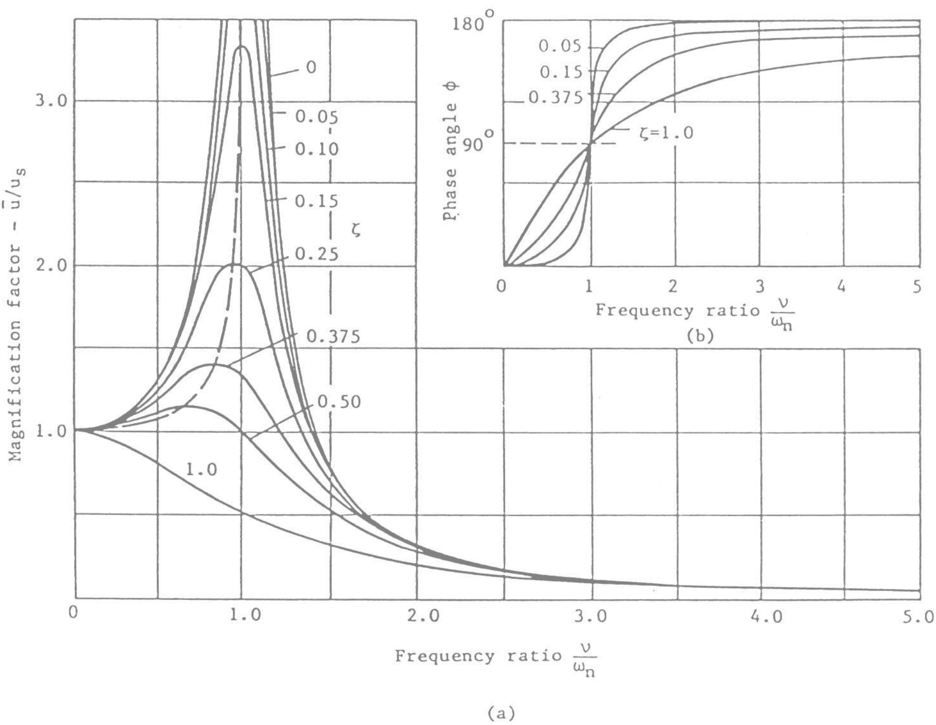


Fig. 1.2.4 Damped harmonic oscillation

where $u_s = f_0/k$ is the static response and $\bar{u} = u_s\omega_n^2/(\omega_n^2 - \nu^2)$ is the response amplitude. The response amplitude is plotted against forcing frequency ν in Fig. 1.2.3.

It should be noted that the amplitude $\bar{u}$ begins with u_s at $\nu = 0$ and increases monotonically when $0 < \nu < \omega_n$. When ν equals the natural frequency ω_n, $\bar{u}$ becomes infinitely large (if physically permissible) and the system is said to be in resonance. Immediately after this, $\bar{u}$ changes sign (i.e., $\bar{u}$ and f_0 are of opposite signs) and $\bar{u}$ is said to be 180° out of phase with f_0, as shown by the dotted curve. However, if only the absolute value of the amplitude is of interest, $\bar{u}$ is plotted with solid curve in the same quadrant.

(ii) Damped harmonic vibration:

Damping ratio $\zeta \neq 0$

Forcing function $f(t) = f_0 \sin \nu t$

The solution is

$$u = u \sin(\nu t - \phi) \tag{1.2.5}$$

where the amplitude $\bar{u} = u_s\omega_n/\sqrt{(\omega_n^2 - \nu^2)^2 + 4\zeta^2\nu^2\omega_n^2}$

the phase angle $\phi = \tan^{-1}[2\zeta\omega_n\nu/(\omega_n^2 - \nu^2)]$

and the static response $u_s = f_0/k$.

From the solution (Eq.(1.2.5)), it can be seen that for the steady state vibration, the response u is lagging behind the forcing function by an angle ϕ. $\bar{u}/u_s$ and ϕ are plotted against the frequency ratio in Figs. 1.2.4a and 1.2.4b respectively.

The dotted curve in Fig. 1.2.4a connects all the points of maximum amplitude. It can be obtained by making $d\bar{u}_{max}/d\nu = 0$, which yields $\nu = \omega_n\sqrt{1 - 2\zeta^2}$. Usually when ζ is small, maximum amplitude occurs when the excitation frequency is equal to the natural frequency of the system, $\nu = \omega_n$, i.e., at resonance. Therefore, the natural frequency is an important design factor for a dynamic system. Unfortunately, it is not a simple matter to determine the natural frequency of a damped system experimentally by means of the u-ν curve, because the peak response spreads over a finite range of ν. Nevertheless, since the phase angle ϕ is always 90° at resonance when $\nu = \omega_n$, it becomes simpler and more accurate to check the phase shift.

