Random Matrix Models and Their Applications

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Alexander R. Its
Editors

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Random matrices arise from, and have important applications to, number theory, probability, combinatorics, representation theory, quantum mechanics, solid-state physics, quantum field theory, quantum gravity, and many other areas of physics and mathematics.

This volume of surveys and research results, based largely on lectures given at the Spring 1999 MSRI program of the same name, covers broad areas such as topologic and combinatorial aspects of random matrix theory; scaling limits, universalities, and phase transitions in matrix models; universalities for random polynomials; and applications to integrable systems. Its stress on the interaction between physics and mathematics will make it a welcome addition to the shelves of graduate students and researchers in both fields, as will its expository emphasis.

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Preface

This volume represents the most recent trends in the random matrix theory with a special emphasis on the exchange of ideas between physical and mathematical communities. The main topics include:

- random matrix theory and combinatorics
- scaling limits; universalities and phase transitions in matrix models
- topologico-combinatorial aspects of the theory of random matrix models
- scaling limit of correlations between zeros on complex and symplectic manifolds

Most contributions are based on talks and series of lectures given by the authors during the MSRI semester "Random Matrix Models and Their Applications" in Spring 1999, and have an expository or pedagogical style.

One of the basic ideas of the MSRI semester was to bring together the leading experts, both physicists and mathematicians, to discuss the latest results in the theory of matrix models and its applications. The book follows this line: it is divided roughly in half between physics and mathematics. The papers by physicists (G. Cicuta; Ph. Di Francesco; V. Kazakov; G. Mahoux, M. Mehta, J.-M. Normand; P. Zinn-Justin) give an overview of different physical problems in which the random matrix theory plays a decisive role, along with a rich variety of methods and ideas used to solve the problems. This includes enumeration of Feynman graphs on Riemann surfaces in the context of two-dimensional quantum gravity, spin systems on random surfaces, "meander problem" and random foldings, enumeration of knots and links, phase transitions and critical phenomena in random matrix models, interacting matrix models, etc.

The papers by mathematicians are devoted to recent breakthrough results on the statistics of longest increasing subsequence in random permutations and related problems of representation theory (J. Baik, E. Rains; A. Borodin, G. Olshanski; A. Its, C. Tracy, H. Widom; K. Johansson; A. Okounkov), universality of correlations between zeros on complex and symplectic manifolds (P. Bleher, B. Shiffman, S. Zelditch), applications of Hankel matrices to the theory of random matrices (E. Basor, Y. Chen, H. Widom), orthogonal polynomials (M. Ismail), interpolation properties of the ensembles of random matrices (P. Forrester, E. Rains), and integrable systems in the theory of random matrix models (J. Harnad and P. van Moerbeke). The paper of I. Kostov, I. Krichever, M. Mineev-Vainstein, P. Wiegmann, and A. Zabrodin is written by physicists and mathematicians and it relates conformal maps to integrable systems and matrix models.

We would like to express our gratitude to the MSRI Director, David Eisenbud, and the Deputy Directors, Hugo Rossi and Joe Buhler, for their help and support during the semester. We thank the series editor, Silvio Levy, for suggesting the publication of this volume and for his careful editing.

Our work in organizing the MSRI semester "Random Matrix Models" and the present volume was partially supported by the School of Science of Indiana University – Purdue University Indianapolis and through NSF Grants DMS-9970625 (Bleher) and DMS-9801608 (Its). We gratefully acknowledge this support.

Pavel Bleher
Alexander Its

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Symmetrized Random Permutations

JINHO BAIK AND ERIC M. RAINS

Abstract. Selecting N random points in a unit square corresponds to selecting a random permutation. Placing symmetry restrictions on the points, we obtain special kinds of permutations: involutions, signed permutations and signed involutions. We are interested in the statistics of the length (in numbers of points) of the longest up/right path in each symmetry type as the number of points increases to infinity. The limiting distribution functions are expressed in terms of a Painlevé II equation. In addition to the Tracy–Widom distributions of random matrix theory, we also obtain two new classes of distribution functions interpolating between the GOE and GSE, and between the GUE and GOE^2 Tracy–Widom distribution functions. Applications to random vicious walks and site percolation are also discussed

1. Introduction

Suppose that we are selecting n points, $p_1, p_2, \dots, p_n$, at random in a rectangle, say $R = [0,1] \times [0,1]$ (see Figure 1). We denote by π the configuration of n random points. With probability 1, no two points have same x-coordinates nor y-coordinates. An up/right path of π is a collection of points $p_{i_1}, p_{i_2}, \dots, p_{i_k}$ such that $x(p_{i_1}) < x(p_{i_2}) < \dots < x(p_{i_k})$ and $y(p_{i_1}) < y(p_{i_2}) < \dots < y(p_{i_k})$. The length of such a path is defined by the number of the points in the path. Now we denote by $l_n(\pi)$ the length of the longest up/right path of a random points configuration π.

As one can see from Figure 1, a configuration of n points gives rise to a permutation. For the example at hand, the corresponding permutation is $\binom{1\,2\,3\,4\,5}{5\,1\,3\,2\,4}$. Therefore we can identity random points in R and random permutations, and we use the same notation π. In this identification, $l_n(\pi)$ is the length of the longest increasing subsequence of a random permutation.

The longest increasing subsequence has been of great interest for a long time (see [AD2], [OR], [BDJ1], for example). Especially as $n \to \infty$, it is known that $\mathrm{E}(l_n) \sim 2\sqrt{n}$ [LS], [VK1; VK2] (also [AD1; Se; Jo2]) and $\mathrm{Var}(l_n) \sim c_0 n^{1/3}$

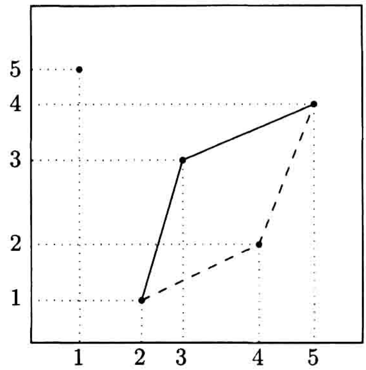

Figure 1. Random points in a rectangle.

[BDJ1] with some numerical constant $c_0 = 0.8132\cdots$. Moreover, the limiting distribution of l_n after proper scaling is obtained in [BDJ1] in terms of the solution to the Painlevé II equation (see Section 3 for precise statement). An interesting feature is that the limiting distribution function above is also the limiting distribution of the (scaled) largest eigenvalue of a random GUE matrix [TW1], the so-called "GUE Tracy–Widom distribution" F_2. In other words, properly centered and scaled, the length of the longest increasing subsequence of a random permutation behaves statistically for large n like the largest eigenvalue of a random GUE matrix. There have been many papers concerning the relations on combinatorics and random matrix theory: we refer the reader to [Re; Ge; Ke; Ra; Jo2; BDJ1; BDJ2; TW3; Bo; Jo1; Jo4; Ok; BOO; TW4; Jo3; BR1; BR2; ITW; St; PS2; PS1; Ba; BR3]. The purpose of this paper is to survey the analytic results of the recent papers [BR1; BR2] and discuss related topics.

In random matrix theory, three ensembles play important roles, GUE, GOE and GSE (see [Me], for example). Since random permutation is related to GUE, it would be interesting to ask which object in combinatorics is related to GOE and GSE. For this purpose, we consider symmetrized permutations. In terms of random points, 5 symmetry types of the rectangle R are considered, denoted by the symbols □, ⧄, ⧅, ⊡, and ⊠. Throughout this paper (and also in [BR1; BR2]), the symbol ⊛ is used to denote an arbitrary choice of the five possibilities above. Let $\delta = \{(t,t) : 0 \le t \le 1\}$, the diagonal line, and $\delta^t = \{(t, 1-t) : 0 \le t \le 1\}$, the anti-diagonal line. Consider the following random points selections:

□ select n points in R at random.

⧄ select n points in $R \setminus \delta$ and m points in δ at random, and add their reflection images about δ.

⧅ select n points in $R \setminus \delta^t$ and m points in δ^t at random, and add their reflection images about δ^t.

⊡ select n points at random in R, and add their rotational images about the center $(1/2, 1/2)$.

$\boxtimes$ select n points in $R \setminus \delta$, m_+ points in δ and m_- points in δ^t at random, and add their reflection images about both δ and δ^t.