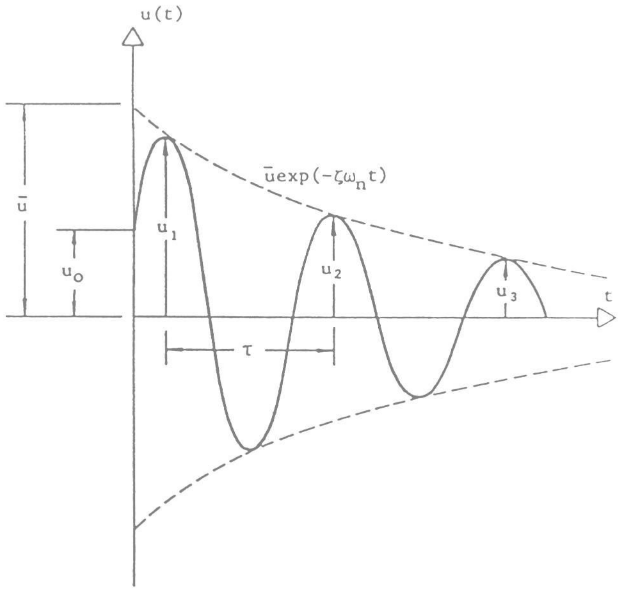
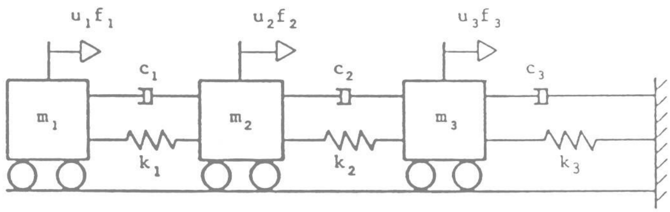
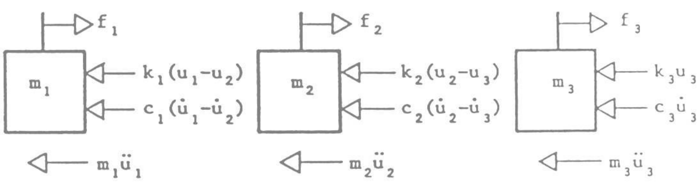


Fig. 1.2.5 Damped free vibration



(a) A multi-degree-of-freedom system



(b) Free body diagrams

Fig. 1.3.1 A three-degree-of-freedom system

(iii) Damped free vibration:

Damping ratio $\zeta \neq 0$

Forcing function $f(t) = 0$

Initial conditions : u_0, $\dot{u}(0) = \dot{u}_0$

The solution is

$$u(t) = \exp(-\zeta\omega_n t)[u_0 \cos\omega_d t + \frac{1}{\omega_d}(u_0\omega_n + \dot{u}_0)\sin\omega_d t]$$

$$= \exp(-\zeta\omega_n t)\bar{u}\sin(\omega_d t + \theta) \qquad (1.2.6)$$

where $\bar{u}^2 = u_0^2 + [(u_0\omega_n + \dot{u}_0)^2/\omega_d^2]$ and θ is an angle dependent on u_0 and $\dot{u}_0$. Eq.(1.2.6) is represented graphically in Fig. 1.2.5. Whenever $\sin(\omega_d t + \theta) = 1$, the response curve is tangential to the exponential envelope curves given by $\pm\bar{u}\exp(-\zeta\omega_n t)$. It should be noted that the tangents are not horizontal at the points of tangency and the points of tangency are not at the points of maximum amplitude exactly. However, the discrepancy is negligible and the amplitude at the point of tangency may be taken as the local maximum amplitude. The logarithmic decrement δ is defined as

$$\delta = \ln\frac{u_1}{u_2} = \ln\frac{\exp(-\zeta\omega_n t_1)}{\exp[-\zeta\omega_n(t_1 + \tau)]} = \ln(\exp\zeta\omega_n\tau) = \zeta\omega_n\tau$$

where τ is the period of damped oscillation and is equal to $2\pi/\omega_d$. Thus finally, the logarithmic decrement is given by $\delta = 2\pi\zeta/\sqrt{1-\zeta^2}$.

Other vibration response, such as the impulsive h(t), and the stepped g(t), can be derived similarly.

For a single-degree-of-freedom system, only one equation of motion is involved, and the decoupling of equations by means of eigenvectors is not required. On the other hand, for multi-degree-of-freedom system, the decoupling of equations is essential. The natural frequencies and modes are important factors which make the modal analysis successful.

1.3. VIBRATION OF MULTI-DEGREE-OF-FREEDOM SYSTEMS

1.3.1. *Introduction*. For a multi-degree-of-freedom system, it is more convenient to use matrix notation to describe the vibrational behaviour. Let $\{\mathbf{u}\}$ denote the displacements of the n mass points, with $u_i (i = 1,2,....,n)$ measured from their static equilibrium positions, then the equations of motion are in the form of