Define the map ι on S_n by $\iota(x) = n + 1 - x$. Let $\mathrm{fp}(\pi)$ denote the number of points satisfying $\pi(x) = x$, and $\mathrm{fpi}(\pi)$ denote the number of points satisfying $\pi(x) = \iota(x)$ (fpi represents negated points: see Remark 3 below). Each of the processes above corresponds to picking a random permutation from each of the following ensembles:

$$\begin{aligned}
S_n^{\square} &= S_n, \\
S_{n,m}^{\boxslash} &= \{\pi \in S_{2n+m} : \pi = \pi^{-1},\ \mathrm{fp}(\pi) = m\}, \\
S_{n,m}^{\boxbslash} &= \{\pi \in S_{2n+m} : \pi = \iota\pi^{-1}\iota,\ \mathrm{fpi}(\pi) = m\}, \\
S_n^{\boxdot} &= \{\pi \in S_{2n} : \pi = \iota\pi\iota\}, \\
S_{n,m_+,m_-}^{\boxtimes} &= \{\pi \in S_{4n+2m_++2m_-} : \pi = \pi^{-1},\ \pi = \iota\pi^{-1}\iota,\ \mathrm{fp}(\pi) = 2m_+,\ \mathrm{fpi}(\pi) = 2m_-\}.
\end{aligned}$$

We denote the length of the longest increasing subsequence (equivalently, the longest up/right path) of π in each of the ensemble respectively by

$$L_n^{\square}, \quad L_{n,m}^{\boxslash}, \quad L_{n,m}^{\boxbslash}, \quad L_n^{\boxdot}, \quad L_{n,m_+,m_-}^{\boxtimes}.$$

REMARK 1. The map $\pi \mapsto \iota^{-1}\pi$ gives a bijection between $S_{n,m}^{\boxslash}$ and $S_{n,m}^{\boxbslash}$. Thus $L_{n,m}^{\boxbslash}$ has the same statistics with the length of the longest *decreasing* subsequence of a random involution with m fixed points taken from $S_{n,m}^{\boxslash}$. From the definition, $L_{n,m}^{\boxslash}$ is the random variable describing the length of the longest *increasing* subsequence of a random involution taken from the same ensemble.

REMARK 2. We may identify S_{2n} with the set of bijections from

$$\{-n, \dots, -2, -1, 1, 2, \dots, n\}$$

onto itself. In this identification, $S_n^{\boxdot}$ becomes the set of signed permutations; $\pi(x) = -\pi(-x)$. The longest increasing subsequence problem of a random signed permutation is considered in [TW3] and [Bo].

REMARK 3. Under the identification in Remark 2, $S_{n,m_+,m_-}^{\boxtimes}$ becomes the set of signed involutions with m_+ fixed points and m_- negated points (we call x a negated point if $\pi(x) = -x$.)

In this paper, we are interested in the statistics of $L^{\circledast}$ as $n \to \infty$. Especially for $\boxslash$, $\boxbslash$ and $\boxtimes$, we are interested in the cases when $m = [\sqrt{2n}\alpha]$ for $\boxslash$, $m = [\sqrt{2n}\beta]$ for $\boxbslash$, and $m_+ = [\sqrt{n}\alpha]$ and $m_- = [\sqrt{n}\beta]$ for $\boxtimes$ with fixed $\alpha, \beta \geq 0$ where $[k]$ denotes the largest integer less than or equal to k. Then for most cases, the expected values have the same asymptotics. Namely, if we set $N = n, 2n + m, 2n + m, 2n, 4n + 2m_+ + 2m_-$ for each of $\square, \boxslash, \boxbslash, \boxdot, \boxtimes$ case respectively, we have

$$\lim_{N\to\infty} \frac{\mathrm{E}(L^{\circledast})}{\sqrt{N}} = 2,$$

when $0 \leq \alpha \leq 1$ and $\beta \geq 0$ are fixed for ⧄, ⧅ and ⊠. When $\alpha > 1$, we have different expected value in the limit (see Section 3.)

On the other hand, the variance behaves asymptotically like $c_0 N^{1/3}$ but now with different constant c_0 depending on the symmetry type. It is because each symmetry type has different limiting distribution: $L_n^{\square}$ has GUE fluctuation, $L_{n,m}^{⧅}$ GOE fluctuation and $L_n^{⊡}$ GUE^2 fluctuation (see Section 3 below for precise statements). Here GUE^2 denotes the statistics of a superimposition of eigenvalues of two random GUE matrices. Similarly for GOE^2. The cases of ⧄ and ⊠ show more interesting features. For ⧄, the limiting distribution function changes depending on the value of $\alpha = m/\sqrt{2n}$. The fluctuation is GSE when $\alpha < 1$, GOE when $\alpha = 1$ and Gaussian when $\alpha > 1$. By taking suitable scaling limit $\alpha \to 1$, we can find a certain smooth transition between GSE and GOE. For ⊠, the value $\alpha = m_+/\sqrt{n}$ determines the limiting distribution ; the value m_- plays no role in the transition. The fluctuation is GUE when $\alpha < 1$, GOE^2 when $\alpha = 1$, and Gaussian when $\alpha > 1$.

In Section 2, we define the Tracy–Widom distributions for GUE, GOE and GSE as well as new classes of distribution functions describing the transition around $\alpha = 1$. Main results are stated in Section 3, and Section 4 includes some applications and the related problems. Most of the results in this article are taken from [BR1; BR2]. Theorems 3.1 and 3.5 for □ were first proved in [BDJ1], and Theorem 3.1 for ⊡ was first obtained in [TW4; Bo]. The only new result is Theorem 4.2.

2. Tracy–Widom Distribution Functions

Let $u(x)$ be the solution of the Painlevé II (PII) equation,

$$u_{xx} = 2u^3 + xu, \tag{2–1}$$

with the boundary condition

$$u(x) \sim -\operatorname{Ai}(x) \quad \text{as} \quad x \to +\infty, \tag{2–2}$$

where Ai is the Airy function. The proof of the (global) existence and the uniqueness of the solution was first established in [HM]: the asymptotics as $x \to -\infty$ are (see [HM; DZ2], for example)

$$u(x) = -\operatorname{Ai}(x) + O\left(\frac{e^{-(4/3)x^{3/2}}}{x^{1/4}}\right), \qquad \text{as } x \to +\infty, \tag{2–3}$$

$$u(x) = -\sqrt{\frac{-x}{2}}\left(1 + O\left(\frac{1}{x^2}\right)\right), \qquad \text{as } x \to -\infty. \tag{2–4}$$

Recall that $\mathrm{Ai}(x) \sim \dfrac{e^{-(2/3)x^{3/2}}}{2\sqrt{\pi}x^{1/4}}$ as $x \to +\infty$. Define

$$v(x) := \int_{\infty}^{x} u(s)^2\, ds, \tag{2–5}$$

so that $v'(x) = u(x)^2$.

We introduce the Tracy–Widom distributions. (Note that $q := -u$, which Tracy and Widom used in their papers, solves the same differential equation with the boundary condition $q(x) \sim +\mathrm{Ai}(x)$ as $x \to \infty$.)

DEFINITION (TRACY–WIDOM DISTRIBUTION FUNCTIONS). Set

$$F(x) := \exp\left(\frac{1}{2}\int_x^{\infty} v(s)\, ds\right) = \exp\left(-\frac{1}{2}\int_x^{\infty} (s-x)u(s)^2\, ds\right),$$
$$E(x) := \exp\left(\frac{1}{2}\int_x^{\infty} u(s)\, ds\right),$$

and set

$$F_2(x) := F(x)^2 = \exp\left(-\int_x^{\infty} (s-x)u(s)^2\, ds\right),$$
$$F_1(x) := F(x)E(x) = F_2(x)^{1/2} e^{\frac{1}{2}\int_x^{\infty} u(s)\, ds},$$
$$F_4(x) := F(x)\big(E(x)^{-1} + E(x)\big)/2 = F_2(x)^{1/2}\, \frac{e^{-\frac{1}{2}\int_x^{\infty} u(s)\, ds} + e^{\frac{1}{2}\int_x^{\infty} u(s)\, ds}}{2}.$$

In [TW1; TW2], Tracy and Widom proved that under proper centering and scaling, the distribution of the largest eigenvalue of a random GUE/GOE/GSE matrix converges to $F_2(x)$ / $F_1(x)$ / $F_4(x)$ as the size of the matrix becomes large. We note that from the asymptotics (2–3) and (2–4), for some positive constant c,

$$F(x) = 1 + O\big(e^{-cx^{3/2}}\big) \quad \text{as } x \to +\infty, \tag{2–6}$$
$$E(x) = 1 + O\big(e^{-cx^{3/2}}\big) \quad \text{as } x \to +\infty, \tag{2–7}$$
$$F(x) = O\big(e^{-c|x|^3}\big) \quad \text{as } x \to -\infty, \tag{2–8}$$
$$E(x) = O\big(e^{-c|x|^{3/2}}\big) \quad \text{as } x \to -\infty. \tag{2–9}$$

Hence in particular, $\lim_{x\to+\infty} F_\beta(x) = 1$ and $\lim_{x\to-\infty} F_\beta(x) = 0$, $\beta = 1, 2, 4$. Monotonicity of $F_\beta(x)$ follows from the fact that $F_\beta(x)$ is the limit of a sequence of distribution functions. Therefore $F_\beta(x)$ is indeed a distribution function.

As indicated in Introduction, we need new classes of distribution functions to describe the phase transitions from GSE to GOE and from GUE to GOE^2. First we consider the Riemann–Hilbert problem (RHP) for the Painlevé II equation

[FN; JMU]. Let Γ be the real line $\mathbb{R}$, oriented from $+\infty$ to $-\infty$. Let $m(\cdot\,;x)$ be the solution of the following RHP:

$$\begin{cases} m(z;x) & \text{is analytic in } z\in\mathbb{C}\setminus\Gamma, \\ m_+(z;x)=m_-(z;x)\begin{pmatrix} 1 & -e^{-2i(\frac{4}{3}z^3+xz)} \\ e^{2i(\frac{4}{3}z^3+xz)} & 0\end{pmatrix} & \text{for } z\in\Gamma, \\ m(z;x)=I+O(\frac{1}{z}) & \text{as } z\to\infty. \end{cases} \tag{2–10}$$

Here $m_+(z;x)$ and m_- are the limits of $m(z';x)$ as $z'\to z$ from the left and right of the contour Γ: $m_\pm(z;x)=\lim_{\varepsilon\downarrow 0} m(z\mp i\varepsilon;x)$. Relation (2–10) corresponds to the RHP for the PII equation with the special monodromy data $p=-q=1, r=0$ (see [FN; JMU], also [FZ; DZ2]). In particular if the solution is expanded at $z=\infty$,

$$m(z;x)=I+\frac{m_1(x)}{z}+O\Big(\frac{1}{z^2}\Big), \qquad \text{as } z\to\infty, \tag{2–11}$$

we have

$$2i(m_1(x))_{12}=-2i(m_1(x))_{21}=u(x),$$
$$2i(m_1(x))_{22}=-2i(m_1(x))_{11}=v(x),$$

where $u(x)$ and $v(x)$ are defined in Equations (2–1) to (2–5). Therefore the Tracy–Widom distributions above are expressed in terms of the residue at ∞ of the solution to the RHP (2–10). It is noteworthy that the new distributions below which interpolate the Tracy–Widom distributions require additional information of the solution of RHP.

Definition. Let $m(z;x)$ be the solution of RHP (2–10) and denote by $m_{jk}(z;x)$ the (jk)-entry of $m(z;x)$. For $w>0$, define

$$F^{\boxslash}(x;w):=F(x)$$
$$\times\Big(\big(m_{22}(-iw;x)-m_{12}(-iw;x)\big)E(x)^{-1}+\big(m_{22}(-iw;x)+m_{12}(-iw;x)\big)E(x)\Big)/2,$$

and for $w<0$, define

$$F^{\boxslash}(x;w):=e^{\frac{8}{3}w^3-2xw}F(x)$$
$$\times\Big(\big(-m_{21}(-iw;x)+m_{11}(-iw;x)\big)E(x)^{-1}-\big(m_{21}(-iw;x)+m_{11}(-iw;x)\big)E(x)\Big)/2.$$

Also define

$$F^{\boxtimes}(x;w):=m_{22}(-iw;x)F_2(x), \qquad w>0,$$
$$F^{\boxtimes}(x;w):=-e^{\frac{8}{3}w^3-2xw}m_{21}(-iw;x)F_2(x), \qquad w<0.$$

First $F^{\boxslash}(x;w)$ and $F^{\boxtimes}(x;w)$ are real from Lemma 2.1(i) below. Note that $F^{\boxslash}(x;w)$ and $F^{\boxtimes}(x;w)$ are continuous at $w=0$ since at $z=0$, the jump condition of the RHP (2–10) implies

$$(m_{12})_+(0;x)=-(m_{11})_-(0;x),$$
$$(m_{22})_+(0;x)=-(m_{21})_-(0;x).$$

In fact, $F^{\boxslash}(x;w)$ and $F^{\boxtimes}(x;w)$ are entire in $w \in \mathbb{C}$ from the RHP (2–10).

From (2–6)–(2–9) and Lemma 2.1(ii) below, we see that

$$\lim_{x\to+\infty} F^{\boxslash}(x;w), F^{\boxtimes}(x;w) = 1, \qquad \lim_{x\to-\infty} F^{\boxslash}(x;w), F^{\boxtimes}(x;w) = 0$$

for any fixed $w \in \mathbb{R}$. Also Theorem 3.3 below shows that $F^{\boxslash}(x;w)$ and $F^{\boxtimes}(x;w)$ are limits of distribution functions, implying that they are monotone in x. Therefore, $F^{\boxslash}(x;w)$ and $F^{\boxtimes}(x;w)$ are indeed distribution functions for each $w \in \mathbb{R}$.

We close this section summarizing some properties of $m(-iw;x)$ in the following lemma. In particular the lemma implies that $F^{\boxslash}(x;w)$ interpolates between $F_4(x)$ and $F_1(x)$, and $F^{\boxtimes}(x;w)$ interpolates between $F_2(x)$ and $F_1(x)^2$ (see Corollary 2.2).

LEMMA 2.1. *Let* $\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, *and set* $[a,b] = ab - ba$.

(i) *For real* w, $m(-iw;x)$ *is real.*

(ii) *For fixed* $w \in \mathbb{R}$, *we have*

$$m(-iw;x) = \left(I + e^{-cx^{3/2}}\right)\begin{pmatrix} 1 & -e^{\frac{8}{3}w^3-2xw} \\ 0 & 1 \end{pmatrix}, \qquad w>0,\ x\to+\infty,$$

$$m(-iw;x) = \left(I + e^{-cx^{3/2}}\right)\begin{pmatrix} 1 & 0 \\ -e^{-\frac{8}{3}w^3+2xw} & 1 \end{pmatrix}, \qquad w<0,\ x\to+\infty,$$

$$m(-iw;x) \sim \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} e^{(-\frac{4}{3}w^3+xw)\sigma_3} e^{(\frac{\sqrt{2}}{3}(-x)^{3/2}+\sqrt{2}w^2(-x)^{1/2})\sigma_3}, \qquad w>0,\ x\to-\infty,$$

$$m(-iw;x) \sim \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} e^{(-\frac{4}{3}w^3+xw)\sigma_3} e^{(-\frac{\sqrt{2}}{3}(-x)^{3/2}-\sqrt{2}w^2(-x)^{1/2})\sigma_3}, \qquad w<0,\ x\to-\infty.$$

(iii) *For any* x, *we have*

$$\lim_{w\to0^+} m(-iw;x) = \lim_{w\to0^-} \sigma_1 m(-iw;x)\sigma_1 = \begin{pmatrix} \frac{1}{2}\left(E(x)^2+E(x)^{-2}\right) & -E(x)^2 \\ \frac{1}{2}\left(-E(x)^2+E(x)^{-2}\right) & E(x)^2 \end{pmatrix}. \tag{2–12}$$

(iv) *For fixed* $w \in \mathbb{R}\setminus\{0\}$, $m(-iw;x)$ *solves the differential equation*

$$\frac{d}{dx}m = w[m,\sigma_3] + u(x)\sigma_1 m,$$

where $u(x)$ *is the solution of the PII equation* (2–1), (2–2).

COROLLARY 2.2. *We have*

$$F^{\boxslash}(x;0) = F_1(x), \qquad \lim_{w\to\infty} F^{\boxslash}(x;w) = F_4(x), \qquad \lim_{w\to-\infty} F^{\boxslash}(x;w) = 0,$$

$$F^{\boxtimes}(x;0) = F_1(x)^2, \qquad \lim_{w\to\infty} F^{\boxtimes}(x;w) = F_2(x), \qquad \lim_{w\to-\infty} F^{\boxtimes}(x;w) = 0.$$

PROOF. The values at $w=0$ follow from (2–12). For $w\to\pm\infty$, note that from the RHP (2–10), we have $\lim_{z\to\infty} m(z;x) = I$. □

3. Main Results

As in the Introduction, let N denote $n, 2n+m, 2n+m, 2n, 4n+2m_+ + 2m_-$ for each of $\square, \boxslash, \boxbslash, \boxdot, \boxtimes$ case respectively. We scale the random variables: for permutations and involutions,

$$\chi_n^{\square} = \frac{L_n^{\square} - 2\sqrt{N}}{N^{1/6}}, \qquad \chi_{n,m}^{\boxslash} = \frac{L_{n,m}^{\boxslash} - 2\sqrt{N}}{N^{1/6}}, \qquad \chi_{n,m}^{\boxbslash} = \frac{L_{n,m}^{\boxbslash} - 2\sqrt{N}}{N^{1/6}},$$

and for signed permutations and signed involutions,

$$\chi_n^{\boxdot} = \frac{L_n^{\boxdot} - 2\sqrt{N}}{2^{2/3}N^{1/6}}, \qquad \chi_{n,m_+,m_-}^{\boxtimes} = \frac{L_{n,m_+,m_-}^{\boxtimes} - 2\sqrt{N}}{2^{2/3}N^{1/6}}.$$

All the results in this section are taken from [BR2] which utilizes the algebraic work of [BR1].

First, we state the results for random permutations and random signed permutations. The result for random permutations was first obtained in [BDJ1], and the result for random signed permutations in [TW4; Bo].

THEOREM 3.1. *For fixed $x \in \mathbb{R}$,*

$$\lim_{n\to\infty} \Pr\big(\chi_n^{\square} \le x\big) = F_2(x),$$
$$\lim_{n\to\infty} \Pr\big(\chi_n^{\boxdot} \le x\big) = F_2(x)^2.$$

For the involution cases, we have the following limits.

THEOREM 3.2. *For each fixed α and β, and for fixed $x \in \mathbb{R}$, we have: for $\boxslash$,*

$$\lim_{n\to\infty} \Pr\big(\chi_{n,[\sqrt{2n}\alpha]}^{\boxslash} \le x\big) = F_4(x), \qquad 0 \le \alpha < 1,$$
$$\lim_{n\to\infty} \Pr\big(\chi_{n,[\sqrt{2n}]}^{\boxslash} \le x\big) = F_1(x),$$
$$\lim_{n\to\infty} \Pr\big(\chi_{n,[\sqrt{2n}\alpha]}^{\boxslash} \le x\big) = 0, \qquad \alpha > 1;$$

for $\boxbslash$,

$$\lim_{n\to\infty} \Pr\big(\chi_{n,[\sqrt{2n}\beta]}^{\boxbslash} \le x\big) = F_1(x), \qquad \beta \ge 0;$$

and for $\boxtimes$,

$$\lim_{n\to\infty} \Pr\big(\chi_{n,[\sqrt{n}\alpha],[\sqrt{n}\beta]}^{\boxtimes} \le x\big) = F_2(x), \qquad 0 \le \alpha < 1,\ \beta \ge 0,$$
$$\lim_{n\to\infty} \Pr\big(\chi_{n,[\sqrt{n}],[\sqrt{n}\beta]}^{\boxtimes} \le x\big) = F_1(x)^2, \qquad \beta \ge 0,$$
$$\lim_{n\to\infty} \Pr\big(\chi_{n,[\sqrt{n}\alpha],[\sqrt{n}\beta]}^{\boxtimes} \le x\big) = 0, \qquad \alpha > 1,\ \beta \ge 0.$$

This theorem shows that for $\boxslash$ and $\boxtimes$, the limiting distributions differ depending on the value of α. As indicated earlier in the Introduction, as $\alpha \to 1$ at a certain rate, we obtain smooth transitions. From Corollary 2.2, the following results are consistent with Theorem 3.2.